DECOMPOSITION ALGORITHMS POTENTIALS FOR THE NON-HOMOGENEOUS GENERALIZED NETWORKED PROBLEMS OF LINEAR-FRACTIONAL PROGRAMMING

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Abstract: We use potentials for calculate a reduced costs in the increment of the objective function for the linear-fractional non-homogeneous flow programming optimization problem with additional constraints of general kind. The effective algorithm for solution of the system of potentials for a sparse matrix is considered.

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Key Words: linear-fractional programming, two-component sparse matrix, increment of the objective function, decomposition, potentials, generalized multi-network

1. Introduction

We consider the linear-fractional non-homogeneous flow programming optimization problem with additional constraints of general kind:

$$f(x) = \frac{p(x)}{q(x)} = \frac{\sum_{(i,j) \in U} \sum_{k \in K(i,j)} p_{ij}^k x_{ij}^k + \beta}{\sum_{(i,j) \in U} \sum_{k \in K(i,j)} q_{ij}^k x_{ij}^k + \gamma} \longrightarrow \max, \quad (1)$$

$$\sum_{j \in I_i^+(U^k)} x_{ij}^k - \sum_{j \in I_i^-(U^k)} \mu_{ji}^k x_{ji}^k = a_i^k, i \in I^k, k \in K; \quad (2)$$

$$\sum_{(i,j) \in U} \sum_{k \in K(i,j)} \lambda_{ij}^{kp} x_{ij}^k = \alpha_p, p = \overline{1, l}; \quad (3)$$

$$\sum_{k \in K_0(i,j)} x_{ij}^k \leq d_{ij}^0, x_{ij}^k \geq 0, k \in K_0(i,j), (i,j) \in U_0; \quad (4)$$

$$0 \leq x_{ij}^k \leq d_{ij}^k, k \in K_1(i,j), (i,j) \in U; \quad (5)$$

$$x_{ij}^k \geq 0, k \in K(i,j) \setminus K_1(i,j), (i,j) \in U \setminus U_0, \quad (6)$$

where $G = (I, U)$ is a finite oriented connected multigraph (multinetwork) without multiple arcs and loops, I is a set of nodes and $U \subset I \times I$ is a set of multiarcs. The finite non-empty set $K = \{1, \dots, |K|\}$ is the set of different products (commodities) transported through the multinetwork G . Let us denote a connected network corresponding to a certain type of flow $k \in K$: $G^k = (I^k, U^k)$, $I^k \subseteq I$, $U^k = \{(i,j)^k : (i,j) \in \hat{U}^k\}$, $\hat{U}^k \subseteq U$ – a set of arcs of the multinetwork G carrying the flow of type $k \in K$, $I^k = I(U^k)$, $I(U^k) = \{i \in I : i \in I^k\}$ is the set of nodes used for transporting (producing/consuming/transiting) the k^{th} product. In order to distinguish the products, which can simultaneously pass through an multiarc $(i,j) \in U$, we introduce the set $K(i,j) = \{k \in K : (i,j)^k \in U^k\}$. Similarly, $K(i) = \{k \in K : i \in I^k\}$ is the set of products simultaneously transported through a node $i \in I$. Let's define a set U_0 as an arbitrary subset of multiarcs of the multinetwork G , $U_0 \subseteq U$. Each multiarc $(i,j) \in U_0$ has an aggregate capacity constraint for a total amount of transported products from a subset $K_0(i,j) \subseteq K(i,j)$, $|K_0(i,j)| > 1$. For all multiarcs $(i,j) \in U$ we assume the amount of each product $k \in K(i,j)$ to be non-negative. For a set $K_1(i,j)$ are true the following conditions: $K_1(i,j) = K(i,j) \setminus K_0(i,j)$, if $(i,j) \in U_0$ and $K_1(i,j) \subseteq K(i,j)$, if $(i,j) \in U \setminus U_0$. Moreover, each multiarc $(i,j) \in U$ can be equipped with carrying capacities for products from a set $K_1(i,j)$, where $K_1(i,j) \subseteq K(i,j)$ is an arbitrary subset of products transported through the multiarc (i,j) . $I_i^+(U^k) = \{j \in I^k : (i,j) \in U^k\}$, $I_i^-(U^k) = \{j \in I^k : (j,i) \in U^k\}$; x_{ij}^k – amount of the k^{th} product transported through an multiarc (i,j) ; d_{ij}^k – carrying capacity of an multiarc (i,j) for the k^{th} product; d_{ij}^0 – aggregate capacity of an multiarc $(i,j) \in U_0$ for a total amount of products $K_0(i,j)$; λ_{ij}^{kp} – weight of a unit of the k^{th} product transported through an multiarc (i,j) in the p^{th} additional constraint; μ_{ij}^k – a flow transformation coefficient for arc $(i,j)^k$, $\mu_{ij}^k \in [0, 1]$; α_p – total weighted amount of products imposed by

the p^{th} additional constraint; a_i^k – intensity of a node i for the k^{th} product, $p_{ij}^k, q_{ij}^k, \beta, \gamma \in \mathbf{R}$.

