International Journal of Pure and Applied Mathematics
Volume 82 No. 3 2013, 365-375
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu
ijpam.eu

ON (E,1)(C,1) SUMMABILITY OF FOURIER SERIES AND ITS CONJUGATE SERIES

Hare Krishna Nigam[1], Kusum Sharma[2]

Department of Mathematics
Faculty of Engineering and Technology
Mody Institute of Technology and Science (Deemed University)
Laxmangarh, 332311, Sikar, Rajasthan, INDIA

Abstract: Several researchers like Singh [7], Khare [3], Mittal and Kumar [5], singh and singh [8], Pandey [6] and Jadia [2] have studied (N, p_n), (N, p, q), almost (N, p, q) and matrix summability methods of Fourier series and its conjugate series using different conditions. But nothing seems to have been done so far to study $(E, 1)(C, 1)$ product summability of Fourier series and its conjugate series. Therefore, in this paper, two theorems on $(E, 1)(C, 1)$ summability of Fourier series and its conjugate series under a general condition have been proved.

AMS Subject Classification: 42B05, 42B08
Key Words: degree of approximation, $(C, 1)$ summability, $(E, 1)$ summability, $(C, 1)\,(E, 1)$ product summability, Fourier series, Lebesgue integral

1. Introduction

Let $\sum_{n=0}^{\infty} u_n$ be a given infinite series with sequence of its n^{th} partial sum $\{s_n\}$.

The (C,1) transform is defined as the n^{th} partial sum of (C,1) summability and is given by

$$t_n = \frac{s_0 + s_1 + s_2 + ...s_n}{n+1} = \frac{1}{n+1}\sum_{k=0}^{n} s_n \to s \ as \ n \to \infty \tag{1.1}$$

then the infinite series $\sum_{n=0}^{\infty} u_n$ is summable to the definite number s by (C,1)

Received: June 5, 2012

method. If

$$(E,1)=\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}s_k\to s \text{ as } n\to\infty \tag{1.2}$$

then the infinite series $\sum_{n=0}^{\infty}u_n$ is said to be summable $(E,1)$ to a definite number s (Hardy[1]). The (E,1) transform of (C,1) transform defines (E,1)(C,1) transform and we denote it by $(EC)_n^1$.

Thus if

$$(EC)_n^1=\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}C_k^1\to s \text{ as } n\to\infty. \tag{1.3}$$

Then the series $\sum_{n=0}^{\infty}u_n$ is said to be summable by $(E,1)\,(C,1)$ means or summable $(E,1)\,(C,1)$ to a definite number s. Therefore, we can write $(EC)_n^1\to s$ as $n\to\infty$.

The method $(E,1)\,(C,1)$ is regular and this case is supposed throughout this paper.

Let $f\,(x)$ be a 2π-periodic function of x and integrable over $[-\pi,\pi]$ in the sense of Lebesgue. The Fourier series $f\,(x)$ is given by

$$f\,(x)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx))\equiv\sum_{n=1}^{\infty}A_n\,(x) \tag{1.4}$$

The conjugate series of Fourier series (1.4) is given by

$$\sum_{n=1}^{\infty}(a_n\cos nx-b_n\sin nx)\equiv\sum_{n=1}^{\infty}B_n\,(x) \tag{1.5}$$

We use the following notations:

$$\phi\,(t)=f\,(x+t)+f\,(x-t)-2f\,(x)$$
$$\psi\,(t)=f\,(x+t)+f\,(x-t)$$
$$K_n\,(t)=\frac{1}{\pi\;2^{n+1}}\sum_{k=0}^{n}\left\{\binom{n}{k}\frac{1}{(1+k)}\sum_{\nu=0}^{k}\frac{\sin\left(\nu+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}$$
$$\bar{K}_n\,(t)=\frac{1}{\pi\;2^{n+1}}\sum_{k=0}^{n}\left\{\binom{n}{k}\frac{1}{(1+k)}\sum_{\nu=0}^{k}\frac{\cos\left(\nu+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}$$

$\tau=\left[\frac{1}{t}\right]$, where τ denotes the greatest integer not greater than $\frac{1}{t}$.

