International Journal of Pure and Applied Mathematics
Volume 82 No. 3 2013, 503-512
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu

ijpam.eu

DYNAMIC PROGRAMMING IN STRUCTURAL AND PARAMETRIC OPTIMIZATION

E.M. Strakhov

Institute of Mathematics, Economics and Mechanics
Department of Optimal Control and Economical Cybernetics
Odessa I.I. Mechnikov National University
Dvoryans'ka st., 2, 65082, Odessa, UKRAINE

Abstract: This paper deals with substantiation of dynamic programming method for a structural and parametric optimization problem with floating switching points.

AMS Subject Classification: 49L20, 49K30, 93B12, 93B50
Key Words: structural optimization, parametric optimization, dynamic programming

1. Introduction

Structural and parametric optimization finds expanding applications in the mathematical modeling of complex engineering systems. Examples of optimal control problems for structured dynamical systems, such as accelerating and focusing systems, are given in [4, 8]. An important class of applications requires the control to belong to a region of admissible values, e.g. bang-bang control [11, 10]. Structural optimization on the basis of variational method is described in [4]. This approach involves construction of the gradient procedures with respect to unknown parameters or switching points. Dynamic programming method [3, 6, 9] is also widely used for the considered problems. Basically, dynamic programming approach divides into two main ways: the first is construction of an optimal structure using the Bellman's principle of

Received: December 4, 2012

optimality [2, 7], the second involves studying of solutions to Hamilton-Jacobi-Bellman equation [1, 15]. However, a large array of problems are now pending to be solved, e.g. construction of optimal structure for systems with feedback control. In the papers [12, 13] we proved the principle of optimality and obtained Bellman equations for structural and parametric optimization problems with a feedback control and fixed switching points. Paper [14] substantiates existence and uniqueness conditions of the solutions and sufficient optimality conditions for these problems. In this paper we give justification of dynamic programming method and prove sufficient optimality conditions for a structural and parametric optimization problem with floating switching points.

2. Main Results

Let us consider an optimal control problem

$$J\left(x,\, u\right) = \int_{t_0}^{T} f_0\left(x\left(t\right),\, u\left(t\right),\, t\right)\, dt + \Phi\left(x\left(T\right)\right) \;\rightarrow\; \inf, \tag{1}$$

$$\frac{dx}{dt} = f\left(x,\, u,\, t\right),\; x\left(t_0\right) = x_0, \tag{2}$$

where $t \in [t_0,\, T]$, $x\left(t\right) \in X\left(t\right) \subseteq \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ denotes a control vector.

Suppose $f_0\left(x,\, u,\, t\right)$, $\Phi\left(x\right)$ are continuous functions, vector function $f\left(x,\, u,\, t\right)$ satisfies the conditions: it is continuous with respect to x for any fixed $t \in [t_0,\, T]$; it is measurable with respect to t for any fixed $x \in X$; there exists an integrable function $p\left(t\right)$ such that $\|f\left(x,\, u,\, t\right)\| \leq p\left(t\right)$, $t \in [t_0,\, T]$; it satisfies Lipschitz condition with respect to x and u at the interval $[t_0,\, T]$.

Define the control in (1)–(2) as follows

$$u\left(t\right) = \Psi_r\left(t,\, b_r,\, x\left(t\right)\right), \quad t \in [t_i,\, t_{i+1}), \tag{3}$$

where by $t_0 < t_1 < ... < t_N = T$ we denote unknown switching points; the structures $\Psi_r(t, b_r, x)$, $r = 1, ..., p$ are specified, but their order is unknown; parameters $b_r \in M_r \subset \mathbb{R}^{k_r}$, $k_r \in \mathbb{N}$, $r = 1, ..., p$ for each structure are unknown.

Suppose the functions $\Psi_r\left(t,\, b_r,\, x\right)$, $r = 1, ..., p$ are continuous with respect to t and b_r at any interval $[t_i,\, t_{i+1})$ and satisfy Lipschitz condition with respect to x at $[t_0,\, T]$. Note that the conditions for existence and uniqueness of the solution to Cauchy problem in Caratheodory form is satisfied for the system (2) with the control functions (3) [5].

We say that the problem (1)–(3) is the structural and parametric optimization problem with the control from the structural class (3). The problem is to determine switching points and an order of structures with their parameters so that to minimize the cost function (1).

