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A SELF-STARTING LINEAR MULTISTEP METHOD FOR DIRECT SOLUTION OF INITIAL VALUE PROBLEMS OF SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Abstract: A Multistep collocation technique is used in this paper for direct solution of second order ordinary differential equations. The approach is used to obtain Multiple Finite Differential Methods (MFDMs) which are combined as simultaneous numerical integrators to form some block methods which are self-starting. The Stability and Convergence of the block methods are investigated, and the methods are found to be zero-stable, consistent and hence convergent. The block methods derived are tested on standard electric circuit and mechanical problems to illustrate the accuracy and desirability of our new method.

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1. Introduction

A great many physical problems give rise to differential equations. Traditionally, solutions to these differential equations can be obtained using analytical methods. However, solutions to certain differential equations are very difficult by any means other than an approximate solution by the application of numerical methods.These methods are classified into two thus: One-step and Multistep methods.

The second order ordinary differential equation of the form

$$y'' = f(x, y, y') \tag{1.1}$$

$$y(a) = y_0, \quad y'(a) = z_0, \quad a \leq x \leq b$$

Where f is a continuous function, is conventionally solved by first reducing it to a system of order differential equations and then applying the various methods available for solving systems of first order Initial Value Problems (IVPs). This approach is extensively discussed in the literature and a few notable ones are Lambert [28], [29], [30], Fatunla [13-16], Jennings [25] and Jator [24]. Although there has been tremendous success with this approach, it has certain draw backs. For instance, computer programs associated with the methods are often complicated especially when incorporating subroutines to supply the starting values for the methods resulting in longer computer time and more computational work. In addition Vigo-Agular and Ramos [40], [41] stated that these methods do not utilize additional information associated with a specific ordinary differential equation, such as the oscillatory nature of the solution.

A lot of efforts have been devoted to the development of various methods for solving

$$y'' = f(x, y), \quad y(a) = y_0, \quad y'(a) = z_0, \tag{1.2}$$

directly without first reducing it to a system of first order Ordinary Differential Equations (ODEs). To mention but few are Twizell and Khaliq [25], Yusuph and Onumanyi [42], Simos [31], Fatunla [13-17], Henrici [21] and Lambert [28-30].

Several methods have also been proposed in the literature for solving (1.1) directly without first reducing it to equivalent first order systems. For instance, Hairer and Wanner [10] proposed Nystrom type methods and stated order conditions for determining the parameters of the methods. Linear Multistep Methods (LMMs) have been considered by Vigo-Aguiar and Ramos [40-41] and Awoyemi [7-9]. In Awoyemi [7-9] LMMs were proposed and implemented in a predictor-corrector scheme using the Taylor series algorithm to supply the starting values. Although, the implementation of the methods yielded good accuracy but the procedure is more costly to implement.

In this paper, we propose a self-starting LMM for direct solution of initial value problems of second oder ordinary differential equations and is applied as a block method. We also show that the block method is zero-stable and consistent, hence convergent.

2. The method of solution

In this section, the development of A Self-Starting Linear Multistep Method (ASSLMM) for direct solution of second order ODEs of the form (1.1) is presented. The proposed method is to solve the second order above directly without reducing it to system of first order ODEs. Linear multistep methods with continuous coefficients of the form

$$\bar{y}(x) = \sum_{j=0}^{k-1} \alpha_j(x) y_{n+j} + h^2 \sum_{j=0}^{k} \beta_j(x) f_{n+j} \tag{2.1}$$

is presented, where $x \in [a, b]$, the positive integer $k \geq 2$ denotes the step number. In the light of this, we seek a solution on $\tau_N : a = x_0 < x_{1/2} < x_{3/4} < x_N = b$,$h = x_{n+1} - x_n$,$n = 0, 1, \cdots, N$ where τ_N is a partition of $[a, b]$,h is the constant step-size of the partition,$\alpha_j(x)$ and $\beta_j(x)$ are the continuous coefficients of the method. We define $\alpha_j(x)$ and $\beta_j(x)$ as

$$\alpha_j(x) = \sum_{i=0}^{t+m-1} \alpha_{j,i+1} x^j ; j \in \{0, 1, ..., t-1\} \tag{2.2}$$

and

$$\beta_j(x) = \sum_{i-1}^{t+m-1} h^2 \beta_{j,i+1} x^j ; j \in \{0, 1, 2, ..., m-1\} \tag{2.3}$$

x_{n+j}: $j = 0, 1, 2, ..., t-1$ in (2.1) are $(0 \leq t \leq k)$ arbitrary chosen interpolation points taken from $\{x_n, ..., x_{n+k}\}$ and $\bar{x}_j$: $j = 0, 1, ..., m-1$ are the m collocation points belonging to $\{x_n, ..., x_{n+k}\}$.

