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TUPLES WITH HEREDITARILY HYPERCYCLIC PROPERTY

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Abstract: In this paper we characterize some necessary and sufficient conditions for a tuple of operators to be hereditarily hypercyclic.

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1. Introduction

Definition 1.1. Let $\mathcal{T} = (T_1, T_2, ..., T_n)$ be an n-tuple of operators acting on an infinite dimensional Banach space X. We will let

$$\mathcal{F}_{\mathbf{T}} = \{{T_1}^{k_1}{T_2}^{k_2}, ..., {T_n}^{k_n} : k_i \geq 0,\ i = 1, ..., n\}$$

be the semigroup generated by $\mathcal{T}$. For $x \in X$, the orbit of x under the tuple $\mathcal{T}$ is the set $Orb(\mathcal{T}, x) = \{Sx : S \in \mathcal{F}_{\mathbf{T}}\}$. A vector x is called a hypercyclic vector for $\mathcal{T}$ if $Orb(\mathcal{T}, x)$ is dense in X and in this case the tuple $\mathcal{T}$ is called hypercyclic.

Note that if $T_1, T_2, ..., T_n$ are commutative bounded linear operators on a Banach space X, and $\{m_j(i)\}_j$, is a sequence of natural numbers for $i = 1, ..., n$, then we say $\{{T_1}^{m_j(1)}{T_2}^{m_j(2)}...{T_n}^{m_j(n)} : j \geq 0\}$ is hypercyclic if there exists $x \in X$ such that $\{{T_1}^{m_j(1)}{T_2}^{m_j(2)}...{T_n}^{m_j(n)}x : j \geq 0\}$ is dense in X.

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Definition 1.2. We say that a tuple $\mathcal{T} = (T_1, T_2, ..., T_n)$ is topologically transitive with respect to a tuple of nonnegative integer sequences

$$(\{k_{j(1)}\}_j, \{k_{j(2)}\}_j, ..., \{k_{j(n)}\}_j),$$

if for every nonempty open subsets U, V of X there exists $j_0 \in \mathbb{N}$ such that $T_1^{k_{j_0(1)}} T_2^{k_{j_0(2)}} ... T_n^{k_{j_0(n)}}(U) \cap V \neq \emptyset$. Also, we say that an n-tuple $\mathcal{T}$ is topologically transitive if it is topologically transitive with respect an n-tuple of nonnegative integer sequences.

Definition 1.3. We say that a pair $\mathcal{T} = (T_1, T_2, ...T_n)$ is hereditarily hypercyclic with respect to a tuple of nonnegative increasing sequences

$$(\{k_{j(1)}\}_j, \{k_{j(2)}\}_j, ..., \{k_{j(n)}\}_j)$$

of integers provided for all tuple of subsequences $(\{k_{j_i(1)}\}_i, \{k_{j_i(2)}\}_i, ..., \{k_{j_i(n)}\}_i)$ of $(\{k_{j(1)}\}_j, \{k_{j(2)}\}_j, ..., \{k_{j(n)}\}_j)$, the sequence $\{T_1^{k_{j_i(1)}} T_2^{k_{j_i(2)}} ... T_n^{k_{j_i(n)}} : i \geq 1\}$ is hypercyclic. We say that an n-tuple $\mathcal{T}$ is hereditarily hypercyclic, if it is hereditarily hypercyclic with respect to an n-tuple of nonnegative increasing sequences of integers.

A nice criterion namely the Hypercyclicity Criterion is used in the proof of our main theorem. It was developed independently by Kitai, Gethner and Shapiro. This criterion has been used to show that hypercyclic operators arise within the class of composition operators, weighted shifts and adjoints of multiplication operators. For some source on this topics see $[1 - 18]$. Note that, all operators in this paper are commutative operator.

2. Main Results

Note that a tuple $\mathcal{T} = (T_1, T_2, ..., T_n)$ is said to satisfy the Hypercyclicity Criterion if it holds in the hypothesis of the following theorem.

