NORM INEQUALITIES FOR SEQUENCES OF BOUNDED LINEAR OPERATORS IN HILBERT SPACES

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Abstract: Let $B(H)$ be the space of all bounded linear operators on a complex Hilbert space H , and let $A_1, \dots, A_n \in B(H)$. In this article, we obtain new upper bounds for $\left\| \sum_{j=1}^n A_j \right\|$. Moreover, we establish and generalize inequalities for the operator norm of sums of bounded linear operators in Hilbert spaces.

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1. Introduction

Let H be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and let $B(H)$ be the space of all bounded linear operators on H . For $A \in B(H)$, let A^* denote the adjoint operator of A , and $\|A\|$ denote the usual operator norm given by

$$\|A\| = \sup_{\|x\|=1} \|Ax\|,$$

where $\|Ax\| = \langle Ax, Ax \rangle^{\frac{1}{2}}$.

For $A_1, \dots, A_n \in B(H)$, several inequalities that provide alternative upper bounds for $\left\| \sum_{j=1}^n A_j \right\|$ have been established by many authors (we refer the read-

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ers to [1], [2], [3], as well as [5], and the references therein). For example, [3] established various inequalities which give upper bounds for $\left\| \sum_{j=1}^n A_j \right\|$. Subsequently, [2] used a worthy technique to obtain new upper bounds for $\left\| \sum_{j=1}^n A_j \right\|$. Recently, [1] applied different approaches to improve these upper bounds. Moreover, [1] got two refinements of the generalized triangle inequality for $\left\| \sum_{j=1}^n A_j \right\|$. Specifically, he proved the following inequalities:

$$\left\| \sum_{k=1}^n A_k \right\|^2 \leq \left\| \sum_{k=1}^n A_k^* A_k \right\| + \frac{1}{2} \left\| \sum_{j=1}^n A_j^* \sum_{k=1}^n A_k - \sum_{k=1}^n A_k^* A_k \right\|^2 + \frac{1}{2}, \quad (1)$$

$$\left\| \sum_{k=1}^n A_k \right\|^2 \leq \left\| \sum_{k=1}^n A_k^* A_k \right\| + \frac{1}{2} \left\| (n-2) \sum_{k=1}^n A_k^* A_k + \sum_{k=1}^n A_k^* \sum_{k=1}^n A_k \right\|, \quad (2)$$

and

$$\left\| \sum_{k=1}^n A_k \right\|^2 \leq \left\| \sum_{k=1}^n A_k^* A_k \right\| + \frac{1}{4} \left\| \sum_{1 \leq j \neq k \leq n} A_j^* A_k + I \right\|^2. \quad (3)$$

The purpose of this work is to establish some new inequalities for the operator norm of sums of bounded linear operators in Hilbert spaces. Also, we generalize the inequalities (1) and (2). Moreover, we use a tranquil technique to attain a refinement of [Inequality (5), [7]].

2. The Main Results

In this section, we introduce and prove some new inequalities for the operator norm of sums of bounded linear operators in Hilbert spaces. Also, we improve and generalize some known results concerning $\left\| \sum_{j=1}^n A_j \right\| \leq \sum_{j=1}^n \|A_j\|$. The proofs of our sequels mainly depend on [1] as well as the following lemmas.

Lemma 1 ([4]). *For $a, b \geq 0$, $0 \leq \alpha \leq 1$, and $r \neq 0$. Let $M_r(a, b, \alpha) = (\alpha a^r + (1 - \alpha)b^r)^{\frac{1}{r}}$ and $M_0(a, b, \alpha) = a^\alpha b^{1-\alpha}$. Then*

$$M_r(a, b, \alpha) \leq M_s(a, b, \alpha) \quad \text{for } r \leq s.$$

Lemma 2 ([6]). *Let $A \in B(H)$ be a positive operator, and let $x \in H$ be a unit vector. Then*

$$\langle Ax, x \rangle^r \leq \langle A^r x, x \rangle \quad \text{for } r \geq 1.$$

Let us use this lemma to generalize inequality (1).