2. Main Theorems

We will prove the following theorems.

Theorem 1. *Let $\{p_n\}$ be a positive, monotonic, non-increasing sequence of real constants such that*

$$P_n = \sum_{\nu}^{n} p_\nu \to \infty \ as \ n \to \infty$$

If

$$\Phi(t) = \int_0^t |\phi(u)|\, du = o\left[\frac{t}{\alpha\left(\frac{1}{t}\right).P_\tau}\right] \ as \ t \to +0, \tag{2.1}$$

where $\alpha(t)$ is a positive, monotonic and non-increasing function of t. and

$$\log(n+1) = O\left[\{\alpha(n+1)\}.P_{n+1}\right] \text{ as } n \to \infty \tag{2.2}$$

then the Fourier series (1.4) is summable $(E,1)(C,1)$ to $f(x)$.

Theorem 2. *Let $\{p_n\}$ be a positive, monotonic, non-decreasing sequence of real constants such that*

$$P_n = \sum_{\nu}^{n} p_\nu \to \infty \ as \ n \to \infty$$

If

$$\Psi(t) = \int_0^t |\psi(u)|\, du = o\left[\frac{t}{\alpha\left(\frac{1}{t}\right).P_\tau}\right] \ as \ t \to +0, \tag{2.3}$$

where $\alpha(t)$ is a positive, monotonic and non-increasing function of t and condition (2.2), then the conjugate series (1.5) is summable to

$$\bar{f}(x) = -\frac{1}{2\pi}\int_0^{2\pi} \psi(t)\ cot\left(\frac{t}{2}\right)\ dt$$

at every point where this integral exists.

3. Some Lemmas

For the proof of our theorems, following lemmas are required:

Lemma 1.

$$|K_n(t)| = O(n+1) \;\; for \; 0 \le t \le \frac{1}{n+1}.$$

Proof. For $0 \le t \le \frac{1}{n+1}$, $\sin nt \le n \sin t$

$$|K_n(t)| \le \frac{1}{\pi\ 2^{n+1}} \left| \sum_{k=0}^{n} \left[\binom{n}{k} \frac{1}{(1+k)} \sum_{\nu=0}^{k} \frac{\sin\left(\nu+\frac{1}{2}\right)t}{\sin\frac{t}{2}} \right] \right|$$

$$\le \frac{1}{\pi\ 2^{n+1}} \left| \sum_{k=0}^{n} \left[\binom{n}{k} \frac{1}{(1+k)} \sum_{\nu=0}^{k} \frac{(2\nu+1)\sin\frac{t}{2}}{\sin\frac{t}{2}} \right] \right|$$

$$= \frac{1}{\pi\ 2^{n+1}} \left| \sum_{k=0}^{n} \left[\binom{n}{k} (k+1) \right] \right|$$

$$= O\left[\frac{(n+1)}{2^n} \sum_{k=0}^{n} \left\{ \binom{n}{k} \right\} \right]$$

$$= O(n+1) \qquad \text{since} \sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

□

Lemma 2.

$$|K_n(t)| = O\left(\frac{1}{t}\right) \;\; for \; \frac{1}{n+1} \le t \le \pi.$$

Proof. For $\frac{1}{n+1} \le t \le \pi$, by applying Jordan's lemma, $\sin\left(\frac{t}{2}\right) \ge \frac{t}{\pi}$ and $\sin nt \le 1$.