Theorem 2.1. (The Hypercyclicity Criterion) *Suppose that X is a separable infinite dimensional Banach space and $\mathcal{T} = (T_1, T_2, ..., T_n)$ be the n-tuple of operators T_1, T_2, ..., T_n acting on X. If there exist two dense subsets Y and Z in X, and strictly increasing sequences $\{m_{j(i)}\}_j$ for $i = 1, ..., n$ such that:*

1. $T_1^{m_{j(1)}} ... T_n^{m_{j(n)}} y \to 0$ for all $y \in Y$,

2. There exist a sequence of functions $\{S_j : Z \to X\}$ such that for every $z \in Z$, $S_j z \to 0$, and $T_1^{m_{j(1)}} ... T_n^{m_{j(n)}} S_j z \to z$,
then $\mathcal{T}$ is a hypercyclic tuple.

Theorem 2.2. *A tuple $\mathcal{T} = (T_1, T_2, ..., T_n)$ is hereditarily hypercyclic with respect to a tuple of increasing sequences of non-negative integers*

$$(\{k_{j(1)}\}_j, \{k_{j(2)}\}_j, ..., \{k_{j(n)}\}_j)$$

if and only if for all given any two open sets U, V, there exist some positive integers M_i such that $T_1^{m_1} T_2^{m_2} ... T_n^{m_n}(\mathcal{U}) \bigcap \mathcal{V} \neq \emptyset$ for any $m_i > M_i$ and $i = 1, ..., n$.

Proof. Let $\mathcal{T} = (T_1, T_2, ..., T_n)$ be hereditarily hypercyclic with respect to a tuple of increasing sequences of non-negative integers

$$(\{k_{j(1)}\}_j, \{k_{j(2)}\}_j, ..., \{k_{j(n)}\}_j).$$

Suppose that there exist some open sets U, V such that

$$T_1^{k_{j_i(1)}} T_2^{k_{j_i(2)}} ... T_n^{k_{j_i(n)}}(U) \bigcap V = \emptyset$$

for some subsequence $\{k_{j_i(m)}\}_i$ of $\{k_{j(m)}\}_j$ for $m = 1, ..., n$. Since the n-tuple $\mathcal{T} = (T_1, T_2, ..., T_n)$ is hereditarily hypercyclic with respect to

$$(\{k_{j(1)}\}_j, \{k_{j(2)}\}_j, ..., \{k_{j(n)}\}_j),$$

thus $\{T_1^{k_{j_i(1)}} T_2^{k_{j_i(2)}} ... T_n^{k_{j_i(n)}}\}$ is hypercyclic and so we get a contradiction.

Conversely, suppose that $\{k_{j_i(m)}\}_i$ is an arbitrary subsequences of $\{k_{j(m)}\}_j$ for $m = 1, ..., n$, and let U, V be open sets in X satisfying

$$T_1^{k_{j(1)}} T_2^{k_{j(2)}} T_n^{k_{j(n)}}(U) \bigcap V \neq \emptyset$$

for any $k_{j(m)} > M_m$ for $m = 1, ..., n$. Clearly, there exists i large enough such that $k_{j_i(m)} > M_m$ for $m = 1, ..., n$ and we have

$$T_1^{k_{j_i(1)}} T_2^{k_{j_i(2)}} ... T_n^{k_{j_i(n)}}(U) \bigcap V \neq \emptyset.$$

This implies that $\{T_1^{k_{j_i(1)}} T_2^{k_{j_i(2)}} ... T_n^{k_{j_i(n)}}\}$ is hypercyclic and so the n-tuple $\mathcal{T} = (T_1, T_2, ..., T_n)$ is indeed hereditarily hypercyclic with respect to the sequences $(\{k_{j(1)}\}_j, \{k_{j(2)}\}_j, ..., \{k_{j(n)}\}_j)$. This completes the proof. □

Theorem 2.3. *If a tuple $\mathcal{T} = (T_1, T_2, ..., T_n)$ is hereditarily hypercyclic with respect to a tuple of increasing sequences of non-negative integers*

$$(\{k_{j(1)}\}_j, \{k_{j(2)}\}_j, ..., \{k_{j(n)}\}_j)$$

with $sup_j(k_{(j+1)(i)} - k_{j(i)}) < \infty$ $(i = 1, ..., n)$, then the tuple $\mathcal{T}$ is hereditarily hypercyclic with respect to the n-tuple of entire sequences.