We then construct a k-step multistep collocation method of the form (2.1) by imposing the following conditions.

$$\bar{y}(x_{n+j}) = y_{n+j}, j = 0, 1, 2, \ldots, t-1 \tag{2.4}$$

$$\bar{y}''(x_{n+j}) = f_{n+j}, j = 0, 1, 2, \dots, m-1 \tag{2.5}$$

To get $\alpha_j(x)$ and $\beta_j(x)$, Sirisena [33] and [35] arrived at a matrix equation of the form

$$DC = I \tag{2.6}$$

where I is the identity matrix of dimension $(t+m)\times(t+m)$ while D and C are matrices defined as

$$D = \begin{pmatrix} 1 & x_n & x_n^2 & \cdots\cdots & x_n^{t+m-1} \\ 1 & x_{n+\frac{1}{2}} & x_{n+\frac{1}{2}}^2 & \cdots & x_{n+\frac{1}{2}}^{t+m-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & x_{n+t-1} & x_{n+t-1}^2 & \cdots & x_{n+t-1}^{t+m-1} \\ 0 & 0 & 2 & \cdots & (t+m-2)\overline{x}_0^{t+m-3} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 2 & \cdots & (t+m-2)\overline{x}_{m-1}^{t+m-3} \end{pmatrix}. \tag{2.7}$$

The above (2.7) is the multistep collocation matrix of dimension $(t+m)\times(t+m)$ and

$$C = \begin{pmatrix} \alpha_{0,1} & \alpha_{1,1} & \cdots & \alpha_{t-1,1} & h^2\beta_{0,1} & \cdots & h^2\beta_{m-1,1} \\ \alpha_{0,2} & \alpha_{1,2} & \cdots & \alpha_{t-1,2} & h^2\beta_{0,2} & \cdots & h^2\beta_{m-1,2} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ \alpha_{0,t+m} & \alpha_{1,t+m} & \cdots & \alpha_{t-1,t+m} & h^2\beta_{0,t+m} & \cdots & h^2\beta_{m-1,t+m} \end{pmatrix}. \tag{2.8}$$

We define t and m as the number of interpolation points and the number of collocation points used respectively. The columns of the matrix $C = D^{-1}$ give the continuous coefficients $\alpha_j(x)$ *and* $\beta_j(x)$ with the following conditions:

$$\left\{t = 3, x_0 = x_n, x_1 = x_{n+\frac{1}{3}}, x_{\frac{3}{2}} = x_{n+\frac{3}{4}}\right\};$$

$$\left\{m = 3, \overline{x}_n = x_n, \overline{x}_1 = x_{n+\frac{1}{2}}, \overline{x}_{\frac{3}{2}} = x_{n+\frac{3}{4}}\right\}$$

and equation (2.1) becomes

$$\begin{aligned} \overline{y}(x) &= \alpha_0(x)y_n + \alpha_{\frac{1}{2}}(x)y_{n+\frac{1}{2}} + \alpha_{\frac{3}{4}}(x)y_{n+\frac{3}{4}} \\ &\quad + h^2\left[\beta_0(x)f_n + \beta_{\frac{1}{2}}(x)f_{n+\frac{1}{2}} + \beta_{\frac{3}{4}}(x)f_{n+\frac{3}{4}} + \beta_1(x)f_{n+1}\right]. \end{aligned} \tag{2.9}$$

Thus, using the elements of $C = D^{-1}$ equations (2.2) and (2.3) become,

$$\alpha_0(x) = \frac{1}{75h^5}[1536(x-x_n)^5 - 3840(x-x_n)^4h + 2560(x-x_n)^3h^2 - 406(x-x_n)h^4 + 75h^5]$$

$$\alpha_{\frac{1}{2}}(x) = \frac{1}{25h^5}[-1536(x-x_n)^5+3840(x-x_n)^4h-2560(x-x_n)^3h^2+306(x-x_n)h^4]$$

$$\alpha_{\frac{3}{4}}(x) = \frac{1}{75h^5}[3072(x-x_n)^5-7680(x-x_n)^4h+5120(x-x_n)^3h^2-512(x-x_n)h^4]$$

$$\beta_0(x) = \frac{1}{1200h^3}[-672(x-x_n)^5 + 1880(x-x_n)^4h - 1720(x-x_n)^3h^2 + 600(x-x_n)^2h^3 - 63(x-x_n)h^4]$$

$$\beta_{\frac{1}{2}}(x) = \frac{1}{600h^3}[2016(x-x_n)^5+4840(x-x_n)^4h-2960(x-x_n)^3h^2+261(x-x_n)h^4]$$

$$\beta_1(x) = \frac{1}{1200h^3}[96(x-x_n)^5 - 40(x-x_n)^4h - 40(x-x_n)^3h^2 + 9(x-x_n)h^4]$$