$$|K_n(t)| \le \frac{1}{\pi\ 2^{n+1}} \left| \sum_{k=0}^{n} \left[\binom{n}{k} \left(\frac{1}{1+k}\right) \sum_{\nu=0}^{k} \frac{\sin\left(\nu+\frac{1}{2}\right)t}{\sin\frac{t}{2}} \right] \right|$$

$$\le \frac{1}{\pi\ 2^{n+1}} \left| \sum_{k=0}^{n} \left[\binom{n}{k} \left(\frac{1}{1+k}\right) \sum_{\nu=0}^{k} \left(\frac{1}{t/\pi}\right) \right] \right|$$

$$= \frac{1}{2^{n+1}\ t} \left| \sum_{k=0}^{n} \left[\binom{n}{k} \left(\frac{1}{1+k}\right) \sum_{\nu=0}^{k} (1) \right] \right|$$

$$= \frac{1}{2^{n+1}\ t}\left|\sum_{k=0}^{n}\left[\left(\begin{array}{c} n \\ k \end{array}\right)\right]\right|$$

$$= O\left(\frac{1}{t}\right) \qquad \text{since} \sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right) = 2^n. \qquad \square$$

Lemma 3.

$$\bar{K}_n\left(t\right) = O\left[\frac{1}{t}\right] \ for\ 0 \leq t \leq \frac{1}{n+1}.$$

Proof. $0 \leq t \leq \frac{1}{n+1}$, $\sin\left(\frac{t}{2}\right) \geq \frac{t}{\pi}$ and $|\cos nt| \leq 1$.

$$\begin{aligned}
\left|\bar{K}_n\left(t\right)\right| &= \frac{1}{\pi\ 2^{n+1}}\left|\sum_{k=0}^{n}\left\{\left(\begin{array}{c} n \\ k \end{array}\right)\frac{1}{(1+k)}\sum_{\nu=0}^{k}\frac{\cos\left(\nu+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}\right| \\
&\leq \frac{1}{\pi\ 2^{n+1}}\sum_{k=0}^{n}\left\{\left(\begin{array}{c} n \\ k \end{array}\right)\frac{1}{(1+k)}\sum_{\nu=0}^{k}\frac{\left|\cos\left(\nu+\frac{1}{2}\right)t\right|}{\left|\sin\frac{t}{2}\right|}\right\} \\
&= \frac{1}{2^{n+1}\ t}\sum_{k=0}^{n}\left\{\left(\begin{array}{c} n \\ k \end{array}\right)\frac{1}{(1+k)}\sum_{\nu=0}^{k}\ (1)\right\} \\
&= \frac{1}{2^{n+1}\ t}\sum_{k=0}^{n}\left\{\left(\begin{array}{c} n \\ k \end{array}\right)\right\} \\
&= \frac{1}{2^{n+1}\ t}2^n \\
&= O\left[\frac{1}{t}\right] \qquad \text{since} \sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right) = 2^n.
\end{aligned}$$

$\square$

Lemma 4. *For $0 \leq a \leq b \leq \infty$, $0 \leq t \leq \pi$ and any n, we have*

$$\left|\bar{K}_n\left(t\right)\right| = O\left[\frac{1}{t}\right].$$

Proof. For $0 \leq \frac{1}{n+1} \leq t \leq \pi$, $\sin\left(\frac{t}{2}\right) \geq \frac{t}{\pi}$.

$$\begin{aligned}
\left|\bar{K}_n\left(t\right)\right| &= \frac{1}{\pi\ 2^{n+1}}\left|\sum_{k=0}^{n}\left\{\left(\begin{array}{c} n \\ k \end{array}\right)\frac{1}{(1+k)}\sum_{\nu=0}^{k}\frac{\cos\left(\nu+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}\right| \\
&\leq \frac{1}{2^{n+1}\ t}\left|\sum_{k=0}^{n}\left[\left(\begin{array}{c} n \\ k \end{array}\right)\frac{1}{(1+k)}\ Re\left\{\sum_{\nu=0}^{k}\ e^{i\left(\nu+\frac{1}{2}\right)t}\right\}\right]\right|
\end{aligned}$$

$$\leq \frac{1}{2^{n+1}\ t}\left|\sum_{k=0}^{n}\left[\left(\begin{array}{c} n \\ k \end{array}\right)\frac{1}{(1+k)}\ Re\left\{\sum_{\nu=0}^{k}\ e^{i\nu t}\right\}\right]\right|\left|e^{i\frac{t}{2}}\right|$$

