STABILITY ANALYSIS OF A LASER WITH A MODULATED SATURABLE ABSORBER

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Abstract: The stability analysis of the model for a laser system with a modulated saturable absorber is performed. The model is based in three equations describing the temporal evolution of the photon flux, the population inversion in the active media and the saturation coefficient of the saturable absorber. The system dynamics is discussed in order to find the best control regions of the system.

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1. Introduction

The dynamics of a laser system with a saturable absorber can be described by a model based on the Stats-DeMarz equations which originally were developed to describe oscillations in a Maser [1]. This model has undergone many modifications to be adopted for laser systems [2] and [3]. For a complete phenomenological description of a laser with a saturable absorber, only three equations are needed: the photon-flux equation, an equation for the population inversion...
density on the active medium and the saturable population inversion equation that gives the saturation coefficient. Therefore the Stats De Mars equations for a three level laser system with a saturable absorber without modulation are written as follows.

\[
\begin{align*}
\frac{dS}{dt} &= \Gamma \nu \sigma NS - \Gamma \nu \frac{L_\alpha}{L_m} k_\alpha S - \frac{1}{T} S, \\
\frac{dN}{dt} &= -\beta \frac{\sigma}{\hbar \omega} NS + \frac{N_0 - N}{\tau}, \\
\frac{dk_\alpha}{dt} &= -\frac{2\sigma_\alpha k_\alpha S}{\hbar \omega} + \frac{k_\alpha - k_\alpha}{\tau_\alpha}.
\end{align*}
\]

where \( S \) is the emitted photon density, \( N \) is the population inversion of the active medium, \( k_\alpha \) is the resonant absorption of the saturable absorber, \( \sigma_\alpha \) is the saturable absorber cross-section and \( N_\alpha \) the population inversion of the saturable absorber (\( k_\alpha = -\sigma_\alpha N_\alpha \)). \( \Gamma, \nu, \sigma, \) and \( T \) stand, respectively, for cavity filling coefficient, optical frequency, active medium cross-section, and photon lifetime in the cavity, \( \beta \) is the coefficient which accounts for the difference in population inversion caused by lasing, \( L_m \) and \( L_\alpha \) are, respectively, the active medium and the saturable absorber lengths, \( k_{0\alpha} \) is the linear resonant saturable absorber absorption coefficient without lasing, \( N_0 \) is the population inversion in the active medium without radiation, \( \tau \) and \( \tau_\alpha \) stand for relaxation time in the active medium and in the saturable absorber, respectively, and finally \( \hbar \omega \) is the photon energy.

Defining the next adimensional parameters and variables: \( \dot{t} = t/\tau \), \( G = \tau/T \), \( \delta = \tau/\tau_\alpha \), \( \rho = 2\sigma_\alpha/\beta \sigma, \alpha = \Gamma \nu \sigma TN \) and \( \alpha_\alpha = -\Gamma \nu T k_\alpha L_\alpha/L_m = -\Gamma \nu T \sigma_\alpha n_\alpha/L_m \); \( n(\dot{t}) = \Gamma \nu \sigma T N(\dot{t}), n_\alpha(\dot{t}) = -\Gamma \nu T k_\alpha(\dot{t}) L_\alpha/L_m \) and \( m(\dot{t}) = \beta B \tau S(\dot{t})/\nu = \beta \sigma T S(\dot{t})/\hbar \omega \). The above system can be rewritten as:

\[
\begin{align*}
\frac{dm}{dt} &= Gm(n + n_\alpha - 1), \\
\frac{dn}{dt} &= \alpha - n(m + 1), \\
\frac{dn_\alpha}{dt} &= \delta \alpha_\alpha - n_\alpha(\rho m + \delta).
\end{align*}
\]

All the parameters used to define the saturable absorber are fixed, except for \( \alpha_\alpha \), which includes a measure of the active centers absorbent density, for this reason, \( \alpha_\alpha \) is used as the saturable absorbers identifying parameter. Adding an external linear sinusoidal modulation (e.g. using an Electro Optic Modulator (EOM)) directly into the saturable absorber through its main parameter (i.e.
the above equations may be transform into:

\[
\begin{align*}
\frac{dm}{dt} &= Gm(n + n_\alpha - 1), \\
\frac{dn}{dt} &= \alpha - n(m + 1), \\
\frac{dn_\alpha}{dt} &= \delta_\alpha \left[ \frac{1 + \cos(\omega_c t)}{2} \right] - n_\alpha(\rho m + \delta).
\end{align*}
\]

where \(\omega_c\) stands for the external modulation frequency applied to the EOM.