On substituting the above into (2.9), we obtained the continuous scheme as follows:

$$\begin{aligned}\bar{y}(x) = {} & \frac{1}{75h^5}[1536(x-x_n)^5-3840(x-x_n)^4h+2560(x-x_n)^3h^2-406(x-x_n)h^4 + 75h^5]y_n \\ & +\frac{1}{25h^5}[-1536(x-x_n)^5+3840(x-x_n)^4h-2560(x-x_n)^3h^2+306(x-x_n)h^4]y_{n+\frac{1}{2}} \\ & +\frac{1}{75h^5}[3072(x-x_n)^5-7680(x-x_n)^4h+5120(x-x_n)^3h^2-512(x-x_n)h^4]y_{n+\frac{3}{4}} \\ & +\frac{1}{1200h^3}[-672(x-x_n)^5+1880(x-x_n)^4h-1720(x-x_n)^3h^2+600(x-x_n)^2h^3 - 63(x-x_n)h^4]f_n \\ & +\frac{1}{600h^3}[2016(x-x_n)^5+4840(x-x_n)^4h-2960(x-x_n)^3h^2+261(x-x_n)h^4]f_{n+\frac{1}{2}} \\ & +\frac{1}{1200h^3}[96(x-x_n)^5-40(x-x_n)^4h-40(x-x_n)^3h^2+9(x-x_n)h^4]f_{n+1} \qquad (2.10)\end{aligned}$$

On evaluating (2.10) at $x = x_{n+1}$, the first derivative of (2.10) at $x = x_n$ and collocating at $x = x_{n+\frac{3}{4}}$, the following discrete schemes were obtained which constitute the block method:

$$y_{n+1} - 2y_{n+\frac{1}{2}} + y_n = \frac{h^2}{48}[f_n + 10f_{n+\frac{1}{2}} + f_{n+1}] \tag{2.11}$$

$$y_{n+\frac{3}{4}} - \frac{3}{2}y_{n+\frac{1}{2}} + \frac{1}{2}y_n = \frac{h^2}{384}[4f_n + 39f_{n+\frac{1}{2}} - 10f_{n+\frac{3}{4}} + 3f_{n+1}] \tag{2.12}$$

$$\frac{512}{75}y_{n+\frac{3}{4}} - \frac{306}{25}y_{n+\frac{1}{2}} + \frac{406}{75}y_n + hy'_n = \frac{h^2}{1200}[-63f_n + 522f_{n+\frac{1}{2}} + 9f_{n+1}] \tag{2.13}$$

where

$$f_n \equiv f(x_n, y_n, z_n), y'_n = z_n$$

$$f_{n+\frac{1}{2}} \equiv f(x_{n+\frac{1}{2}}, y_{n+\frac{1}{2}}, z_{n+\frac{1}{2}}), y'_{n+\frac{1}{2}} = z_{n+\frac{1}{2}}$$

$$f_{n+\frac{3}{4}} \equiv f(x_{n+\frac{3}{4}}, y_{n+\frac{3}{4}}, z_{n+\frac{3}{4}}), y'_{n+\frac{3}{4}} = z_{n+\frac{3}{4}}$$

and

$$f_{n+1} \equiv f(x_{n+1}, y_{n+1}, z_{n+1}), y'_{n+1} = z_{n+1}$$

Equation (2.11) is an improved popular two-step Numerov direct method for the special case of (1.1) when y' is not existing as in equation (1.2).

Equations (2.11), (2.12) and (2.13) constitute the members of a zero-stable block integrator of order four with $C_{p+2} = \begin{pmatrix} -\frac{1}{15360} \\ -\frac{21}{655360} \\ -\frac{3}{128000} \end{pmatrix}$.

On differentiating through (2.10) and then evaluate the resulting equation at $x = x_{n+1}$ to get

$$hy'_{n+1} = -\frac{406}{75}y_n + \frac{306}{25}y_{n+\frac{1}{2}} - \frac{512}{75}y_{n+\frac{3}{4}} + \frac{h^2}{1200}[137f_n + 1322f_{n+\frac{1}{2}} + 209f_{n+1}] \tag{2.14}$$

(2.14) is used for evaluating the derivative term in (2.13), is of order four and has error constant $C_6 = -\frac{427}{7200}$.

In order to handle the general case, i.e. when y' is not lacking as we have in equation (1.1), we consider the integration of the derivative IVP

$$y' = z, y(a) = y_0 \tag{2.15}$$

by considering a k-step interpolant defined in $[x_n, x_{n+k}]$ as follows

$$y_1 = C_0(x)y_{n+k-1} + h\sum_{r=1}^{k+1} C_r(x)z_{n+r-1} \tag{2.16}$$

where $h = x_{n+1} - x_n$ is a variable step-size and the continuous coefficients are assumed polynomials of the form

$$C_r(x) = \sum_{j=0}^{k+1} C_{rj}x^j \tag{2.17}$$

such that for step number $k \geq 1$

(i) $y(x_{n+k-1}) = y_{n+k-1}$

(ii) $y'(\bar{x}_j) = z_{n+j-1}, j = 1, \ldots, k+1$ (iii) $\bar{x}_1 = x_n, \bar{x}_2 = x_{n+1}, \ldots, \bar{x}_{k+1} = x_{n+k}$

