ON THE TOTAL NEGATION OF RIGIDITY

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Abstract: In this paper we investigate the total negation of rigidity namely anti-rigidity. A topological space \( X \) is anti-rigid if there is no rigid subspace of \( X \) with more than one point. In particular, we establish the relationships of anti-rigidity with metric spaces, scattered spaces and ordered sets with order topology etc.

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1. Introduction

In 1979, Bankston [1] introduced the notion of anti-properties in topology. He defined the total negation of a topological property \( P \) as follows. If \( P \) is any topological property, the spectrum of \( P \), denoted by \( \text{Spec}(P) \) is the class of all cardinal numbers \( \alpha \) such that any topological space of cardinality \( \alpha \) has the property \( P \). Now a topological space \((X, T)\) is said to have the property anti-\( P \) if a subspace of it has the property \( P \) only if the cardinality of the subspace is an element of \( \text{Spec}(P) \). Authors like I. L. Reily and M. K. Vamanamurthy investigated the anti-properties arising from compactness conditions. They also characterized anti-normality in [2]. In 1990, Julie Matier and Brian M. McMaster in [9] showed that the anti-first countable or anti-completely separable topological spaces are precisely those that are finite, but that the anti-separable
spaces exhibit several properties resembling those of the anti-compact ones. Also, they examined the minimality of anti-separable topologies. In [13], P. T. Ramachandran proved that anti-homogeneity is equivalent to hereditary rigidity. He started the investigation on anti-rigid spaces in [14]. P. T. Ramachandran and V. Kannan characterised hereditarily homogeneous spaces in [6]. The present author [15], examined several examples of anti-rigid spaces and generalized the relations between anti-rigidity with homogeneity and filters.

In this paper, we prove that any countable metric space is anti-rigid. A characterisation of an anti-rigid scattered space is also provided. Then we consider anti-rigidity of ordered spaces.

2. Scatteredness and Anti-Rigid Spaces

A topological space \((X, T)\) is said to be scattered if every nonempty subset of it contains a point which is isolated with respect to the relative topology. Scattered spaces have a remarkable role in the theory of topological spaces. Some investigations on these spaces can be found in [4], [5] and [12]. Here we prove a characterisation of anti-rigidity among scattered spaces.

**Theorem 1.** A scattered space is anti-rigid if and only if it is \(T_1\).

**Proof.** Consider a scattered anti-rigid space \((X, T)\) and let \(x, y \in X\). As \((X, T)\) is anti-rigid, \(\{x, y\}\) has either discrete topology or indiscrete topology. Also, as \((X, T)\) is scattered, \(\{x, y\}\) must contain an isolated point. Since \((X, T)\) is both anti-rigid and scattered, \(\{x, y\}\) must be the discrete space. Hence there exists open sets \(U, V \in T\) such that \(x \in U, y \in V, x /\in V, y /\in U\). This shows that \((X, T)\) is \(T_1\).

Conversely, let \((X, T)\) be scattered and \(T_1\). Let \(A\) be any subspace of \((X, T)\) with more than one point. Then there exist \(x \in A\) such that \(\{x\}\) is open in \(A\). Let \(B = A - \{x\}\) and let \(y \in B\) be such that \(\{y\}\) is open in \(B\). Now, take \(U, V \in T\) such that \(U \cap A = \{x\}\) and \(V \cap B = \{y\}\). As \((X, T)\) is \(T_1\) there exist \(G, H \in T\) such that \(x \in G, y /\in G, y \in H, x /\in H\). Define \(K = U \cap G, L = V \cap H\). Then both \(K, L \in T\) and \(x \in K, y /\in K, y \in L, x /\in L\) implies that \(K \cap A = \{x\}\) and \(L \cap A = \{y\}\) by construction of \(K\) and \(L\). Thus \(\{x\}\) and \(\{y\}\) are open in \(A\).

Let \(h : A \to A\) by \(h = (x, y)\). Then \(h = h^{-1}\). Also \(h\) is one-one and onto. Let \(U\) be an open subset of \(A\). If \(x, y \in U\) or \(x, y /\in U\), then \(h^{-1}(U) = U\). If \(x \in U, y /\in U\), then \(h^{-1}(U) = (U - \{x\}) \cup \{y\}\) which is open in \(A\) as \(\{x\}\) is closed in \(A\) since \(A\) is \(T_1\). The case \(x /\in U, y \in U\) is similar to the last discussed
one. Hence \( h \) is continuous and so it is a homeomorphism on \( A \) other than identity homeomorphism. Thus \( A \) is not rigid. As \( A \subseteq X \) is arbitrary, \((X, T)\) is anti-rigid.

**Remark 1.** There exist \( T_1 \) anti-rigid spaces which are not scattered.

Eg: The set of all rational numbers with the relative topology from the usual topology on the set of all real numbers is \( T_1 \) but not scattered. But it is anti-rigid which is shown in the next section.

### 3. Countable Anti-Rigid Metric Spaces

In this section we consider countable anti-rigid metric spaces.