These three differential equations compose the working system, it must be noted that, in absence of modulation frequency applied to the EOM, \(\omega_c\), the system returns to Eqs. (2), i.e. rate equations for a laser with a passive saturable absorber [4].

2. Linear Stability Analysis

Linear Stability Analysis is used to understand the system dynamics [5] and [6]. The analysis is based on the linear disturbance equations, these equations are derived from the original equations. As is well known the method consist in linearize the describing equations, obtain the initial state condition (i.e. when the derivatives are zero), expand the system about the initial state condition, construct the Jacobian matrix and find the eigenvectors and eigenvalues with the determinant equal to zero. This gives as a result the fixed points of the equations system, which must be analyzed in order to know what type of points are (i.e. fixed, source, saddle, etc.). The equations of interest are Eqs. 3, these equations are non-autonomous due to the explicit time-dependence found in the cosine of the third equation. To be able to perform the Linear Stability Analysis that dependence must be eliminated, to do that, a variable change must be applied, giving as a result, the next equations system:

\[
\begin{align*}
\frac{dm}{dt} &= Gm(n + n_\alpha - 1), \\
\frac{dn}{dt} &= \alpha - n(m + 1), \\
\frac{dn_\alpha}{dt} &= \delta_\alpha \left[ \frac{1 + \cos(x)}{2} \right] - n_\alpha(\rho m + \delta), \\
\frac{dx}{dt} &= \omega_c.
\end{align*}
\]
Naming the Eqs. 4 as:

\[
\begin{align*}
    f(m, n, n_\alpha, x) &= \frac{dm}{dt} = Gm(n + n_\alpha - 1), \\
    g(m, n, n_\alpha, x) &= \frac{dn}{dt} = \alpha - n(m + 1), \\
    h(m, n, n_\alpha, x) &= \frac{dn_\alpha}{dt} = \delta \alpha \left[ \frac{1 + \cos(x)}{2} \right] - n_\alpha (\rho m + \delta), \\
    j(m, n, n_\alpha, x) &= \frac{dx}{dt} = \omega_c.
\end{align*}
\]

(5)

Assuming that \((m^*, n^*, n_\alpha^*, x^*)\) is the steady state, that is \(f(m^*, n^*, n_\alpha^*, x^*) = 0, g(m^*, n^*, n_\alpha^*, x^*) = 0, h(m^*, n^*, n_\alpha^*, x^*) = 0\) and \(j(m^*, n^*, n_\alpha^*, x^*) = 0\). In order to find if the steady state is stable or unstable, a small perturbation (represented by the subscript “\(p\)”) must be added to it,

\[
\begin{align*}
    f &= f^* + f_p, \\
    g &= g^* + g_p, \\
    h &= h^* + h_p, \\
    j &= j^* + j_p.
\end{align*}
\]

(6)

with \((f_p, g_p, h_p, j_p) \ll 1\).

Now, the main question for practical purposes is: Will the perturbations grow (steady state unstable) or decay (steady state stable)? To be able to observe if the perturbations grow or decay, the perturbations derivatives must be found.

\[
\begin{align*}
    \frac{df_p}{dt} &= \frac{dm}{dt} = f(m, n, n_\alpha, x) = f(m^* + f_p, n^* + g_p, n_\alpha^* + h_p, x^* + j_p) \\
    &= f(m^*, n^*, n_\alpha^*, x^*) + \frac{\delta}{\delta m} f(m^*, n^*, n_\alpha^*, x^*) f_p \\
    &\quad + \frac{\delta}{\delta n} f(m^*, n^*, n_\alpha^*, x^*) g_p + \frac{\delta}{\delta n_\alpha} f(m^*, n^*, n_\alpha^*, x^*) h_p \\
    &\quad + \frac{\delta}{\delta x} f(m^*, n^*, n_\alpha^*, x^*) j_p + \text{high order terms}
\end{align*}
\]

(7)

Following the Taylor series expansion shown in Eqs. (7) the derivatives for each
perturbation are:

\[
\frac{df_p}{dt} = \frac{\delta}{\delta m} f \cdot f_p + \frac{\delta}{\delta n} f \cdot g_p + \frac{\delta}{\delta n_\alpha} f \cdot h_p + \frac{\delta}{\delta x} f \cdot j_p,
\]

\[
\frac{dg_p}{dt} = \frac{\delta}{\delta m} g \cdot f_p + \frac{\delta}{\delta n} g \cdot g_p + \frac{\delta}{\delta n_\alpha} g \cdot h_p + \frac{\delta}{\delta x} g \cdot j_p,
\]

\[
\frac{dh_p}{dt} = \frac{\delta}{\delta m} h \cdot f_p + \frac{\delta}{\delta n} h \cdot g_p + \frac{\delta}{\delta n_\alpha} h \cdot h_p + \frac{\delta}{\delta x} h \cdot j_p,
\]

\[
\frac{dj_p}{dt} = \frac{\delta}{\delta m} j \cdot f_p + \frac{\delta}{\delta n} j \cdot g_p + \frac{\delta}{\delta n_\alpha} j \cdot h_p + \frac{\delta}{\delta x} j \cdot j_p.
\]