$$\leq \frac{1}{2^{n+1}\ t}\left|\sum_{k=0}^{n}\left[\left(\begin{array}{c} n \\ k \end{array}\right)\frac{1}{(1+k)}\ Re\left\{\sum_{\nu=0}^{k}\ e^{i\nu t}\right\}\right]\right|$$

$$\leq \frac{1}{2^{n+1}\ t}\left|\sum_{k=0}^{\tau-1}\left[\left(\begin{array}{c} n \\ k \end{array}\right)\frac{1}{(1+k)}\ Re\left\{\sum_{\nu=0}^{k}\ e^{i\nu t}\right\}\right]\right|$$

$$+\frac{1}{2^{n+1}\ t}\left|\sum_{k=\tau}^{n}\left[\left(\begin{array}{c} n \\ k \end{array}\right)\frac{1}{(1+k)}\ Re\left\{\sum_{\nu=0}^{k}\ e^{i\nu t}\right\}\right]\right| \qquad (3.1)$$

Now considering first term of (3.1),

$$\frac{1}{2^{n+1}\ t}\left|\sum_{k=0}^{\tau-1}\left[\left(\begin{array}{c} n \\ k \end{array}\right)\frac{1}{(1+k)}\ Re\left\{\sum_{\nu=0}^{k}\ e^{i\nu t}\right\}\right]\right|$$

$$\leq \frac{1}{2^{n+1}\ t}\left|\sum_{k=0}^{\tau-1}\left[\left(\begin{array}{c} n \\ k \end{array}\right)\frac{1}{(1+k)}\ \left\{\sum_{\nu=0}^{k}\ 1\right\}\right]\right|\left|e^{i\nu t}\right|$$

$$\leq \frac{1}{2^{n+1}\ t}\left|\sum_{k=0}^{\tau-1}\left[\left(\begin{array}{c} n \\ k \end{array}\right)\right]\right| \qquad (3.2)$$

Now considering second term of (3.1) and using Abel's lemma,

$$\frac{1}{2^{n+1}\ t}\left|\sum_{k=\tau}^{n}\left[\left(\begin{array}{c} n \\ k \end{array}\right)\frac{1}{(1+k)}\ Re\left\{\sum_{\nu=0}^{k}\ e^{i\nu t}\right\}\right]\right|$$

$$\leq \frac{1}{2^{n+1}\ t}\sum_{k=\tau}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)\frac{1}{(1+k)}\ \begin{array}{c} max \\ 0\leq m\leq k \end{array}\ \left|\sum_{\nu=0}^{k} e^{i\nu t}\right|$$

$$\leq \frac{1}{2^{n+1}\ t}\sum_{k=\tau}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)\frac{1}{(1+k)}(1+k)$$

$$= \frac{1}{2^{n+1}\ t}\sum_{k=\tau}^{n}\left(\begin{array}{c} n \\ k \end{array}\right) \qquad (3.3)$$

Combining (3.1), (3.2) and (3.3), we get

$$\left|\bar{K}_n\left(t\right)\right| \leq \frac{1}{2^{n+1}\ t}\sum_{k=0}^{\tau-1}\left(\begin{array}{c} n \\ k \end{array}\right)+\frac{1}{2^{n+1}\ t}\sum_{k=\tau}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)$$

$$= O\left[\frac{1}{t}\right].$$ □

4. Proofs of Theorems

Proof of Theorem 1. Following Titchmarsh [9] and using Riemann-Lebesgue theorem, $s_n(f;x)$ the series (1.4) is given by

$$s_n\left(f;x\right) - f\left(x\right) = \frac{1}{2\pi}\int_0^\pi \phi\left(t\right)\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{t}{2}}dt$$

Therefore using (1.1) the $(C,1)$ transform C_n^1 of $s_n\left(f;x\right)$ is given by

$$C_n^1 - f\left(x\right) = \frac{1}{2\pi\ \left(n+1\right)}\int_0^\pi \phi\left(t\right)\sum_{k=0}^{n}\frac{\sin\left(k+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\ dt$$