Denote by $\{\Psi_r\}$ an ordered set of structures $\Psi_r(t, b_r, x)$, $(r = 1, ..., p)$, $\{b_r\}$ sets of parameters of these structures respectively, and $\{t_j\}$ an ordered set of switching points t_j, $(j = 1, ..., N-1)$. Now we shall give the following definition.

Definition 1. Three objects $\left(\{\Psi_r^*\}, \{b_r^*\}, \left\{t_j^*\right\}\right)$ provide the infimum of the cost function (1) in the class of structural controls (3) under conditions (2) and phase constraints represents *the optimal control of the structural and parametric optimization problem* (1)–(3).

Suppose a solution of the problem (1)–(3) exists in the class (3) with finite number of switching points. Let us fix a moment $s \in [t_0, T]$. Consider an auxiliary problem

$$J_s(x, u) = \int_s^T f_0(x(t), u(t), t)\, dt + \Phi(x(T)) \ \rightarrow \ \inf, \tag{4}$$

$$\frac{dx}{dt} = f(x, u, t), \, x(s) = x^*(s), \tag{5}$$

where the functions f, f_0, and Φ satisfy the same conditions as they do in the problem (1)–(3). The control is given as follows

$$u(t) = \Psi_r(t, b_r, x(t)), \quad t \in [t_i, t_{i+1}) \subset [s, T], \tag{6}$$

where t_i are unknown switching points. As before, we specify set of structures $\{\Psi_r\}$, but parameters $b_r \in M_r \subset \mathbb{R}^{k_r}$, $(k_r \in \mathbb{N}, r = 1, ..., p)$ are unknown for each structure. The functions Ψ_r satisfy the same conditions as they do in the problem (1)–(3).

Theorem 2. (Bellman's Principle of Optimality) *Let the set of structures* $\{\tilde{\Psi}_k\}$, *their parameters* $\{\tilde{b}_k\}$ *and switching points* $\{\tilde{t}_j\}$ *be optimal in the problem* (4)–(6). *Then this set represents the optimal solution for the problem* (1)–(3) *at the interval* $t \in [s, T]$.

Proof. Assume the converse, i.e. an optimal control for the problem (4)–(6) is not optimal for the problem (1)–(3) at the interval $t \in [s, T]$. Thus, we have inequality $J_s(x^*, u^*) > J_s(\tilde{x}, \tilde{u})$, where $x^*(t)$, $u^*(t)$ – the optimal trajectory and the optimal control for the problem (1)–(3), and $\tilde{x}(t)$, $\tilde{u}(t)$ – the optimal trajectory and the optimal control for the problem (4)–(6) respectively.

Let us build a control

$$\hat{u}(t) = \begin{cases} \Psi_r^*(t, b_r^*, x^*(t)), \ t \in \left[t_i^*, t_{i+1}^*\right] \subset [t_0, s], \\ \tilde{\Psi}_k\left(t, \tilde{b}_k, \tilde{x}(t)\right), \ t \in \left[\tilde{t}_j, \tilde{t}_{j+1}\right] \subset (s, T]. \end{cases}$$

Corresponding trajectory for $\hat{u}(t)$ will be as follows

$$\hat{x}(t) = \begin{cases} x^*(t), & t \in [t_0, s], \\ \tilde{x}(t), & t \in (s, T]. \end{cases}$$

Thus,

$$\begin{aligned} J(\hat{x}, \hat{u}) = \int_{t_0}^{T} f_0(\hat{x}(t), \hat{u}(t), t)\, dt + \Phi(\hat{x}(T)) = \\ = \sum_i \int_{t_i^*}^{t_{i+1}^*} f_0(x^*(t), \Psi_r^*(t, b_r^*, x^*(t)), t)\, dt + \\ + \sum_j \int_{\tilde{t}_j}^{\tilde{t}_{j+1}} f_0\left(\tilde{x}(t), \tilde{\Psi}_k\left(t, \tilde{b}_k, \tilde{x}(t)\right), t\right) dt + \Phi(\tilde{x}(T)) < \\ < \sum_i \int_{t_i^*}^{t_{i+1}^*} f_0(x^*(t), \Psi_r^*(t, b_r^*, x^*(t)), t)\, dt + J_s(x^*, u^*) = \\ = J(x^*, u^*), \end{aligned}$$

and it means that the control $u^*(t)$ is not optimal. This contradiction proves the theorem. □

