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# ORDER OF MAGNITUDE OF MULTIPLE WALSH-FOURIER COEFFICIENTS OF FUNCTIONS OF BOUNDED $p$-VARIATION

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**Abstract:** For a Lebesgue integrable complex-valued function $f$ defined over the $n$-dimensional torus $\mathbb{I}^n := [0,1)^n$ $(n \in \mathbb{N})$, let $\hat{f}(\mathbf{k})$ denote the multiple Walsh-Fourier coefficient of $f$, where $\mathbf{k} = (k_1, \ldots, k_n) \in (\mathbb{Z}^+)^n$, $\mathbb{Z}^+ = \mathbb{N} \cup \{0\}$. The Riemann-Lebesgue lemma shows that $\hat{f}(\mathbf{k}) = o(1)$ as $|\mathbf{k}| \to 0$ for any $f \in \mathrm{L}^1(\mathbb{I}^n)$. However, it is known that, these Fourier coefficients can tend to zero as slowly as we wish. When $n = 1$ the definitive results are due to B. L. Ghodadra and J. R. Patadia [J. Inequal. Pure Appl. Math., 9 (2) (2008), Article 44] for functions of certain classes of functions of generalized bounded variation. Ghodadra [Acta Math. Hungar 128 (4), 2010, 328–343] defined the notion of bounded $p$-variation $(p \geq 1)$ for a function from a rectangle $[a_1, b_1] \times \cdots \times [a_n, b_n]$ to $\mathbb{C}$ and obtained definitive results for the order of magnitude of multiple trigonometric Fourier coefficients. In this paper, such definitive results for the order of magnitude of multiple Walsh-Fourier coefficients for a function of bounded $p$-variation are obtained.

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## 1. Introduction

In 1949, N. J. Fine [2] proved using the second mean value theorem that if $f$ is of bounded variation on $[0,1]$ and if $\hat{f}(n)$ denotes its (one dimensional) Walsh-Fourier coefficient, then $\hat{f}(n) = O(\frac{1}{n})$, for all $n \neq 0$. In [6] we have studied the order of magnitude of Walsh-Fourier coefficients of functions of various classes of generalized bounded variation and extended the result of Fine to these classes. Further in [5], we have defined the notion of bounded $p$-variation ($p \geq 1$) for a function from a rectangle $[a_1, b_1] \times \cdots \times [a_n, b_n]$ to $\mathbb{C}$ and studied the order of magnitude of trigonometric Fourier coefficients of such functions from $[0, 2\pi]^n$ to $\mathbb{C}$. Here we study the order of magnitude of Walsh-Fourier coefficients for a function of bounded $p$-variation from $[0,1]^n$ to $\mathbb{C}$ and obtain analogous results. For $n = 1$, our new results give our earlier result [6, Theorem 2.1]. Also, for $p = 1$, our results give the Walsh analogue of the results of Móricz [7] and Fülöp and Móricz [3], except possibly for the exact constant in their case.

## 2. Notation and Definitions

In [5] we have defined two concepts of bounded p-variation for functions of several variables that generalize the definitions of bounded variation for functions of several variables given by Vitali and by Hardy. For the sake of completeness, here we rewrite those definitions.

Let $R$ be the rectangle $R = [a_1, b_1] \times \cdots \times [a_n, b_n]$. By a (finite) partition $\mathcal{P}$ of $R$ we mean the set $\mathcal{P} = \{R_1, \ldots, R_m\}$, in which $R_i$'s are pairwise disjoint (no two have common interior) subrectangles of $R$ having their sides (faces) parallel to the standard coordinate hyperplanes and whose union is $R$. Let $f = f(x_1, \ldots, x_n)$ be a real or complex-valued function on $R$. For any subrectangle $R' = [\alpha_1, \beta_1] \times \cdots \times [\alpha_n, \beta_n]$ of $R$ with $a_i \leq \alpha_i < \beta_i \leq b_i$ for all $i = 1, 2, \ldots, n$, we define $\Delta f(R')$ as follows: When $n = 2$ we put

$$\begin{aligned}\Delta f(R') :&= \Delta f([\alpha_1, \beta_1] \times [\alpha_2, \beta_2]) \\ &= f(\beta_1, \beta_2) - f(\beta_1, \alpha_2) - f(\alpha_1, \beta_2) + f(\alpha_1, \alpha_2);\end{aligned}$$