**Lemma 1.** The space of rational numbers \( Q \) is anti-rigid.

**Proof.** Let \( A \subseteq Q \) be any subspace containing more than one element of \( Q \). If \( A \) is finite, \((A, T_A)\) will be the discrete space and hence not rigid. Let \( A \) be infinite. As \( Q \) is \( T_2 \), any of its subspaces having more than one isolated point will not be rigid. If \( A \) contains no isolated point, then it is self dense and hence homeomorphic to \( Q \), by [7] and hence not rigid. Now, let \( a \in A \) be a unique isolated point of \( A \). Then as \( A \) is \( T_2 \), \( A - \{a\} \) does not have isolated points, and hence self dense and not rigid as discussed above. The topology \((A, T_A)\) is the topological sum of the two non-rigid topological spaces \( \{a\} \) and \( A \), and hence is not rigid. Thus, in any case, \( A \) is not rigid and hence \( Q \) is anti-rigid.

**Theorem 2.** Every countable metric space is anti-rigid.

**Proof.** By [7], any countable metric space \( X \) is homeomorphic to a subspace of \( Q \), the space of rational numbers and hence anti-rigid by Lemma 1.

**Remark 2.**

a) There exist countable non metrisable spaces which are not anti-rigid.

Eg: Consider \( N \), the set of all natural numbers and define a topology \( T \) on \( N \) by \( T = \{\emptyset, N, \{1\}, \{1, 2\}, \{1, 2, 3\}, \ldots\} \). Then \((N, T)\) is countable, nonmetrisable and not anti-rigid.

b) There exist countable nonmetrisable spaces which are anti-rigid.

Eg: A countable set with co-finite topology is countable, nonmetrisable and anti-rigid.

c) There exist uncountable metrisable spaces which are not anti-rigid.
Eg: The real line with usual topology is uncountable, metrisable but not anti-rigid.

4. Anti-Rigid Order Topologies

Here we discuss the relation between anti-rigidity and order topology. Let $X$ be a set having a simple order relation $<$. Let $\mathcal{B}$ be the collection of all subsets of $X$ of the following types.

1. All open intervals $(a, b)$ in $X$, where $(a, b) = \{x : a < x < b\}$.

2. All intervals of the form $[a_0, b)$, where $a_0$ is the smallest element (if any) of $X$, where $[a_0, b) = \{x : a_0 \leq x < b\}$.

3. All intervals of the form $(a, b_0]$ where $b_0$ is the largest element (if any) of $X$, where $(a, b_0] = \{x : a < x \leq b\}$.

The collection $\mathcal{B}$ is a basis for a topology on $X$, which is called the order topology on $X$. A linearly ordered set is said to be well-ordered if every non-empty subset of it has a first element. A linearly ordered set is said to be dually well-ordered if every non-empty subset of it has a last element. A linearly ordered set is said to be semi well-ordered if every non-empty subset of it has either a first element or a last element. (see [13]).

Theorem 3. Any countable linearly ordered set with order topology is anti-rigid.

Proof. Every ordered space is regular and $T_1$. Hence, a countable ordered space is $T_1$, regular and second countable and hence metrisable by Urysohn Metrisation Theorem (see [3]). Thus a countable ordered space is a countable metrisable space and hence anti-rigid by Theorem 2.

Theorem 4. Any well-ordered set with order topology is anti-rigid.

Proof. Let $(X, T)$ be a well ordered set with order topology. Consider a nonempty subset $A$ of $X$. Then there exist a first element $a \in A$. Also let $b$ be the first element of $A - \{a\}$. Now, take the initial segment $B$ of $b \in X$, which is open in $X$. We have $B \cap A = \{a\}$, and hence $a$ is an isolated point of $A$. Thus every subspace of $X$ contains an isolated point, ie, $X$ is scattered. Since an order topology is $T_1$, $(X, T)$ is anti-rigid by Theorem 6.

Theorem 5. Any dually well-ordered set with order topology is anti-rigid.
Proof. Let \((X, T)\) be a dually well ordered set with order topology. Consider a nonempty subset \(A\) of \(X\). Then there exist a last element \(a \in A\) of \(A\). Also let \(b\) be the last element of \(A - \{a\}\). Now, take the final segment \(B\) of \(b \in X\), which is open in \(X\). We have \(B \cap A = \{a\}\), and hence \(a\) is an isolated point of \(A\). Thus every subspace of \(X\) contains an isolated point, ie, \(X\) is scattered. Since an order topology is \(T_1\), \((X, T)\) is anti-rigid by Theorem 6.

Let us combine above two theorems to get the following theorem.

**Theorem 6.** Any semi well-ordered set with order topology is anti-rigid.

Proof. Let \(A\) and \(B\) be two disjoint linearly ordered sets with the linear orders \(R\) and \(S\) respectively. Then by \(A + B\) we denote the set \(A \cup B\) with the linear order \(R \cup S \cup \{(a, b) : a \in A, b \in B\}\) on it.

By [13] every semi well ordered set \(X\) can be written in the form \(A + B\) where \(A\) is a well ordered set and \(B\) is a dually well ordered set.