Using $\{x_n, x_{n+\frac{1}{2}}, x_{n+\frac{3}{4}}, x_{n+1}\}$ as collocation points we have from (2.16) the following set of algebraic equations to solve for $\underline{C} = (c_1, c_2, c_3, c_{4,}c_5)^T$ constant column vector.

$$\underline{DC} = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 \\ 0 & 1 & 2x_{n+\frac{1}{2}} & 3x^2_{n+\frac{1}{2}} & 4x^3_{n+\frac{1}{2}} \\ 0 & 1 & 2x_{n+\frac{3}{4}} & 3x^2_{n+\frac{3}{4}} & 4x^3_{n+\frac{3}{4}} \\ 0 & 1 & 2x_{n+1} & 3x^2_{n+1} & 4x^3_{n+1} \end{bmatrix} \underline{C} = \underline{F} \tag{2.18}$$

$$\underline{F} = (y_{n+1}, z_n, z_{n+\frac{1}{2}}, z_{n+\frac{3}{4}}, z_{n+1})^T \tag{2.19}$$

The matrix inversion yields $\underline{C}$ uniquely to obtain

$$\bar{y}(x) = \alpha_0(x)y_n + \sum_{k=0}^{3} \beta_k(x)z_n \tag{2.20}$$

where D is invertible, $DC = I$ and elements of D^{-1} are:

$$D_{11} = 1, D_{21} = D_{31} = D_{41} = D_{51} = 0$$

$$D_{12} = -\frac{1}{6h^3}(6x_nh^3 + 13x_n^2h^2 + 12x_n^3h + 4x_n^4)$$

$$D_{22} = \frac{1}{3h^3}(3h^3 + 13x_nh^2 + 18x_n^2h + 8x_n^3)$$

$$D_{32} = -\frac{1}{6h^3}(13h^2 + 36x_nh + 24x_n^2)$$

$$D_{42} = \frac{2}{3h^3}(3h + 4x_n)$$

$$D_{52} = -\frac{2}{3h^3}$$

$$D_{13} = \frac{2}{3h^3}(9x_n^2h^2 + 14x_n^3h + 6x_n^4)$$

$$D_{23} = -\frac{4}{h^3}(3x_nh^2 + 7x_n^2h + 4x_n^3)$$

$$D_{33} = \frac{2}{h^3}(3h^2 + 14x_nh + 12x_n^2)$$

$$D_{43} = -\frac{4}{3h^3}(7h + 12x_n)$$

$$D_{53} = \frac{4}{h^3}$$

$$D_{14} = -\frac{16}{3h^3}(x_n^4 + 2x_n^3h + x_n^2h^2)$$

$$D_{24} = \frac{32}{3h^3}(x_nh^2 + 3x_n^2h + 2x_n^3)$$

$$D_{34} = -\frac{16}{3h^3}(h^2 + 6x_nh + 6x_n^2)$$

$$D_{44} = \frac{32}{3h^3}(2x_n + h)$$

$$D_{54} = -\frac{16}{3h^3}$$

$$D_{15} = \frac{1}{6h^3}(9x_n^2h^2 + 20x_n^3h + 12x_n^4)$$

$$D_{25} = -\frac{1}{h^3}(3x_nh^2 + 10x_n^2h + 8x_n^3)$$

$$D_{35} = \frac{1}{2h^3}(3h^2 + 20x_nh + 24x_n^2)$$

$$D_{45} = -\frac{2}{3h^3}(12x_n + 5h)$$

$D_{55} = \frac{2}{h^3}$ and $D_{55} = \frac{2}{h^3}$

Using the elements of D^{-1} above, we have

$$\alpha_0(x) = 1$$

$$\beta_0(x) = \frac{1}{6h^3}[-4(x-x_n)^4 + 12(x-x_n)^3h - 13(x-x_n)^2h^2 + 6(x-x_n)h^3]$$

$$\beta_{\frac{1}{2}}(x) = \frac{1}{3h^3}[12(x-x_n)^4 - 28(x-x_n)^3h + 18(x-x_n)^2h]$$

$$\beta_{\frac{3}{4}}(x) = \frac{1}{3h^3}[-16(x-x_n)^4 + 32(x-x_n)^3h - 16(x-x_n)^2h^2]$$

$$\beta_1(x) = \frac{1}{6h^3}[12(x-x_n)^4 - 20(x-x_n)^3h + 9(x-x_n)^2h^2]$$

Putting the above into (2.20) to get the continuous schemes

$$\begin{aligned}\bar{y}(x) = y_n &+ \frac{1}{6h^3}[-4(x-x_n)^4 + 12(x-x_n)^3h - 13(x-x_n)^2h^2 + 6(x-x_n)h^3]z_n \\ &+ \frac{1}{3h^3}[12(x-x_n)^4 - 28(x-x_n)^3h + 18(x-x_n)^2h]z_{n+\frac{1}{2}} \\ &+ \frac{1}{3h^3}[-16(x-x_n)^4 + 32(x-x_n)^3h - 16(x-x_n)^2h^2]z_{n+\frac{3}{4}} \\ &+ \frac{1}{6h^3}[12(x-x_n)^4 - 20(x-x_n)^3h + 9(x-x_n)^2h^2]z_{n+1} \end{aligned} \tag{2.21}$$