2. Sparse Systems for a Potentials

The formula of the increment of the objective function (1) for the extreme linear-fractional non-homogeneous problem of flow programming (1)–(6) with additional constraints has the following kind:

$$\Delta f = \frac{\sum_{k \in K} \sum_{(\tau, \rho)^k \in U_N^k} \tilde{\Delta}^k(\tau, \rho) \Delta x_{\tau\rho}^k + \sum_{(i, j) \in U^*} r_{ij} \Delta z_{ij}}{\sum_{k \in K} \sum_{(\tau, \rho)^k \in U^k \setminus U_L^k} \Delta_Q^k(\tau, \rho) \left(x_{\tau\rho}^k + \Delta x_{\tau\rho}^k \right) + Q + \gamma}, \quad (7)$$

where

$$\tilde{\Delta}^k(\tau, \rho) = \Delta^k(\tau, \rho) - \sum_{p=1}^l r_p \Lambda_{\tau\rho}^{kp} - \sum_{(i, j) \in U^*} r_{ij} \delta_{ij}(B_{\tau\rho}^k), \quad (8)$$

$$\Delta^k(\tau, \rho) = \Delta_P^k(\tau, \rho) - f(x) \Delta_Q^k(\tau, \rho),$$

$$\Delta_P^k(\tau, \rho) = p_{\tau\rho}^k + \sum_{(i, j)^k \in U_L^k} p_{ij}^k \delta_{ij}^k(\tau, \rho),$$

$$\Delta_Q^k(\tau, \rho) = q_{\tau\rho}^k + \sum_{(i, j)^k \in U_L^k} q_{ij}^k \delta_{ij}^k(\tau, \rho),$$

$$\Lambda_{\tau\rho}^{kp} = \lambda_{\tau\rho}^{kp} + \sum_{(i, j)^k \in U_L^k} \lambda_{ij}^{kp} \delta_{ij}^k(\tau, \rho), (\tau, \rho)^k \in U^k \setminus U_L^k, p = \overline{1, l},$$

$$\delta_{ij}(B_{\tau\rho}^k) = \begin{cases} \delta_{ij}^k(\tau, \rho), k \in K_0(i, j), & 0, k \notin K_0(i, j), \\ (i, j) \in U_0, (\tau, \rho)^k \in U^k \setminus U_L^k, k \in K. \end{cases}$$

$$r_p = \sum_{k \in K} \sum_{(\tau, \rho)^k \in U_B^k} \Delta^k(\tau, \rho) \nu_{t(\tau, \rho)^k, p}, p = \overline{1, l},$$

$$r_{ij} = \sum_{k \in K} \sum_{(\tau, \rho)^k \in U_B^k} \Delta^k(\tau, \rho) \nu_{t(\tau, \rho)^k, l + \xi(i, j)}, (i, j) \in U^*,$$

where $x = (x_{ij}^k, (i, j) \in U, k \in K(i, j))$ be a multifold of the problem (1)–(6) i. e. components of the vector x meet the conditions (2)–(6). Along with the

multiflow x let us define support multiflow $\{x, U_S\}$ as a pair [1], containing of an arbitrary multiflow x and a support [1, 4] U_S of multigraph $G = \{I, U\}$ of the problem (1)-(6), $U_S = \{U_S^k, k \in K, U^*\}$, $U_S^k \subset U^k, k \in K$; $U^* \subseteq \overline{U}_0, \overline{U}_0 = \{(i, j) \in U_0 : |K_S^0(i, j)| > 1\}$, $K_S(i, j) = \{k \in K(i, j) : (i, j)^k \in U_S^k\}$, $(i, j) \in U$, $K_S^0(i, j) = K_S(i, j) \cap K_0(i, j)$, $(i, j) \in U_0$ of the problem (1)-(6). A support U_S of multigraph $G = \{I, U\}$ of the problem (1)-(6) includes a support $U_L = \{U_L^k, k \in K\}$ for system (2) and the set $U_B = \{U_B^k, k \in K\}$ of bicycling arcs [1]-[4].