Definition 3. Let us define a function

$$\begin{aligned} B(z, s) = \inf_{\{b_r\}, \{\Psi_r\}, \{t_j\}} \Bigg\{ \int_s^{t_k} f_0(x(t), \Psi_{r_k}(t, b_{r_k}, x(t)), t)\, dt + \\ + \sum_{j=k}^{N-1} \int_{t_j}^{t_{j+1}} f_0\left(x(t), \Psi_{r_j}\left(t, b_{r_j}, x(t)\right), t\right) dt + \Phi(x(T)) \Bigg\}, \end{aligned} \tag{7}$$

on the solutions of the system (2) under initial condition $x(s) = z$. Denote by $t_k \in \{t_j\}$ the nearest switching point such that $t_k > s$, $r_k, r_j \in \{1, ..., p\}$, $z \in X(s)$, and $x(\cdot)$ the solution of the system (2) under an admissible control $u(t), t \in [s, T]$. The function $B(z, s)$ is called *Bellman function* of the problem (1)–(3).

According to this definition the value $B(x_0, t_0)$ is equal to the optimal value of the functional (1) for the problem (1)–(3) with fixed left endpoint $x(t_0) = x_0$, i.e. $B(x_0, t_0) = J(x^*, u^*)$.

From the definition of Bellman function it follows that

$$B(z, t) = \inf_{\{b_r\}, \{\Psi_r\}, \{t_j\}} \left\{ \int_t^{t_k} f_0(x(\tau), \Psi_{r_k}(\tau, b_{r_k}, x(\tau)), \tau)\, d\tau + \right.$$

$$+\sum_{j=k}^{N-1}\int_{t_j}^{t_{j+1}} f_0\left(x(\tau),\, \Psi_{r_j}\left(\tau,\, b_{r_j},\, x(\tau)\right),\, \tau\right)\, d\tau + \Phi\left(x(T)\right)\Bigg\} =$$

$$= \int_t^{t_k^*} f_0\left(x^*(\tau),\, \Psi_{r_k}^*\left(\tau,\, b_{r_k}^*,\, x^*(\tau)\right),\, \tau\right)\, d\tau +$$

$$+\sum_{j=k}^{N-1}\int_{t_j^*}^{t_{j+1}^*} f_0\left(x^*(\tau),\, \Psi_{r_j}^*\left(\tau,\, b_{r_j}^*,\, x^*(\tau)\right),\, \tau\right)\, d\tau + \Phi\left(x^*(T)\right) =$$

$$= \int_t^{t+\Delta t} f_0\left(x^*(\tau),\, \Psi_{r_k}^*\left(\tau,\, b_{r_k}^*,\, x^*(\tau)\right),\, \tau\right)\, d\tau +$$

$$+ \int_{t+\Delta t}^{t_k^*} f_0\left(x^*(\tau),\, \Psi_{r_k}^*\left(\tau,\, b_{r_k}^*,\, x^*(\tau)\right),\, \tau\right)\, d\tau +$$

$$+\sum_{j=k}^{N-1}\int_{t_j^*}^{t_{j+1}^*} f_0\left(x^*(\tau),\, \Psi_{r_j}^*\left(\tau,\, b_{r_j}^*,\, x^*(\tau)\right),\, \tau\right)\, d\tau + \Phi\left(x^*(T)\right).$$

Here $\Delta t > 0$ is such that $t + \Delta t < t_k^*$, and $\left(x^*(t),\, u^*(t)\right)$ is the solution of the problem (1)–(3). Using Bellman's principle of optimality, we get

$$\int_{t+\Delta t}^{t_k^*} f_0\left(x^*(\tau),\, \Psi_{r_k}^*\left(\tau,\, b_{r_k}^*,\, x^*(\tau)\right),\, \tau\right)\, d\tau +$$

$$+\sum_{j=k}^{N-1}\int_{t_j^*}^{t_{j+1}^*} f_0\left(x^*(\tau),\, \Psi_{r_j}^*\left(\tau,\, b_{r_j}^*,\, x(\tau)\right),\, \tau\right)\, d\tau + \Phi\left(x^*(T)\right) =$$