Now on evaluating (2.21) at $x = x_{n+\frac{1}{2}}, x = x_{n+\frac{3}{4}}, x = x_{n+1}$, we have

$$y_{n+\frac{1}{2}} = y_n + \frac{h}{96}[16z_n + 56z_{n+\frac{1}{2}} - 32z_{n+\frac{3}{4}} + 8z_{n+1}] \tag{2.22}$$

$$y_{n+\frac{3}{4}} = y_n + \frac{h}{1536}[252z_n + 1080z_{n+\frac{1}{2}} - 288z_{n+\frac{3}{4}} + 108z_{n+1}] \tag{2.23}$$

$$y_{n+1} = y_n + \frac{h}{6}[z_n + 4z_{n+\frac{1}{2}} + z_{n+1}] \tag{2.24}$$

Equations (2.22), (2.23) and (2.24) are of order four with the error constants $\begin{pmatrix} \frac{-31}{92160} \\ \frac{-51}{163840} \\ \frac{-1}{2880} \end{pmatrix}$.

On solving equations (2.22) - (2.24) simultaneously for $z_{n+\frac{1}{2}}, z_{n+\frac{3}{4}}$ and z_{n+1} give the following first derivative approximation schemes from

$$\begin{bmatrix} z_{n+\frac{1}{2}} \\ z_{n+\frac{3}{4}} \\ z_{n+1} \end{bmatrix} = A^{-1}R \tag{2.25}$$

where

$$A = \begin{pmatrix} 56 & -32 & 8 \\ 1080 & -288 & 108 \\ 4 & 0 & 1 \end{pmatrix}$$

and

$$R = \frac{1}{h}\begin{bmatrix} 96(y_{n+\frac{1}{2}} - y_n) - 16z_n \\ 1536(y_{n+\frac{3}{4}} - y_n) - 252z_n \\ 6(y_{n+1} - y_n) - z_n \end{bmatrix}$$

Thus, (2.25) becomes

$$z_{n+1} = \frac{1}{108h}(864y_{n+1} - 1536y_{n+\frac{3}{4}} + 864y_{n+\frac{1}{2}} - 192y_n) - \frac{z_n}{3} \tag{2.26}$$

$$z_{n+\frac{3}{4}} = \frac{1}{576h}(648y_{n+1} + 1536y_{n+\frac{3}{4}} - 2592y_{n+\frac{1}{2}} + 408y_n) + \frac{z_n}{8} \tag{2.27}$$

$$z_{n+\frac{1}{2}} = \frac{1}{432h}(-216y_{n+1} + 1536y_{n+\frac{3}{4}} - 864y_{n+\frac{1}{2}} - 456y_n) - \frac{z_n}{6} \tag{2.28}$$

Also, the order of equations (2.26) - (2.28) is four and having error constants of $\begin{pmatrix} \frac{-1}{90} \\ \frac{27}{160} \\ \frac{-9}{40} \end{pmatrix}$.

3. Stability Property

To analyze the methods for zero-stability, we normalize (2.11) – (2.13) and write them as a block method given by the matrix difference equation

$$A^0 Y_m = \sum_{i=1}^{k} A^i Y_{m-i} + h^2 \sum_{i=0}^{k} B^i F_{m-i} \tag{3.1}$$

h is a fixed mesh size within a block, $A^i, B^i, i = 0(1)k$ are $r \times r$ matrix coefficients, and A^0 is $r \times r$ identity matrix. Y_m, Y_{m-i} and F_{m-i} are vectors of numerical estimates described by

$$Y_m = (y_{n+1}, y_{n+2}, \cdots, y_{n+r})^T,$$

$$Y_{m-i} = (y_{n-r}, \cdots, y_{n+1}, y_n)^T,$$

$F_m = \begin{pmatrix} f_{n+1} \\ f_{n+2} \\ \vdots \\ f_{n+r} \end{pmatrix}$ and $F_{m-i} = (f_{n-r} \cdots, f_{n-r}, f_n)^T$ for $n = mr$, some integer of $m \geq 0$.