for $n = 3$

$$\begin{aligned}\Delta f(R') &:= \Delta f([\alpha_1, \beta_1] \times \cdots \times [\alpha_3, \beta_3]) \\ &= [f(\beta_1, \beta_2, \beta_3) - f(\beta_1, \alpha_2, \beta_3) - f(\alpha_1, \beta_2, \beta_3) + f(\alpha_1, \alpha_2, \beta_3)] \\ &\quad - [f(\beta_1, \beta_2, \alpha_3) - f(\beta_1, \alpha_2, \alpha_3) - f(\alpha_1, \beta_2, \alpha_3) + f(\alpha_1, \alpha_2, \alpha_3)] \\ &= \Delta_{[\alpha_3, \beta_3]} \Delta f([\alpha_1, \beta_1] \times [\alpha_2, \beta_2]), say;\end{aligned}$$

and successively for any $n \geq 3$

$$\begin{aligned}\Delta f(R') :&= \Delta f([\alpha_1, \beta_1] \times \cdots \times [\alpha_n, \beta_n]) \\ &= \Delta_{[\alpha_n, \beta_n]} \Delta f([\alpha_1, \beta_1] \times \cdots \times [\alpha_{n-1}, \beta_{n-1}]).\end{aligned}$$

**Definition V.** For $p \geq 1$ we say that $f$ is of *bounded p-variation over* $R$ (written as $f \in BV_V^{(p)}(R)$) if $V_p(f; R)$, the *total p-variation* of $f$ over $R$, is finite, where

$$V_p(f; R) := \sup \left\{ \sum_{i=1}^{m} |\Delta f(R_i)|^p \right\}^{1/p},$$

in which the supremum is taken over all partitions $\{R_1, \ldots, R_m\}$ of $R$.

**Remark 1.** As noted in [5], for $p = 1$ our Definition V is equivalent to that Vitali (see, for example, [1, 3]). Also, the class $BV_V^{(p)}(R)$ contains functions for which the $n$-dimensional Lebesgue integral over $R$ fails to exist. The following notion of bounded $p$-variation is motivated by this fact.

**Definition H.** In case $n = 2$, we say that a function $f = f(x_1, x_2)$ is of bounded $p$-variation over $R := [a_1, b_1] \times [a_2, b_2]$, in symbol: $f \in BV_H^{(p)}(R)$, if it is in the class $BV_V^{(p)}(R)$ and if the marginal functions $f(x_1, a_2)$ and $f(a_1, x_2)$ are of bounded $p$-variation on the intervals $I_1 := [a_1, b_1]$ and $I_2 := [a_2, b_2]$, respectively in the sense of Wiener [9].

In case $n \geq 3$, the notion of bounded $p$-variation over a rectangle $R$ can naturally be defined by the following recurrence: $f \in BV_H^{(p)}(R)$ if $f \in BV_V^{(p)}(R)$ and each of the marginal functions $f(x_1, \ldots, a_k, \ldots, x_n)$ is in the class $BV_H^{(p)}(R(a_k))$, where $k = 1, \ldots, n$ and

$$\begin{aligned}R(a_k) = \{(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n) \in \mathbb{R}^{n-1} :\ & a_j \leq x_j \leq b_j \\ & \text{for } j = 1, \ldots, k-1, k+1, \ldots, n\}.\end{aligned}$$

This definition can be equivalently reformulated as follows: $f \in BV_H^{(p)}(R)$ if and only if $f \in BV_V^{(p)}(R)$ and for any choice of $(1 \leq) j_1 < \cdots < j_m (\leq n)$, $1 \leq m < n$, the function

$$f(x_1, \ldots, a_{j_1}, \ldots, a_{j_m}, \ldots, x_n)$$

is in the class $BV_V^{(p)}(R(a_{j_1}, \ldots, a_{j_m}))$, where

$$R(a_{j_1}, \ldots, a_{j_m}) := \{(x_{\ell_1}, \ldots, x_{\ell_{n-m}}) \in \mathbb{R}^{n-m} : a_j \leq x_j \leq b_j$$

for $j=\ell_1,\dots,\ell_{n-m}\}$

and $\{\ell_1,\dots,\ell_{n-m}\}$ is the complementary set of $\{j_1,\dots,j_m\}$ with respect to $\{1,\dots,n\}$.

**Remark 2.** When $p=1$ our Definition H is equivalent to the definition given by Hardy (see, for example, [1, 3]).

Let $\{\varphi_n\}$ $(n=0,1,2,3,\dots)$ denote the complete orthonormal Walsh system [8], where the subscript denotes the number of zeros (that is, sign-changes) in the interior of the interval $[0,1]$. For a periodic $f=f(x_1,\dots,x_n)$ with period 1 in each variable and Lebesgue integrable over the $n$-dimensional torus $\mathbb{I}^n:=[0,1)^n$, in symbol $f\in L^1(\mathbb{I}^n)$, its Walsh-Fourier coefficients (see, for example, [4]) are defined by

$$\hat{f}(\mathbf{k})=\int_{\mathbb{I}^n} f(x_1,\dots,x_n)\varphi_{k_1}(x_1)\dots\varphi_{k_n}(x_n)dx_1\dots dx_n, \tag{1}$$

where $\mathbf{k}=(k_1,\dots,k_n)\in(\mathbb{Z}^+)^n$.