(8)

The system presented in Eqs. (8) can be rewritten in the matrix form:

\[
\begin{pmatrix}
\frac{d}{dt} f_p \\
\frac{d}{dt} g_p \\
\frac{d}{dt} h_p \\
\frac{d}{dt} j_p
\end{pmatrix}
= \begin{pmatrix}
\frac{\delta}{\delta m} f \\
\frac{\delta}{\delta n} f \\
\frac{\delta}{\delta m_\alpha} f \\
\frac{\delta}{\delta x} f
\end{pmatrix}
\begin{pmatrix}
\delta \\
\delta \\
\delta \\
\delta
\end{pmatrix}
\begin{pmatrix}
f_p \\
g_p \\
h_p \\
j_p
\end{pmatrix}
= J
\begin{pmatrix}
f_p \\
g_p \\
h_p \\
j_p
\end{pmatrix},
\]

where \(J\) denotes the Jacobian matrix of the original system at the steady state. Substituting the values in the Jacobian function and the next matrix is obtained:

\[
\begin{pmatrix}
G(n + n_\alpha - 1) & Gm & Gm & 0 \\
-n & -m - 1 & 0 & 0 \\
-\rho n_\alpha & 0 & -\rho m - \delta - \frac{\delta n_\alpha}{2} \sin(x) & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

(10)

The next step is to find the eigenvectors and eigenvalues of the system \((J - \lambda I) \frac{d}{dt} = 0\), so the determinant would be:

\[
|J - \lambda I| = 0.
\]

(11)

The matrix for the former equation is:

\[
\det \begin{pmatrix}
G(n + n_\alpha - 1) - \lambda & Gm & Gm & 0 \\
-n & -m - 1 - \lambda & 0 & 0 \\
-\rho n_\alpha & 0 & -\rho m - \delta - \lambda - \frac{\delta n_\alpha}{2} \sin(x) & 0 \\
0 & 0 & 0 & -\lambda
\end{pmatrix} = 0,
\]

(12)

The solutions for the perturbed steady-state of the original system are:

\[
m^* = 0,
\]

\[
n^* = n_\alpha,
\]

\[
n_\alpha^* = n_\alpha,
\]

\[
x^* = 0.
\]

(13)
Substituting the former solutions in Eqs. (12), the matrix transforms into:

$$
\begin{pmatrix}
G(\alpha + \alpha_\alpha - 1) - \lambda & 0 & 0 & 0 \\
-\alpha & -1 - \lambda & 0 & 0 \\
-\rho \alpha_\alpha & 0 & -\delta - \lambda & 0 \\
0 & 0 & 0 & -\lambda 
\end{pmatrix} = 0,
$$

Matrix (14) has the following characteristic equation:

$$
[G(\alpha + \alpha_\alpha - 1) - \lambda][-1 - \lambda][-\delta - \lambda][-\lambda] = 0,
$$

where $\lambda_1 = G(\alpha + \alpha_\alpha - 1)$, $\lambda_2 = -1$, $\lambda_3 = -\delta$ and $\lambda_4 = 0$ are eigenvalues which are all real; being $\lambda_2$ and $\lambda_3$ always negative (i.e. the perturbation will decay) and $\lambda_4$ is critically stable. Therefore, the stability condition is defined only by the sign of $\lambda_1$, i.e. the fixed point is a source when $\alpha_\alpha + \alpha > 1$ as shown in Fig. 1.

3. Conclusions

The stability analysis of the model for a laser system with a modulated saturable absorber is performed. The model is based in three equations describing the temporal evolution of the photon flux, the population inversion in the active media and the saturation coefficient of the saturable absorber. The stability condition of the systems is found to depend only on the sign of the first eigenvalue of the system $\lambda$. As is graphically shown, the stability conditions are given by the relation between $\alpha$ and $\alpha_\alpha$. 

![Figure 1: Stability condition given by the relation between $\alpha$ and $\alpha_\alpha$](image)
References