Let's consider some other multiflow

$$\overline{x} = (\overline{x}_{ij}^k = x_{ij}^k + \Delta x_{ij}^k : (i, j) \in U, k \in K(i, j))$$

Then $\Delta x = (\Delta x_{ij}^k, (i, j) \in U, k \in K(i, j))$ is the vector of flow increments along the multiarc $(i, j) \in U$,

$$z_{ij} = \sum_{k \in K_0(i, j)} x_{ij}^k, \quad \overline{z}_{ij} = \sum_{k \in K_0(i, j)} \overline{x}_{ij}^k, \quad (9)$$

$$\Delta z_{ij} = \overline{z}_{ij} - z_{ij} = \sum_{k \in K_0(i, j)} \Delta x_{ij}^k, \quad (i, j) \in U_0,$$

$\delta^k(\tau, \rho) = (\delta_{ij}^k(\tau, \rho), (i, j)^k \in U^k)$ – characteristic vector, entailed by arc $(\tau, \rho)^k \in U^k \setminus U_L^k$ concerning a support U_L^k for system (2), $k \in K$ [2],

$$Q = \sum_{k \in K} \sum_{(i, j)^k \in U_L^k} q_{ij}^k \left(\tilde{x}_{ij}^k - \sum_{(\tau, \rho)^k \in U^k \setminus U_L^k} \tilde{x}_{\tau\rho}^k \delta_{ij}^k(\tau, \rho) \right), \quad (10)$$

where $\tilde{x}^k = (\tilde{x}_{ij}^k, (i, j)^k \in U^k)$ – is partial solution of the nonhomogeneous system (2) and $\delta^k(\tau, \rho) = (\delta_{ij}^k(\tau, \rho), (i, j)^k \in U^k)$, $(\tau, \rho)^k \in U^k \setminus U_L^k, k \in K$ is the system of characteristic vectors, entailed by an arc $(\tau, \rho)^k \in U^k \setminus U_L^k, k \in K$ for the fixed $k \in K$ [2, 4].

Remark. We use the partial solution

$$\tilde{x}^k = (\tilde{x}_{ij}^k, (i, j)^k \in U^k), k \in K$$

which is constructed to the following rules: non-supporting elements $(\tau, \rho)^k \in U^k \setminus U_L^k, k \in K$ are equal to zeros and supporting elements $(i, j)^k \in U_L^k, k \in K$ satisfy system (2).

For calculation of reduced costs (8) we will use the sparse system potentials. Let's write down system (11)-(13) of potentials $r, u^k, k \in K$:

$$r = (r_p : p = \overline{1, l}; r_{ij}, (i, j) \in U^*),$$

$$u^k = (u_i^k, i \in I^k), k \in K,$$

for a support U_S [5] of the multigraph G for a problem (1)-(6):

$$\begin{aligned} u_i^k - \mu_{ij}^k u_j^k + \sum_{p=1}^l \lambda_{ij}^{kp} r_p &= - \left(p_{ij}^k - f(x) q_{ij}^k \right), \\ (i, j) &\in U \setminus U^*, k \in K_S(i, j); \end{aligned} \quad (11)$$

$$\begin{aligned} u_i^k - \mu_{ij}^k u_j^k + \sum_{p=1}^l \lambda_{ij}^{kp} r_p + r_{ij} &= - \left(p_{ij}^k - f(x) q_{ij}^k \right), \\ (i, j) &\in U^*, k \in K_S^0(i, j); \end{aligned} \quad (12)$$

$$\begin{aligned} u_i^k - \mu_{ij}^k u_j^k + \sum_{p=1}^l \lambda_{ij}^{kp} r_p &= - \left(p_{ij}^k - f(x) q_{ij}^k \right), \\ (i, j) &\in U \setminus U^*, k \in K_S(i, j) \text{ or } (i, j) \in U^*, k \in K_S(i, j) \setminus K_S^0(i, j). \end{aligned} \quad (13)$$

Consider the effective algorithm for solving sparse system of potentials which is based on principles of decomposition of sparse system (11)-(13).