## 3. Results

We prove following theorems.

**Theorem 3.** *Let $f:\mathbb{R}^n\to\mathbb{C}$ be 1-periodic in each variable. If $f\in L^p(\mathbb{I}^n)\cap BV_V^{(p)}([0,1]^n)$ $(p\geq 1)$ and $\mathbf{k}=(k_1,\dots,k_n)\in\mathbb{N}^n$, then*

$$\hat{f}(\mathbf{k})=O\left(\frac{1}{\left(\prod_{j=1}^n k_j\right)^{1/p}}\right).$$

**Theorem 4.** *Let $f:\mathbb{R}^n\to\mathbb{C}$ be 1-periodic in each variable. If $f\in BV_H^{(p)}([0,1]^n)$ $(p\geq 1)$ then for any $\mathbf{0}\neq\mathbf{k}=(k_1,\dots,k_n)\in(\mathbb{Z}^+)^n$,*

$$\hat{f}(\mathbf{k})=O\left(\frac{1}{\left(\prod_{j=1,k_j\neq 0}^n k_j\right)^{1/p}}\right).$$

## 4. Proof of the Results

To prove Theorems 3 and 4 we need the following lemmas which are proved in [5].

**Lemma 5.** *If $f \in BV_H^{(p)}(R)$ then $f$ is bounded over $R$.*

**Lemma 6.** *If $f \in BV_H^{(p)}(R)$ then for any arbitrary fixed values $c_{j_1} \in [a_{j_1}, b_{j_1}]$, $\dots$, $c_{j_m} \in [a_{j_m}, b_{j_m}]$, $(1 \le) j_1 < \cdots < j_m (\le n)$, and $1 \le m < n$, the function $f(\cdot, \dots, c_{j_1}, \dots, c_{j_m}, \dots, \cdot)$ is in $BV_H^{(p)}(R(a_{j_1}, \dots, a_{j_m}))$ and that*

$$(V_p(f(\cdot, \dots, c_{j_1}, \dots, c_{j_m}, \dots, \cdot); R(a_{j_1}, \dots, a_{j_m})))^p \le 2^{mp}\Bigg\{(V_p(f;R))^p + \sum_{k=1}^{m} \sum_{\substack{s_1<\cdots<s_k,\\ s_1,\dots,s_k \in \\ \{j_1,\dots,j_m\}}} (V_p(f(\cdot, \dots, a_{s_1}, \dots, a_{s_k}, \dots, \cdot); R(a_{s_1}, \dots, a_{s_k})))^p\Bigg\}.$$

**Lemma 7.** *Let $f \in BV_V^{(p)}(R)$, where $R = [a_1, b_1] \times \cdots \times [a_n, b_n]$. Let $\{R_1, \dots, R_m\}$ be a partition of $R$. Then $f \in BV_V^{(p)}(R_i)$ for each $i = 1, \dots, m$, and that*

$$\sum_{i=1}^{m} (V_p(f; R_i))^p \le (V_p(f; R))^p.$$

**Lemma 8.** *Let $f \in BV_H^{(p)}(R)$, where $R = [a_1, b_1] \times \cdots \times [a_n, b_n]$. Then the discontinuities of $f$ are located on a countable number of $(n-1)$-dimensional hyperplanes parallel to some of the coordinate hyperplanes.*

**Proof of Theorem 3.** For the sake of simplicity in writing, we carry out the proof in the case for $n = 2$, and we write $(x, y)$ and $(k, \ell)$ in place of $(x_1, x_2)$ and $(k_1, k_2)$ respectively.