$$= \inf_{\{b_r\},\{\Psi_r\},\{t_j\}}\left\{\int_{t+\Delta t}^{t_k} f_0\left(x(\tau),\, \Psi_{r_k}\left(\tau,\, b_{r_k},\, x(\tau)\right),\, \tau\right)\, d\tau +\right.$$

$$\left.+\sum_{j=k}^{N-1}\int_{t_j}^{t_{j+1}} f_0\left(x(\tau),\, \Psi_{r_j}\left(\tau,\, b_{r_j},\, x(\tau)\right),\, \tau\right)\, d\tau + \Phi\left(x(T)\right)\right\} =$$

$$= B\left(x\left(t+\Delta t\right),\, t+\Delta t\right).$$

Therefore

$$B(z,\, t) = \inf_{\{b_r\},\{\Psi_r\},\{t_j\}}\left\{\int_t^{t+\Delta t} f_0\left(x(\tau),\, \Psi_{r_k}\left(\tau, b_{r_k},\, x(\tau)\right),\, \tau\right)\, d\tau + \right. \\ \left. + B\left(x\left(t+\Delta t\right),\, t+\Delta t\right)\right\}. \tag{8}$$

The equation (8) is called *integral Bellman equation.*

Let us consider the equation (8) now. Assume that Bellman function is continuously differentiable. Since $x(t) = z$ we have

$$\begin{gathered} B\left(x\left(t+\Delta t\right),\, t+\Delta t\right) = B\left(z,\, t\right) + \tfrac{\partial B(z,t)}{\partial t}\Delta t + \\ + \langle grad_z B(z,t), x(t+\Delta t) - z\rangle + o(\Delta t). \end{gathered} \tag{9}$$

Substituting (9) in (8), we get

$$B\left(z,\, t\right) = \inf_{\{b_r\},\{\Psi_r\},\{t_j\}} \left\{ \int_t^{t+\Delta t} f_0\left(x\left(\tau\right), \Psi_{r_k}\left(\tau, b_{r_k}, x\left(\tau\right)\right), \tau\right) d\tau + \right.$$

$$\left. + B\left(z,\, t\right) + \frac{\partial B(z,t)}{\partial t}\Delta t + \langle grad_z B(z,t), x(t+\Delta t) - z\rangle + o(\Delta t) \right\}.$$

It now follows that

$$\frac{\partial B(z,t)}{\partial t} + \inf_{\{b_r\},\{\Psi_r\},\{t_j\}} \left\{ \frac{1}{\Delta t} \int\limits_t^{t+\Delta t} f_0\left(x\left(\tau\right), \Psi_{r_k}\left(\tau, b_{r_k}, x\left(\tau\right)\right), \tau\right) d\tau + \right.$$

$$\left. + \left\langle grad_z B(z,t), \frac{x(t+\Delta t) - z}{\Delta t} \right\rangle + \frac{o(\Delta t)}{\Delta t} \right\} = 0. \tag{10}$$

Denote $G(t) = \int_{t_0}^{t} f_0\left(x\left(\tau\right),\, \Psi_{r_k}\left(\tau, b_{r_k},\, x\left(\tau\right)\right),\, \tau\right)\, d\tau$. Since

$$\begin{gathered} \frac{1}{\Delta t} \int\limits_t^{t+\Delta t} f_0\left(x\left(\tau\right), \Psi_{r_k}\left(\tau, b_{r_k}, x\left(\tau\right)\right), \tau\right) d\tau = \\ = \frac{1}{\Delta t}\left[G(t+\Delta t) - G(t)\right] \to \frac{d}{dt}G(t) \end{gathered}$$

under $\Delta t \to 0$ and $\frac{dG(t)}{dt} = f_0(z, \Psi_{r_k}\left(t, b_{r_k},\, z\right), t)$, we have

$$\frac{1}{\Delta t} \int\limits_t^{t+\Delta t} f_0\left(x\left(\tau\right), \Psi_{r_k}\left(\tau, b_{r_k}, x\left(\tau\right)\right), \tau\right) d\tau \to f_0(z, \Psi_{r_k}\left(t, b_{r_k}, z\right), t).$$