Let \(X\) be any semi well ordered set with order topology. Then there exist a well ordered set \(A\) and dually well ordered set \(B\) such that \(X = A + B\). By Theorem 4 and Theorem 5, both \(A\) and \(B\) are anti-rigid in the order topology. Let \(P\) be any subset of \(X\). Then there exist well ordered set \(G\) and dually well ordered set \(H\) such that \(P = G + H\), where \(G = P \cap A\) and \(H = P \cap B\).

If either \(G = \emptyset\) or \(H = \emptyset\), then \(P\) is well ordered or dually well ordered and hence contain isolated points as we have seen in the proof of Theorem 7. Now let \(x \in G\) be the smallest element of \(G\). If \(|G| \geq 2\), let \(y\) be the smallest element of \(G - \{x\}\) and hence \((-\infty, y) \cap G = \{y\} = (-\infty, y) \cap P\). Thus \(x\) is an isolated point of \(P\). Now, if \(G = \{x\}\), \(G \cup H\) can be considered as a dually well ordered set where \(x \leq y\) for all \(y \in G\) and hence contain an isolated point \(b\) as we have seen in the proof of theorem 8. Thus in any case, \(P\) contains an isolated point. Since \(P\) is arbitrary, \(X\) is scattered. Since an order topology is \(T_1\), \((X, T)\) is anti-rigid by Theorem 6.

**Remark 3.**

1. There exists an uncountable linearly ordered set with order topology which is not anti-rigid, if we assume the axiom of choice.

Eg: The real line with usual topology is the order topology with the usual ordering, which is not anti-rigid by assuming the axiom of choice (see [8]).

2. The Cantor set, which is homeomorphic to \(Z_2^N\) is an uncountable linearly ordered set with order topology which is not anti-rigid. (see proof of Theorem 7).
3. There exists anti-rigid order topologies which are not semi well-ordered.

Eg: Consider the set of all rational numbers with usual topology. It coincides with the set of all rational numbers with usual ordering having order topology, which is not well-ordered, dually well-ordered or semi well-ordered. We have proved in Lemma 1 that this space is anti-rigid.

5. Products and Quotients

In this section we prove that anti-rigidity is not preserved by countable products and quotients.

**Definition 1.** (see [3]) A subset $A$ of a topological space $(X, T)$ is said to be clopen if it is both open and closed.

**Theorem 7.** In ZFC, anti-rigidity is not countably productive.

*Proof.* By [8], there exists a rigid subspace $K$ of $\mathbb{R}$. Then $K$ is separable and second countable. Also, since $K$ is rigid, it does not contain any interval, for otherwise we can find non-trivial homeomorphisms. Hence $K$ is zero dimensional, since a subset of real line is zero dimensional if it does not contain any non degenerate interval. Thus we get a clopen base $\mathcal{C} = \{C_1, C_2, \ldots, C_n, \ldots\}$ for $K$ which is countable since $K$ is second countable and let $\mathcal{F} = \{f_k : C_k \to \{0, 1\} : k \in \mathbb{N}\}$ be the family of characteristic functions of $C_k, k = 1, 2, \ldots n, \ldots$, where $\{0, 1\}$ with discrete topology and $\mathbb{N}$ is the set of all natural numbers. Then each $f_k$ is continuous as $C_k$ is clopen. Let $C$ be any closed subset of $K$ and let $x$ be any point of $K$ which do not belong to $C$. Then $K - C$ is an open set containing $x$ and hence there exists $C_k \in \mathcal{C}$ such that $x \in C_k$ and $C_k \subset K - C$. Then $f_k(x) = 1$ and $f_k(y) = 0$ for all $y \in C$. Thus $\mathcal{F}$ distinguishes points from closed sets. Also, as $K$ is $T_1$, each singleton set is closed and hence $\mathcal{F}$ separates points also. Hence by embedding lemma (see [3]), $K$ can be embedded in $\{0, 1\}^\mathbb{N}

Thus we have proved that a rigid space can be embedded in a countable product of anti-rigid spaces. This proves the result. □

**Definition 2.** (see [3]) A topological property is said to be divisible if whenever a space has it, so does every quotient space of it.

**Theorem 8.** Anti-rigidity is not divisible.

*Proof.* Let $X = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}, T = \{A \subset X : 0 \notin A\} \cup \{A \subset X : 0 \in A, X - A \text{ is finite}\}$. Then $(X, T)$ is anti-rigid as it is of the form $P(X - A) \cup \mathcal{F}, \cap \mathcal{F} = \{a\}$ (see [15]).
Now, let $A = \{0, 1\}$ and define $P : X \to A$ by $P(\frac{1}{n}) = 1 \quad \forall n \in \mathbb{N}, \quad P(0) = 0$. Then the quotient topology on $A$ defined by the map $P$ is $S = \{\emptyset, \{1\}, \{0, 1\}\}$, which makes $A$ a rigid space. Thus $(A, S)$ is a rigid space which is a quotient space of an anti-rigid space and hence the proof.

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References


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