Let $\mathbf{k} = (k, \ell) \in \mathbb{N}^2$. Let $s, t \in \mathbb{Z}^+$ be such that $2^s \le k < 2^{s+1}$ and $2^t \le \ell < 2^{t+1}$. For each $i = 0, 1, 2, 3, \dots, 2^s$ and $j = 0, 1, 2, 3, \dots, 2^t$ put $a_i = (i/2^s)$, $b_j = (j/2^t)$. Then by definition of Walsh functions, $\varphi_k$ takes the value 1 on one half of each of the intervals $(a_{i-1},\ a_i)$ and the value $-1$ on the other half, and hence

$$\int_{a_{i-1}}^{a_i} \varphi_k(x)dx = 0 \quad (i = 1, 2, 3, \dots, 2^s). \tag{2}$$

Similarly, the function $\varphi_\ell$ takes the value 1 on one half of each of the intervals

$(b_{j-1},\ b_j)$ and the value $-1$ on the other half, and hence

$$\int_{b_{j-1}}^{b_j} \varphi_\ell(y)dy = 0 \quad (j = 1, 2, 3, \dots, 2^t). \tag{3}$$

Define three functions $f_1, f_2, f_3$ on $\mathbb{I}^2$ by setting

$$f_1(x, y) = f(a_{i-1}, y)$$

for $a_{i-1} \le x < a_i$, $0 \le y < 1$, $i = 1, 2, 3, \dots, 2^s$;

$$f_2(x, y) = f(x, b_{j-1})$$

for $0 \le x < 1$, $b_{j-1} \le y < b_j$, $j = 1, 2, 3, \dots, 2^t$; and

$$f_3(x, y) = f(a_{i-1}, b_{j-1})$$

for $a_{i-1} \le x < a_i$, $b_{j-1} \le y < b_j$, $i = 1, 2, 3, \dots, 2^s$, $j = 1, 2, 3, \dots, 2^t$. Then in view of Fubini's theorem and relations (2) and (3) we have

$$\begin{aligned}\int_0^1 \int_0^1 & f_1(x, y)\varphi_k(x)\varphi_\ell(y)dxdy \\ &= \int_0^1 \left[\sum_{i=1}^{2^s} f(a_{p-1}, y) \int_{a_{i-1}}^{a_i} \varphi_k(x)dx\right]\varphi_\ell(y)dy = 0,\end{aligned}$$

$$\begin{aligned}\int_0^1 \int_0^1 & f_2(x, y)\varphi_k(x)\varphi_\ell(y)dxdy \\ &= \int_0^1 \left[\sum_{j=1}^{2^t} f(x, b_{j-1}) \int_{b_{j-1}}^{b_j} \varphi_\ell(y)dy\right]\varphi_k(x)dx = 0,\end{aligned}$$

and

$$\begin{aligned}\int_0^1 \int_0^1 & f_3(x, y)\varphi_k(x)\varphi_\ell(y)dxdy \\ &= \sum_{i=1}^{2^s}\sum_{j=1}^{2^t} f(a_{i-1}, b_{j-1})\left[\int_{a_{i-1}}^{a_i} \varphi_k(x)dx\right]\left[\int_{b_{j-1}}^{b_j} \varphi_\ell(y)dy\right] = 0.\end{aligned}$$

Using these equations in the definition (1) of $\hat{f}(\mathbf{k})$ for $n = 2$ we get

$$|\hat{f}(\mathbf{k})| = \left|\int_0^1 \int_0^1 f(x, y)\varphi_k(x)\varphi_\ell(y)dxdy\right|$$

$$
\begin{aligned}
&= \left| \int_0^1 \int_0^1 (f - f_1 - f_2 + f_3)(x,y)\varphi_k(x)\varphi_\ell(y)dxdy \right| \\
&\le \int_0^1 \int_0^1 |(f - f_1 - f_2 + f_3)(x,y)|dxdy \\
&\le \left( \int_0^1 \int_0^1 |(f - f_1 - f_2 + f_3)(x,y)|^p dxdy \right)^{1/p} (1)^{2/q},
\end{aligned}
$$

in view of the Hölder's inequality (when $p > 1$) since $f - f_1 - f_2 + f_3 \in L^p(\mathbb{I}^2)$, where $q$ is such that $1/p + 1/q = 1$. Observe that when $p = 1$, we don't use Hölder's inequality and in that case we consider the inequality except last step. In any case, it follows that

$$
\begin{aligned}
|\hat{f}(\mathbf{k})|^p &\le \int_0^1 \int_0^1 |(f - f_1 - f_2 + f_3)(x,y)|^p dxdy \\
&= \sum_{i=1}^{2^s} \sum_{j=1}^{2^t} \int_{a_{i-1}}^{a_i} \int_{b_{j-1}}^{b_j} |(f - f_1 - f_2 + f_3)(x,y)|^p dxdy \\
&= \sum_{i=1}^{2^s} \sum_{j=1}^{2^t} \int_{a_{i-1}}^{a_i} \int_{b_{j-1}}^{b_j} |f(x,y) - f(x,b_{j-1}) \\
&\qquad\qquad - f(a_{i-1},y) + f(a_{i-1},b_{j-1})|^p dxdy \\
&\le \sum_{i=1}^{2^s} \sum_{j=1}^{2^t} (V_p(f;[a_{i-1},a_i] \times [b_{j-1},b_j]))^p \\
&\qquad\qquad \times (a_i - a_{i-1})(b_j - b_{j-1}) \\
&\le \frac{1}{2^s 2^t} \left( V_p \left( f;[0,1]^2 \right) \right)^p \\
&\le \frac{2^2}{k\ell} \left( V_p \left( f;[0,1]^2 \right) \right)^p,
\end{aligned}
$$