Now denoting $(E,1)\left(C,1\right)$ transform of $s_n\left(f;x\right)$ by $(EC)_n^1$, we write

$$(EC)_n^1 - f\left(x\right) = \frac{1}{2^{n+1}\ \pi}\sum_{k=0}^{n}\left[\binom{n}{k}\int_0^\pi \frac{\phi\left(t\right)}{\sin\frac{t}{2}}\left(\frac{1}{k+1}\right)\left\{\sum_{\nu=0}^{k}\sin\left(\nu+\frac{1}{2}\right)t\right\}dt\right]$$
$$= \int_0^\pi \ \phi\left(t\right)\ K_n\left(t\right)dt$$

In order to prove the theorem, we have to show that, under our assumptions

$$\int_0^\pi \ \phi\left(t\right)\ K_n\left(t\right)dt = o\left(1\right)\ \text{ as } n\to\infty$$

For $0<\delta<\pi$, we have

$$\int_0^\pi \ \phi\left(t\right)\ K_n\left(t\right)dt = \left[\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\delta} + \int_\delta^\pi\right]\phi\left(t\right)\ K_n\left(t\right)dt$$
$$= I_{1.1} + I_{1.2} + I_{1.3} \quad \text{(say)} \tag{4.1}$$

We consider,

$$\left|I_{1.1}\right| \le \int_0^{\frac{1}{n+1}} \left|\phi\left(t\right)\right|\left|K_n\left(t\right)\right|dt$$
$$= O\left(n+1\right)\left[\int_0^{\frac{1}{n+1}} \left|\phi\left(t\right)\right|\ dt\right] \qquad \text{(using Lemma 1)}$$

$$= O\left(n+1\right)\left[o\left\{\frac{1}{\left(n+1\right)\ \alpha\left(n+1\right)P_{n+1}}\right\}\right] \text{ by } (2.1)$$
$$= o\left\{\frac{1}{\alpha\left(n+1\right)P_{n+1}}\right\}$$
$$= o\left\{\frac{1}{\log\left(n+1\right)}\right\} \qquad \text{using } (2.2)$$
$$= o(1)\ as\ n \to \infty \tag{4.2}$$

Now we consider,

$$|I_{1.2}| \leq \int_{\frac{1}{n+1}}^{\delta} |\phi\left(t\right)| \left|K_n\left(t\right)\right| dt$$
$$= O\left[\int_{\frac{1}{n+1}}^{\delta} |\phi\left(t\right)| \left(\frac{1}{t}\right) dt\right] \quad \text{(using Lemma 2)}$$
$$= O\left[\left\{\frac{1}{t}\ \Phi\left(t\right)\right\}_{\frac{1}{n+1}}^{\delta} + \int_{\frac{1}{n+1}}^{\delta} \frac{1}{t^2}\ \Phi\left(t\right)\ dt\right]$$
$$= O\left[o\left\{\frac{1}{\alpha\left(t\right)P_\tau}\right\}_{\frac{1}{n+1}}^{\delta} + \int_{\frac{1}{n+1}}^{\delta} o\left(\frac{1}{t\ \alpha\left(\frac{1}{t}\right)P_\tau}\right) dt\right] \text{ by } (2.1)$$
$$= O\left[o\left\{\frac{1}{\alpha\left(n+1\right)P_{n+1}}\right\}_{\frac{1}{n+1}}^{\delta} + \int_{\frac{1}{\delta}}^{n+1} o\left(\frac{1}{u\ \alpha\left(u\right)P_u}\right) du\right]$$
$$= o\left\{\frac{1}{\alpha\left(n+1\right)P_{n+1}}\right\} + o\left\{\frac{1}{\left(n+1\ \right)\alpha\left(n+1\right)P_{n+1}}\right\}\int_{\frac{1}{\delta}}^{n+1} 1.du$$
$$= o\left\{\frac{1}{\log\left(n+1\right)}\right\} + o\left\{\frac{1}{\log\left(n+1\right)}\right\}$$
$$= o(1) + o(1) \text{ as } n \to \infty \quad \text{by } (2.2)$$
$$= o(1) \text{ as } n \to \infty \tag{4.3}$$