Thus, equations (2.11), (2.12) and (2.13) can be expressed in the form of (3.1) to give

$$\begin{pmatrix} -2 & 0 & 1 \\ \frac{-3}{2} & 1 & 0 \\ \frac{-306}{25} & \frac{512}{75} & 0 \end{pmatrix} \begin{pmatrix} y_{n+1/2} \\ y_{n+3/4} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -1/2 \\ 0 & 0 & -406/75 \end{pmatrix} \begin{pmatrix} y_{n-1} \\ y_{n-2} \\ y_n \end{pmatrix}$$

$$+h^2 \begin{pmatrix} \frac{10}{48} & 0 & \frac{1}{48} \\ \frac{39}{384} & \frac{-10}{384} & \frac{3}{384} \\ \frac{522}{1200} & 0 & \frac{9}{1200} \end{pmatrix} \begin{pmatrix} f_{n+1/2} \\ f_{n+3/4} \\ f_{n+1} \end{pmatrix}$$

$$+h^2 \begin{pmatrix} 0 & 0 & \frac{1}{48} \\ 0 & 0 & \frac{4}{384} \\ 0 & 0 & \frac{-63}{1200} \end{pmatrix} \begin{pmatrix} f_{n-1} \\ f_{n-2} \\ f_n \end{pmatrix}.$$

Here

$$A^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A^1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

$$B^0 = \begin{pmatrix} \frac{10}{48} & \frac{39}{384} & \frac{522}{1200} \\ 0 & \frac{-10}{384} & 0 \\ \frac{1}{48} & \frac{3}{384} & \frac{9}{1200} \end{pmatrix} \text{ and } B^1 = \begin{pmatrix} 0 & 0 & \frac{1}{48} \\ 0 & 0 & \frac{4}{384} \\ 0 & 0 & \frac{-63}{1200} \end{pmatrix}.$$

The first characteristic polynomial of the block method is given by

$$\rho(R) = \det(RA^0 - A^1) \tag{3.2}$$

On substituting the $A^0$and A^1into (3.2), we have

$$\rho(R) = \det \left[R \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right] = \det \begin{pmatrix} R & 0 & -1 \\ 0 & R & -1 \\ 0 & 0 & R-1 \end{pmatrix}$$

$=R^2(R-1) \Rightarrow R_1 = R_2 = 0$ or $R_3 = 1$

Following Fatunla [16-17], the block method (3.1) is zero-stable, since from (3.2), $\rho(R) = 0$, satisfy $|R_j| \leq 1, j = 1, \cdots, k$, and for those roots with$|R_j| = 1$, the multiplicity does not exceed two.

The block method (3.1) is consistent as it has order $p > 1$. According to Henrici [21], we can safely assert the convergence of the block method (3.1).

3.1. Stability Regions

In concluding this section, we compute and plot the region of absolute stability of the proposed schemes derived in section two. To achieve this, the schemes are reformulated as general linear methods and expressed as:

$$\begin{bmatrix} y \\ y_{1+i} \end{bmatrix} = \left[\begin{array}{c|c} \mathrm{A} & \mathrm{U} \\ \hline \mathrm{B} & \mathrm{V} \end{array}\right] \begin{bmatrix} \mathrm{h}^2\mathrm{f}\ (\mathrm{y}) \\ \mathrm{y}_{\mathrm{i}-1} \end{bmatrix} \tag{3.3}$$

Here

$$Y = \begin{bmatrix} \mathrm{y}_{\mathrm{n}} \\ y_{n+1/2} \\ \cdot \\ \cdot \\ \cdot \\ y_{n+k} \end{bmatrix} ; \quad \mathrm{A} = \begin{bmatrix} \mathrm{a}_{11} & \cdot & \cdot & \cdot & \mathrm{a}_{15} \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ a_{51} & \cdot & \cdot & \cdot & \mathrm{a}_{55} \end{bmatrix}$$

$$\mathrm{B} = \begin{bmatrix} \mathrm{b}_{11} & \cdot & \cdot & \cdot & \mathrm{b}_{15} \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ b_{k1} & \cdot & \cdot & \cdot & \mathrm{b}_{\mathrm{k}5} \end{bmatrix} \quad Y_{i+1} = \begin{bmatrix} \mathrm{y}_{\mathrm{n}+\mathrm{k}} \\ \cdot \\ \cdot \\ \cdot \\ y_{n+k-1} \end{bmatrix}$$

and $\mathrm{y}_{\mathrm{i}-1} = \begin{bmatrix} \mathrm{y}_{\mathrm{n}+\mathrm{k}-1} \\ \cdot \\ \cdot \\ \cdot \\ y_{n+k-2} \end{bmatrix}$

The elements of U and V are obtained from the interpolation and collocation points respectively. The elements of the matrices A, B, U ands V are substituted into the stability matrix

$$M(Z) = V + ZB(1 - ZA)^{-1}U \tag{3.4}$$

And the stability matrix (3.4) is substituted into the stability function

$$\rho(\eta,\ z) = \ \det\ (\eta I - M(Z)) \tag{3.5}$$

Computing the stability function gives the stability polynomial of the methods, which is plotted to produce the required absolute stability region of the method.