in view of Lemma 7. Thus we get

$$
|\hat{f}(\mathbf{k})| \le \frac{4 \cdot V_p \left( f;[0,1]^2 \right)}{(k\ell)^{1/p}}. \tag{4}
$$

This completes the proof of Theorem 3. □

**Proof of Theorem 4.** Here also we will carry out the proof for $n = 2$ and use notations as in the proof of Theorem 3. Since $f \in BV_H^{(p)}([0,1]^2)$, in view of Lemma 8 (for $n = 2$), the discontinuities of $f$ lie on countable number of parallels to the axes and hence $f$ is measurable over $\mathbb{I}^2$ in the sense of Lebesgue. Further, by Lemma 5, $f$ is bounded over $[0,1]^2$ and hence $f \in L^p(\mathbb{I}^2)$. As $BV_H^{(p)}([0,1]^2) \subset BV_V^{(p)}([0,1]^2)$, $f \in L^p(\mathbb{I}^2) \cap BV_V^{(p)}([0,1]^2)$. Therefore if $\mathbf{k} = (k,\ell) \in \mathbb{N}^2$, by Theorem 3,

$$\hat{f}(\mathbf{k}) = O\left(\frac{1}{(k\ell)^{1/p}}\right).$$

Next, let $\mathbf{k} = (k,\ell) \in (\mathbb{Z}^+)^2$ be such that $k \neq 0$, $\ell = 0$ and let $a_i$'s and $f_2$ be as defined in the proof of Theorem 3. Then we have

$$\begin{aligned}\int_0^1 \int_0^1 & f_2(x,y)\varphi_k(x)dxdy \\ &= \int_0^1 \left(\sum_{i=1}^{2^s} f(a_{i-1},y)\left[\int_{a_{i-1}}^{a_i} \varphi_k(x)dx\right]\right) dy = 0,\end{aligned}$$

in view of Fubini's theorem and (2); and,

$$\begin{aligned}|\hat{f}(\mathbf{k})| &= \left|\int_0^1 \int_0^1 [f(x,y) - f_2(x,y)]\varphi_k(x)dxdy\right| \\ &\le \left(\int_0^1 \int_0^1 |f(x,y) - f_2(x,y)|^p dxdy\right)^{1/p} (1)^{2/q},\end{aligned}$$

in view of Hölder's inequality as in the proof of Theorem 3. Therefore

$$\begin{aligned}|\hat{f}(\mathbf{k})|^p &\le \int_0^1 \left[\sum_{i=1}^{2^s} \int_{a_{i-1}}^{a_i} |f(x,y) - f(a_{i-1},y)|^p dx\right] dy \\ &\le \int_0^1 \left[\sum_{i=1}^{2^s} (V_p(f(\cdot,y);[a_{i-1},a_i]))^p (a_i - a_{i-1})\right] dy \\ &\le \frac{1}{2^s} \int_0^1 (V_p(f(\cdot,y);[0,1]))^p dy \\ &\le \frac{2}{k} \int_0^1 2^p[(V_p(f;[0,1]^2))^p + (V_p(f(\cdot,0);[0,1]))^p] dy \\ &= \frac{2^{p+1}[(V_p(f;[0,2\pi]^2))^p + (V_p(f(\cdot,0);[0,2\pi]))^p]}{k},\end{aligned}$$

in view of Lemma 7 (for a function of one variable) and Lemma 6. Thus we have

$$\hat{f}(\mathbf{k}) = \hat{f}(k,0) = O\left(\frac{1}{k^{1/p}}\right). \tag{5}$$

The case $k = 0$, $\ell \neq 0$, is similar to the above case and in this case we get

$$\hat{f}(0,\ell) = O\left(\frac{1}{\ell^{1/p}}\right). \tag{6}$$

This completes the proof of the Theorem 4. □

**Remark 9.** Theorem 3 or Theorem 4 with $n = 1$ gives our earlier result [6, Thorem 2.1]. (4), (5) and (6) with $p = 1$ give Walsh analogues of the results of Móricz [7] and Fülöp and Móricz [3, for $n = 2$], except possibly for the exact constant in their case.

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