Thus, from (10) we obtain

$$\begin{gathered} \frac{\partial B(z,t)}{\partial t} + \inf_{\{b_r\},\{\Psi_r\},\{t_j\}} \left\{ \left\langle grad_z B(z,t), \frac{dx(t)}{dt} \right\rangle + \right. \\ \left. + f_0(z, \Psi_{r_k}\left(t, b_{r_k},\, z\right), t) \right\} = 0. \end{gathered}$$

Since $\frac{dx(t)}{dt} = f\left(z, \Psi_{r_k}\left(t, b_{r_k}, z\right), t\right)$, then

$$\frac{\partial B(z,t)}{\partial t} + \inf_{\{b_r\},\{\Psi_r\},\{t_j\}} \{\langle grad_z B(z,t), f\left(z, \Psi_{r_k}\left(t, b_{r_k},\ z\right), t\right)\rangle + \\ + f_0(z, \Psi_{r_k}\left(t, b_{r_k},\ z\right), t)\} = 0. \tag{11}$$

By the definition of Bellman function it follows that under $t = T$ we have

$$B(z,T) = \Phi(z). \tag{12}$$

The equation (11) is called *differential Hamilton-Jacobi-Bellman equation for the problem* (1)–(3).

Theorem 4. *Let function $B(z,t)$ be a piecewise continuously differentiable solution of the equation* (11) *under the initial condition* (12). *Suppose the optimal sets of structures $\Psi_s^*, s = 0, ..., N-1$, their parameters $b_s^*, s = 0, ..., N-1$ and switching points $t_1^*, ..., t_{N-1}^*$ satisfy the following conditions*

$$\langle grad_z B(z,t), f\left(z, \Psi_s^*\left(t, b_s^*,\ z\right), t\right)\rangle + f_0(z, \Psi_s^*\left(t, b_s^*,\ z\right), t) = \\ = \inf_{\{b_r\},\{\Psi_r\},\{t_j\}} \{\langle grad_z B(z,t), f\left(z, \Psi_s\left(t, b_s,\ z\right), t\right)\rangle + \\ + f_0(z, \Psi_s\left(t, b_s,\ z\right), t)\}. \tag{13}$$

Finally, suppose the control functions

$$u_s^*\left(z,t\right) = \Psi_s^*\left(t,\ b_s^*,\ z\right),\ t \in [t_s^*, t_{s+1}^*],\ s = 0,\ \dots,\ N-1$$

generate under $z = x$ the unique solution $x^*(\cdot)$ of the system (2). *Then the sets of structures $\Psi_s^*, s = 0, ..., N-1$, parameters $b_s^*, s = 0, ..., N-1$ and switching points $t_1^*, ..., t_{N-1}^*$ represent optimal control for the problem* (1)–(3).

Proof. Since under $z = x$ the sets $\Psi_s^*, s = 0, ..., N-1$, $b_s^*, s = 0, ..., N-1$, $t_1^*, ..., t_{N-1}^*$ are solutions of the problem (13) and for $B(z,t)$ we have (11), then

$$\frac{\partial B(x,t)}{\partial t} + \langle grad_x B(x,t), f\left(x, \Psi_s^*\left(t, b_s^*,\ x\right), t\right)\rangle + \\ + f_0(x, \Psi_s^*\left(t, b_s^*,\ x\right), t) = 0,\ t \in [t_s^*, t_{s+1}^*].$$

Therefore

$$\frac{dB(x^*(t),t)}{dt} + f_0(x^*(t), \Psi_s^*\left(t, b_s^*,\ x^*(t)\right), t) = 0,\ t \in [t_s^*, t_{s+1}^*].$$

Integrating the last expression from t_s^* to t_{s+1}^*, we obtain

$$\int_{t_s^*}^{t_{s+1}^*} f_0(x^*(t), \Psi_s^*\left(t, b_s^*,\ x^*(t)\right), t)dt = -\int_{t_s^*}^{t_{s+1}^*} \frac{dB(x^*(t),t)}{dt}dt =$$
$$= B(x^*(t_s^*), t_s^*) - B(x^*(t_{s+1}^*), t_{s+1}^*).$$