The coefficients of (2.11) – (2.13) are shown below:

$$\begin{bmatrix} y_n \\ y_{n+\frac{1}{2}} \\ y_{n+\frac{3}{4}} \\ y_{n+1} \\ \cdots \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & \vdots & 1 \\ \frac{63}{14688} & \frac{-522}{14688} & 0 & \frac{-9}{14688} & \vdots & \frac{406}{918} \\ \frac{4}{384} & \frac{39}{384} & \frac{-10}{384} & \frac{3}{384} & \vdots & \frac{-1}{2} \\ \frac{1}{48} & \frac{10}{48} & 0 & \frac{1}{48} & \vdots & -1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{48} & \frac{10}{48} & 0 & \frac{1}{48} & \vdots & -1 \end{bmatrix} \begin{bmatrix} f_n \\ f_{n+\frac{1}{2}} \\ f_{n+\frac{3}{4}} \\ f_{n+1} \\ \cdots \\ y_n \end{bmatrix} \tag{3.6}$$

Thus

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{63}{14688} & \frac{-522}{14688} & 0 & \frac{-9}{14688} \\ \frac{4}{384} & \frac{39}{384} & \frac{-10}{384} & \frac{3}{384} \\ \frac{1}{48} & \frac{10}{48} & 0 & \frac{1}{48} \end{bmatrix}$$

$$U = \begin{bmatrix} 1 \\ \frac{406}{918} \\ \frac{-1}{2} \\ -1 \end{bmatrix}, B = \left[\frac{1}{48}, \frac{10}{48}, 0, \frac{1}{48}\right]$$

$$V = [-1]$$

Therefore, (3.6) gives the stability polynomial below:

$$f(z) = \frac{72z^2 + 3412z - 44064 + 27\eta z^2 - 648\eta z - 44064\eta}{27(z^2 - 24z - 1632)} \tag{3.7}$$

On differentiating (3.7), we have

$$f'(z) = \frac{20(257z^2 + 7344z + 331296)}{27(z^2 - 24z - 1632)^2} \tag{3.8}$$

Thus, the graph of the region of absolute stability of the block method is shown below:

4. Numerical problems

In this section, the accuracy of the new block method developed for direct solution of four initial value problems of second order differential equations were illustrated by considering linear, non-linear, stiff and electric circuit problems.

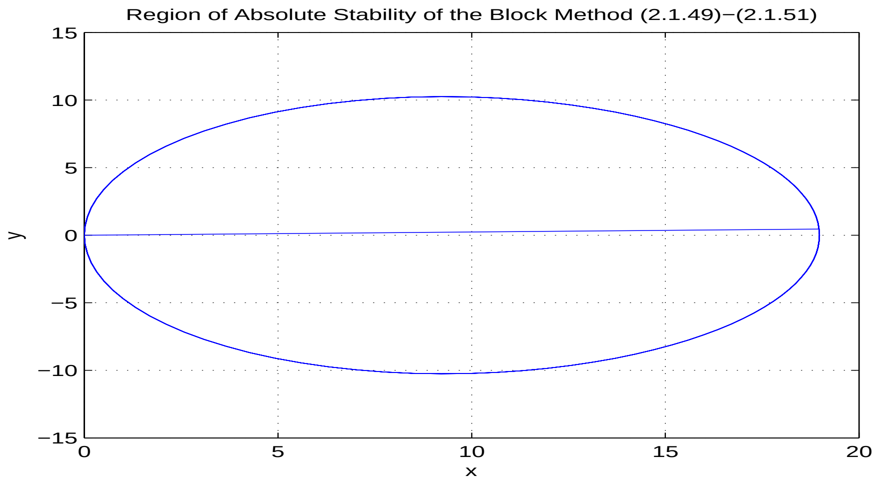

Figure 1: The graph of the region of absolute stability of the block method for second order

Although the errors arising from Problem 4 were not necessarily compared with the errors arising from another different method. However, the errors recorded by the proposed method for Problems 1-3 were compared with errors recorded earlier for the same problems by Adee et al. [1] and Awoyemi [7-9].

4.1. Test Problems

Problem 1. $y'' + y = 0,\ 0 \leq x \leq 0.2,\ y(0) = 1, y'(0) = 1, h = 0.1$

$$y(x) = \cos x + \sin x$$

(see Adee et al. [1]).

Problem 2. $y'' - 100y = 0;\ 0 \leq x \leq 0.02,\ y(0) = 1, y'(0) = -10, h = 0.01$

$$y(x) = \exp(-10x)$$

(see Adee et al. [1]).

Problem 3. $y'' - x(y')^2 = 0, y(0) = 1, y'(0) = {}^1\!/_2, h = 0.003125$

$$y(x) = 1 + {}^1\!/_2 In((2 + x)/(2 - x))$$

[see Awoyemi [7-9]]

x	Adee et al. [1]	ASSLMM
0.1	6.92×10^{-9}	0.0
0.2	1.76×10^{-8}	6.0×10^{-09}
0.3	1.62×10^{-8}	1.8×10^{-08}
0.4	4.37×10^{-8}	2.5×10^{-08}
0.5	1.20×10^{-7}	3.4×10^{-08}
0.6	1.87×10^{-7}	4.7×10^{-08}
0.7	3.07×10^{-7}	5.3×10^{-08}
0.8	4.19×10^{-7}	6.4×10^{-08}
0.9	5.79×10^{-7}	7.5×10^{-08}
1.0	7.27×10^{-7}	8.8×10^{-08}
1.1	9.20×10^{-7}	9.8×10^{-08}
1.2	1.10×10^{-6}	1.11×10^{-07}