Now by Riemann- Lebesgue theorem and by regularity condition of the method of summability, We have

$$|I_{1.3}| \leq \int_{\delta}^{\pi} |\phi\left(t\right)|\ \left|K_n\left(t\right)\right|\ dt$$
$$= o(1), \quad \text{as } n \to \infty \tag{4.4}$$

Combining (4.1) to (4.4), we get

$$(EC)_n^1 - f(x) = o(1) \text{ as } n \to \infty$$

This completes the proof of Theorem 1.

Proof of Theorem 2. Let $\bar{s}_n(f;x)$ denotes the partial sum of series (1.5). Then following Lal [4] and using Riemaann- Lebesgue theorem $\bar{s}_n(f;x)$ of (1.5) is given by

$$\bar{s}_n(f;x) - \bar{f}(x) = \frac{1}{2\pi}\int_0^\pi \psi(t)\ \frac{\cos\left(n+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}\ dt$$

Therefore, using (1.5) the $(C,1)$ transform C_n^1 of $\bar{s}_n(f;x)$ is given by

$$\bar{C}_n^1 - \bar{f}(x) = \frac{1}{2\pi\ (n+1)}\int_0^\pi \psi(t)\sum_{k=0}^{n}\frac{\cos\left(k+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\ dt$$

Now denoting $\overline{(E,1)(C,1)}$ transform of $\bar{s}_n$ by $N_pE_n^q$, we write

$$\begin{aligned}\overline{(EC)}_n^1 - \bar{f}(x) &= \frac{1}{2^{n+1}\ \pi}\sum_{k=0}^{n}\left[\binom{n}{k}\int_0^\pi \frac{\psi(t)}{\sin\frac{t}{2}}\left(\frac{1}{k+1}\right)\left\{\sum_{\nu=0}^{k}\cos\left(\nu+\frac{1}{2}\right)t\right\}dt\right]\\ &= \int_0^\pi \psi(t)\ \bar{K}_n(t)\,dt\end{aligned}$$

In order to prove the theorem, we have to show that, under our assumptions

$$\int_0^\pi \psi(t)\ \bar{K}_n(t)\ dt = o(1) \quad \text{as} \quad n\to\infty$$

For $0<\delta<\pi$, we have

$$\begin{aligned}\int_0^\pi \psi(t)\ \bar{K}_n(t)\ dt &= \left[\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\delta} + \int_\delta^\pi\right]\psi(t)\ \bar{K}_n(t)\ dt\\ &= I_{2.1} + I_{2.2} + I_{2.3} \quad \text{(say)}\end{aligned} \tag{4.5}$$

Now consider,

$$\begin{aligned}|I_{2.1}| &= \int_0^{\frac{1}{n+1}} |\psi(t)|\ \left|\bar{K}_n(t)\right| dt\\ &= O\left[\int_0^{\frac{1}{n+1}} \frac{1}{t}\,|\psi(t)|\ dt\right] \qquad \text{(using Lemma 3)}\\ &= O(n+1)\left[\int_0^{\frac{1}{n+1}} |\psi(t)|\ dt\right]\end{aligned}$$

$$= O(n+1)\left[o\left\{\frac{1}{(n+1)\ \alpha(n+1)P_{n+1}}\right\}\right] \text{ by } (2.3)$$

$$\begin{aligned}
&= o\left\{\frac{1}{\alpha(n+1)P_{n+1}}\right\}\\
&= o\left\{\frac{1}{\log(n+1)}\right\} \quad \text{using } (2.2)\\
&= o(1) \text{ as } n\to\infty
\end{aligned} \tag{4.6}$$