Then the optimal value of the functional (1) equals

$$J\left(u^*, x^*\right) = \int_{t_0}^{T} f_0\left(x^*(\tau), u^*(\tau), \tau\right) d\tau + \Phi\left(x^*(T)\right) =$$
$$= \sum_{s=0}^{N-1} \int_{t_s^*}^{t_{s+1}^*} f_0\left(x^*(\tau), \Psi_s^*\left(\tau, b_s^*,\ x^*\left(\tau\right)\right), \tau\right) d\tau + \Phi\left(x^*(T)\right) =$$
$$= \sum_{s=0}^{N-1} \left[B(x^*(t_s^*), t_s^*) - B(x^*(t_{s+1}^*), t_{s+1}^*)\right].$$

Note that $x^*(t_0) = x_0$ and $B\left(x^*\left(T\right), T\right) = \Phi\left(x^*(T)\right)$. Then we have

$$J\left(u^*, x^*\right) = B\left(x_0, t_0\right). \tag{14}$$

For random sets of structures $\Psi_s, s = 0, ..., N-1$, parameters $b_s, s = 0, ..., N-1$ and switching points $t_1, ..., t_{N-1}$ we can similarly obtain

$$J\left(u, x\right) \geq B\left(x_0, t_0\right). \tag{15}$$

Combining (14) and (15), we obtain $J\left(u, x\right) \geq J\left(u^*, x^*\right)$. This completes the proof of the theorem. □

3. Conclusion

In this paper we have obtained the following results for the structural and parametric problem with floating switching points: Bellman's principle of optimality is established, integral Bellman's equation and differential Hamilton-Jacobi-Bellman equation are developed, sufficient optimality conditions are justified. It enables us to construct numerical algorithms for the specific optimal control problems.

References

[1] M. Bardi, I. Capuzzo Dolcetta, *Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations*, Birkhäuser, Boston (1997).

[2] O.M. Bashnyakov, F.G. Garashchenko, V.V. Pichkur, *Practical Stability, Estimations, and Optimization*, Kyiv University, yiv (2008), In Ukrainian.

[3] D.P. Bertsekas, *Dynamic Programming and Stochastic Control*, Academic Press, New York, San Francisco, London (1976).

[4] B.N. Bublik, F.G. Garashchenko, N.F. Kirichenko, *Structural and Parametric Optimization and Stability of Beam Dynamics*, Naukova dumka, Kyiv (1985), In Russian.

[5] A.F. Filippov, *Differential Equations with Discontinuous Right-hand Sides*, Nauka, Moscow (1985), In Russian.

[6] W.H. Fleming, R.W. Rishel, *Deterministic and Stochastic Optimal Control*, Springer-Verlag, Berlin, Heidelberg, New York (1975).

[7] V.I. Gurman, M.Yu. Ukhin, Optimal Control Synthesis in Systems with an Unbounded Velocity Set, *Differentional Equations*, **41** (2005), 1557-1565.

[8] R.T. Haftka, Z. Gürdal, *Elements of Structural Optimization*, Kluwer Academic Publishers, Dordrecht, Boston, London (1991).

[9] A. Kurzhanski, I. Vályi, *Ellipsoidal Calculus for Estimation and Control*, Birkhäuser, Boston (1997).

[10] B.G. Penev, N.D. Christov, A Fast Time-optimal Control Synthesis Algorithm for a Class of Linear Systems, *Journal of Control Egineering ana Applied Informatics*, **7** (2006), 41-48.

[11] N.N. Moiseev, *Elements of Optimal System Theory*, Nauka, Moscow (1975), In Russian.

[12] V.V. Pichkur, E.M. Strakhov, Dynamic Programming Method in A Discrete Structural and Parametric Optimization Problem, *Journal of Computational and Applied Mathematics*, **3** (2010), 103-109, In Ukrainian.

[13] V.V. Pichkur, E.M. Strakhov, Application of Dynamic Programming Metgod to Structural and Parametric Optimization Problem with Fixed Switching Points, *Visnyk Odes'kogo Natsional'nogo Universytetu, Matematyka i Mekhanika*, **15** (2010), 94-102, In Ukrainian.

[14] V.V. Pichkur, E.M. Strakhov, Sufficient Optimality Conditions in a Structural and Parametric Optimization Problem, *Tavryheskiy Vestnik Informatiki i Matematiki*, **1** (2012), 77-87, In Ukrainian.

[15] A.I. Subbotin, Existence and Uniqueness Results for Hamilton-Jacobi Equations, *Nonlinear Analysis*, **16** (1991), 683-699.