Let's construct a matrix

$$D = \begin{pmatrix} D_1 \\ D_2 \end{pmatrix}, D_1 = \left(\Lambda_{\tau\rho}^{kp}, p = \overline{1, l}, t(\tau, \rho)^k = \overline{1, |U_B|} \right),$$

$$D_2 = \left(\delta_{ij}(B_{\tau\rho}^k), \xi(i, j) = \overline{1, |U^*|}, t(\tau, \rho)^k = \overline{1, |U_B|} \right),$$

$\xi = \xi(i, j)$ – a number of arc $(i, j) \in U^*, \xi \in \{1, 2, \dots, |U^*|\}$, where

$$\Lambda_{\tau\rho}^{kp} = \lambda_{\tau\rho}^{kp} + \sum_{(i,j)^k \in U_L^k} \lambda_{ij}^{kp} \delta_{ij}^k(\tau, \rho), (\tau, \rho)^k \in U^k \setminus U_L^k.$$

$\delta^k(\tau, \rho) = (\delta_{ij}^k(\tau, \rho), (i, j)^k \in U^k)$ – characteristic vector, entailed by arc $(\tau, \rho)^k \in U^k \setminus U_L^k$ concerning a support U_L^k for system (2), $k \in K, \tilde{t} = |U_B|$,

$U_B = \{U_B^k, k \in K\}$ [2, 4].

The vector

$$r = (r_p, p = \overline{1, l}; r_{ij}, (i, j) \in U^*),$$

we compute from the system

$$D'r = \omega \quad (14)$$

where $\omega = (\omega_t, t = \overline{1, \tilde{t}})$,

$$\omega_t = - \sum_{(i,j)^k \in B_{\tau\rho}^k} \left(p_{ij}^k - f(x)q_{ij}^k \right) \delta_{ij}^k(\tau, \rho), t = t(\tau, \rho)^k, k \in K,$$

The system (14) has unique solution, as $\det D \neq 0$ [4].

For each $k \in K$ let us put $u_i^k = 0$ for some $i \in I^k$. The other components of vectors $u^k = (u_i^k : i \in I^k), k \in K$ are uniquely determined by the system (15):

$$\begin{aligned} u_i^k - \mu_{ij}^k u_j^k &= - \sum_{p=1}^l \lambda_{ij}^{kp} r_p - \left(p_{ij}^k - f(x)q_{ij}^k \right), \\ (i, j) &\in U \setminus U^*, k \in K_S(i, j), (i, j)^k \in U_L^k; \\ u_i^k - \mu_{ij}^k u_j^k &= - \sum_{p=1}^l \lambda_{ij}^{kp} r_p - r_{ij} - \left(p_{ij}^k - f(x)q_{ij}^k \right), \\ (i, j) &\in U^*, k \in K_S^0(i, j), (i, j)^k \in U_L^k; \\ u_i^k - \mu_{ij}^k u_j^k &= - \sum_{p=1}^l \lambda_{ij}^{kp} r_p - \left(p_{ij}^k - f(x)q_{ij}^k \right), \\ (i, j) &\in U^*, k \in K_S(i, j) \setminus K_S^0(i, j), (i, j)^k \in U_L^k. \end{aligned} \quad (15)$$

The system (15) consists from $|K|$ independent subsystems. For calculation nonzero components of vector $u^k = (u_i^k : i \in I^k)$ for every independent subsystem (15) for fixed $k \in K$ we are able to take using $O(|I^k|)$ arithmetical operations, where $|I^k|$ – the number of nodes of the graph $G^k = (I^k, U^k)$. The algorithm with $O(n)$ computational complexity in the worst case is used for calculation nonzero component of everyone characteristic vector $\delta^k(\tau, \rho)$, where $n = |I^k|$ [2, 4].

We add to a vector $r = (r_p : p = \overline{1, l}; r_{ij}, (i, j) \in U^*)$ the following components $r_{ij} = 0, (i, j) \in U_0 \setminus U^*$. Let's receive a new vector

$$\tilde{r} = (r_p : p = \overline{1, l}; r_{ij}, (i, j) \in U^*; r_{ij} = 0, (i, j) \in U_0 \setminus U^*).$$

A reduced costs $\tilde{\Delta}_{ij}^k$, we calculate for the arcs $(i, j)^k \in U_N^k$, $U_N^k = U^k \setminus U_S^k$, $k \in K$ and also for the arcs $(i, j)^k$, $k \in K_S^0(i, j)$, $(i, j) \in U^*$, using the formulas:

$$\tilde{\Delta}_{ij}^k = - \left(p_{ij}^k - f(x) q_{ij}^k \right) - (u_i^k - \mu_{ij}^k u_j^k + \sum_{p=1}^l \lambda_{ij}^{kp} r_p).$$

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