Now we consider,

$$|I_{2.2}| = \int_{\frac{1}{n+1}}^{\delta} |\psi(t)|\ \left|\bar{K}_n(t)\right| dt \tag{4.7}$$

$$\begin{aligned}
&= O\left[\int_{\frac{1}{n+1}}^{\delta} |\psi(t)|\ \left(\frac{1}{t}\right) dt\right] \quad \text{(using Lemma 4)}\\
&= O\left[\left\{\frac{1}{t}\ \Psi(t)\right\}_{\frac{1}{n+1}}^{\delta} + \int_{\frac{1}{n+1}}^{\delta} \frac{1}{t^2}\ \Psi(t)\ dt\right]\\
&= O\left[o\left\{\frac{1}{\alpha(t)P_\tau}\right\}_{\frac{1}{n+1}}^{\delta} + \int_{\frac{1}{n+1}}^{\delta} o\left(\frac{1}{t\ \alpha\left(\frac{1}{t}\right)P_\tau}\right) dt\right] \quad \text{by } (2.3)\\
&= O\left[o\left\{\frac{1}{\alpha(n+1)P_{n+1}}\right\}_{\frac{1}{n+1}}^{\delta} + \int_{\frac{1}{\delta}}^{n+1} o\left(\frac{1}{u\ \alpha(u)P_u}\right) du\right]\\
&= o\left\{\frac{1}{\alpha(n+1)P_{n+1}}\right\} + o\left\{\frac{1}{(n+1\ )\alpha(n+1)P_{n+1}}\right\}\int_{\frac{1}{\delta}}^{n+1} 1.du\\
&= o\left\{\frac{1}{\log(n+1)}\right\} + o\left\{\frac{1}{\log(n+1)}\right\}\\
&= o(1) + o(1) \quad \text{as } n\to\infty \quad \text{by } (2.2)\\
&= o(1) \quad \text{as } n\to\infty
\end{aligned} \tag{4.8}$$

Now by Riemann- Lebesgue theorem and by regularity condition of the method of summability, We have

$$\begin{aligned}
|I_{2.3}| &\le \int_{\delta}^{\pi} |\psi(t)|\ \left|\bar{K}_n(t)\right|\ dt\\
&= o(1) \quad \text{as } n\to\infty
\end{aligned} \tag{4.9}$$

Combining (4.5) to (4.9), we get

$$\overline{(EC)_n^1} - \bar{f}\,(x) = o(1) \text{ as } n \to \infty$$

This completes the proof of theorem 2.

Acknowledgments

First author is thankful to his parents for their encouragement and support to this work.

References

[1] G.H. Hardy, *Divergent Series*, First Edition, Oxford University Press (1949).

[2] B.L. Jadiya, On Nörlund summability of conjugate Fourier series, *Indian Journal of Pure and Applied Mathematics*, **13**, No. 11 (1982), 1354-1359.

[3] S.P. Khare, Generalized Nörlund summability of Fourier series and its conjugate series, *Indian Journal of Pure and Applied Mathematics*, **21**, No. 5 (1990), 457-467.

[4] Shyam Lal, On K^λ summability of conjugate series of Fourier series, *Bulletin of Calcutta Math. Soc.*, **89** (1997), 97-104.

[5] M.L. Mittal, Rajesh Kumar, Matrix summability of Fourier series and its conjugate series, *Bull. Call. Math. Soc.*, **82** (1990), 362-368.

[6] G.S. Pandey, On Nörlund summability of Fourier series, *Indian Journal of Pure and Applied Mathematics*, **8** (1977), 412-417.

[7] A.N. Singh, Nörlund summability of Fourier series and its conjugate series, *Bull. Call. Math. Soc.*, **82** (1990), 99-105.

[8] U.N. Singh, V.S. Singh, Almost Nörlund summability of Fourier series and its conjugate series, *Bull. Call. Math. Soc.*, **87** (1995), 57-62.

[9] E.C. Titchmarsh, *The Theory of Functions*, Oxford University Press (1939), 402-403.