Table 1: Comparison of absolute errors for Problem 1

Problem 4. Consider the DE of an electric circuit containing an inductance L(Henrie's), a resistance R(Ohms), a condenser of capacitance C(Faraday's) and an electromotive force E(t) measured in volts: $Lq''(t) + Rq' + \frac{q(t)}{C} = E(t)$ $\frac{1}{20}q''(t) + 5q' + 2500q(t) = 110$, $q(0) = q\prime(0) = 0$, $h = 0.1$ In this equation, $E = 110$ Volts, $R = 5$, $L = \frac{1}{20}$, $C = 4 \times 10^{-4}$ and q is the charge in coulombs.

Theoretical solution: $q(t) = -\frac{11}{250}(cos 50\sqrt{19}t + \frac{sin 50\sqrt{19}t}{\sqrt{19}})\ell^{-50t} + \frac{11}{250}$

4.2. Numerical solutions

See Table 1 – Table 4.

5. Conclusion

An approach for the derivation of ASSLMM for direct solution of second order ODEs has been presented and tested on some IVPs. Numerical evidences as shown in the tables of results in the preceding section clearly show that the proposed integration schemes for IVPs perform better in terms of accuracy than existing methods compared with. Hence, the desirability of our new method.

The computational burdens and wastage of computer time involved in the reduction of higher order differential equations into systems of first order equations are addressed by proposing methods (2.11)-(2.14) and (2.22)-(2.24) combined with (2.26)-(2.28). The continuous coefficients of the methods allowed

x	Adee et al. [1]	ASSLMM
0.01	4.96×10^{-9}	3.0×10^{-10}
0.02	1.49×10^{-8}	1.0×10^{-09}
0.03	6.68×10^{-9}	2.8×10^{-09}
0.04	2.40×10^{-8}	3.0×10^{-09}
0.05	6.77×10^{-7}	5.1×10^{-09}
0.06	1.08×10^{-7}	6.4×10^{-09}
0.07	1.77×10^{-7}	9.4×10^{-09}
0.08	2.32×10^{-7}	1.15×10^{-08}
0.09	3.13×10^{-7}	1.49×10^{-08}
0.10	3.95×10^{-7}	1.77×10^{-08}
0.11	4.96×10^{-7}	2.3×10^{-08}
0.12	6.00×10^{-6}	2.7×10^{-08}

Table 2: Comparison of absolute errors for Problem 2

x	Awoyemi [7] Order $p = 6$ $h = 0.003125$	Awoyemi [8-9] Order $p = 8$ $h = 0.003125$	ASSLMM Order $p = 4$ $h = 0.003125$
0.1	0.26075×10^{-9}	0.66391×10^{-13}	1.60316×10^{-13}
0.2	1.98167×10^{-9}	0.20012×10^{-9}	4.296563×10^{-13}
0.3	6.50741×10^{-9}	1.72007×10^{-9}	2.575717×10^{-14}
0.4	15.5924×10^{-9}	5.89464×10^{-9}	1.424638×10^{-12}
0.5	31.5045×10^{-9}	14.4347×10^{-9}	3.539835×10^{-12}
0.6	56.3746×10^{-9}	41.8664×10^{-9}	6.332490×10^{-12}
0.7	96.1640×10^{-9}	63.1096×10^{-9}	9.675149×10^{-12}
0.8	156.868×10^{-9}	91.1317×10^{-9}	1.475506×10^{-11}
0.9	248.698×10^{-9}	149.242×10^{-9}	2.297873×10^{-11}
1.0	387.984×10^{-9}	237.189×10^{-9}	3.483280×10^{-11}

Table 3: Comparison of absolute errors for Problem 3

the first derivatives of the methods to be determined, hence their applicability to general second order differential equations.

x	Exact	ASSLMM	Error
0.1	0.4427746614E-1	0.4421614030E-1	$6.132584507 \times 10^{-5}$
0.2	0.4399833056E-1	0.4399899230E-1	$6.617440790 \times 10^{-7}$
0.3	0.4400000947E-1	.04400000439E-1	$5.073170307 \times 10^{-9}$
0.4	0.4399999995E-1	0.4399999998E-1	$3.197430515 \times 10^{-11}$
0.5	0.4400000000E-1	0.4400000000E-1	$1.650693471 \times 10^{-13}$
0.6	0.4400000000E-1	0.4400000000E-1	$7.008282843 \times 10^{-16}$
0.7	0.4400000000E-1	0.4400000000E-1	0.0
0.8	0.4400000000E-1	0.4400000000E-1	0.0
0.9	0.4400000000E-1	0.4400000000E-1	0.0
1.0	0.4400000000E-1	0.4400000000E-1	0.0

Table 4: Table of results for Problem 4

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