Proof. Let $\mathcal{T} = (T_1, T_2, ..., T_n)$ be hereditarily hypercyclic with respect to the tuple of sequences

$$(\{k_{j(1)}\}_j, \{k_{j(2)}\}_j, ..., \{k_{j(n)}\}_j)$$

such that $sup_j(k_{(j+1)(i)} - k_{j(i)}) < \infty$ $(i = 1, ..., n)$. We will show that $\mathcal{T}$ is hereditary hypercyclic with respect to the tuple of entire sequences. For this let U and V be two nonempty open sets in X. We will show that there exist integers $M_1, ..., M_n$ such that

$$T_1^{m_1} T_2^{m_2} ... T_n^{m_n}(U) \bigcap V \neq \emptyset$$

for any $m_i > M_i$ for $i = 1, ..., n$, which by Theorem 2.2 implies that $\mathcal{T}$ is indeed hereditarily hypercyclic with respect to the tuple of entire sequences. Put $M_i = sup_j(k_{(j+1)(i)} - k_{j(i)}) < \infty$ for $i = 1, ..., n$. For integers $0 \leq r_i \leq M_i$ $(i = 1, ..., n)$, set $U_{r_1, r_2, ..., r_n} = U$ and $V_{r_1, r_2, ..., r_n} = T_1^{-r_1} T_2^{-r_2} ... T_n^{-r_n}(V)$. Since $\mathcal{T}$ is hereditarily hypercyclic with respect to $(\{k_{j(1)}\}_j, \{k_{j(2)}\}_j, ..., \{k_{j(n)}\}_j)$, by Theorem 2.2, for all integers $0 \leq r_i \leq M_i$ $(i = 1, ..., n)$, there exists $N_0(r_i) \in N$ such that

$$T_1^{k_{j(1)}} T_2^{k_{j(2)}} ... T_n^{k_{j(n)}}(U_{r_1, r_2, ..., r_n}) \bigcap V_{r_1, r_2, ..., r_n} \neq \emptyset$$

for all $j(i) > N_0(r_i)$ and $i = 1, ..., n$. Let $M_o(i) = max\{N_0(r_i) : r_i = 0, 1, 2, ..., M_i\}$ for $i = 1, ..., n$. Then $M_0(i) = k_{j_0(i)}$ for some integer $0 \leq j_0(i) \leq M_i$ for $i = 1, ..., n$. Now we can show that

$$T_1^{m_1} T_2^{m_2} ... T_n^{m_n}(U) \bigcap V \neq \emptyset$$

for all $m_p > M_0(p)$ for $p = 1, ..., n$. In fact if $m_p > M_0(p)$, then there exist $j_p(p) > j_0(p)$ and $0 \leq r_p \leq M_p$ for $p = 1, ..., n$, such that $m_p = m_{j_p(p)} + r_p$ for $p = 1, ..., n$. Note that

$$T_1^{k_{j_1(1)}} T_2^{k_{j_2(2)}} ... T_n^{k_{j_n(n)}}(U_{r_1, r_2, ..., r_n}) \bigcap V_{r_1, r_2, ..., r_n} \neq \emptyset$$

for all $j_p(p) > j_0(p)$, $p = 1, ..., n$. Hence

$$T_1^{k_{j_1(1)}} ... T_n^{k_{j_n(n)}}(U) \bigcap T_1^{-r_1} ... T_n^{-r_n}(V) = T_1^{k_{j_1(1)}+r_1} ... T_n^{k_{j_n(n)}+r_n}(U) \bigcap V \neq \emptyset$$

and so

$$T_1^{m_1} ... T_n^{m_n}(U) \bigcap (V) = T_1^{m_{j_1(1)}+r_1} ... T_n^{m_{j_n(n)}+r_n}(U_{r_1, r_2, ..., r_n}) \bigcap V_{r_1, r_2, ..., r_n} \neq \emptyset$$

for all $m_p > M_0(p)$ and $p = 1, ..., n$. So $\mathcal{T}$ is hereditarily hypercyclic with respect to the n-tuple of entire sequences and the proof is complete. □

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