Lemma 3. *Let $A_1, A_2, \dots, A_n \in B(H)$ and $r \geq 1$. Then*

$$\left\| \sum_{k=1}^n A_k \right\|^2 \leq \left\| \sum_{k=1}^n A_k^* A_k \right\| + \left(\frac{1}{2} + \frac{1}{2} \left\| \sum_{1 \leq j \neq k \leq n} A_j^* A_k \right\|^{2r} \right)^{\frac{1}{r}}.$$

Proof. For any $x \in H$, Lemma 1 yields that

$$\begin{aligned} \left\| \sum_{k=1}^n A_k x \right\|^2 &= \sum_{k=1}^n \langle A_k^* x, A_k x \rangle + \sum_{1 \leq j \neq k \leq n} \operatorname{Re} \langle A_k x, A_j x \rangle \\ &= \sum_{k=1}^n \langle A_k^* x, A_k x \rangle + \operatorname{Re} \left\langle \sum_{1 \leq j \neq k \leq n} A_j^* A_k x, x \right\rangle \\ &\leq \sum_{k=1}^n \langle A_k^* A_k x, x \rangle + \frac{1}{2} \left\| \sum_{1 \leq j \neq k \leq n} A_j^* A_k x \right\|^2 + \frac{\|x\|^2}{2} \\ &\leq \left\langle \sum_{k=1}^n A_k^* A_k x, x \right\rangle + \left(\frac{1}{2} \left\| \sum_{1 \leq j \neq k \leq n} A_j^* A_k x \right\|^{2r} + \frac{1}{2} \|x\|^{2r} \right)^{\frac{1}{r}}. \end{aligned}$$

We complete the proof by taking the supremum over all unit vectors. □

In the above lemma, take $r = 1$, we get inequality (1).

As a direct consequence of Lemma 3 we obtain the following corollary.

Corollary 4. *Let $A \in B(H)$ and $r \geq 1$. Then*

$$\|A\|^2 \leq \left\| \frac{A^* A + A A^*}{2} \right\| + \left(\frac{1}{2} + \frac{1}{4} \|A^* A - A A^*\|^{2r} \right)^{\frac{1}{r}}.$$

Proof. For any bounded linear operator $A \in B(H)$, we let $A = B + iC$ be the cartesian decomposition of A . Since $B^* = B$ and $C^* = C$, we obtain that

$$\|A\|^2 \leq \|B^* B + (iC)^*(iC)\| + \left(\frac{1}{2} + \frac{1}{2} \|B^*(iC) + (iC)^* B\|^{2r} \right)^{\frac{1}{r}}$$

$$\begin{aligned}
&= \|B^2 + C^2\| + \left(\frac{1}{2} + \frac{1}{2}\|BC - CB\|^{2r}\right)^{\frac{1}{r}} \\
&= \left\|\frac{A^*A + AA^*}{2}\right\| + \left(\frac{1}{2} + \frac{1}{4}\|A^*A - AA^*\|^{2r}\right)^{\frac{1}{r}}.
\end{aligned}$$

□

A straightforward technique plus Lemma 1 and Lemma 2, we reach the following theorem.

Theorem 5. Let $A_1, \dots, A_n \in B(H)$, $0 < \alpha < 1$. Then for any $r \geq 1$,

$$\begin{aligned}
\left\|\sum_{k=1}^n A_k\right\|^2 &\leq \left\|\sum_{k=1}^n A_k^* A_k\right\| \\
&+ \left\|\sum_{1 \leq j \neq k \leq n} \left[\alpha \left(\frac{1}{\sqrt{4}}\right)^r I + (1 - \alpha)((A_k^* + A_j^*)(A_k + A_j))^{\frac{r}{1-r}}\right]\right\|^{\frac{1}{r}}.
\end{aligned}$$

Proof. Let $x \in H$ be any vector. By Lemmas 1-2, we acquire that

$$\begin{aligned}
\left\|\sum_{k=1}^n A_k x\right\|^2 &= \sum_{k=1}^n \langle A_k^* A_k x, x \rangle + \frac{1}{4} \sum_{1 \leq j \neq k \leq n} \left(\|(A_k + A_j)x\|^2 - \|(A_k - A_j)x\|^2\right) \\
&\leq \sum_{k=1}^n \langle A_k^* A_k x, x \rangle + \frac{1}{4} \sum_{1 \leq j \neq k \leq n} \|(A_k + A_j)x\|^2 \\
&\leq \sum_{k=1}^n \langle A_k^* A_k x, x \rangle + \sum_{1 \leq j \neq k \leq n} \left(\alpha \left(\frac{1}{\sqrt{4}}\right)^r + (1 - \alpha)\|(A_k + A_j)x\|^{\frac{2r}{1-r}}\right)^{\frac{1}{r}} \\
&= \sum_{k=1}^n \langle A_k^* A_k x, x \rangle \\
&+ \sum_{1 \leq j \neq k \leq n} \left(\alpha \left(\frac{1}{\sqrt{4}}\right)^r + (1 - \alpha)\langle (A_k^* + A_j^*)(A_k + A_j)x, x \rangle^{\frac{r}{1-r}}\right)^{\frac{1}{r}} \\
&\leq \sum_{k=1}^n \langle A_k^* A_k x, x \rangle \\
&+ \sum_{1 \leq j \neq k \leq n} \left(\alpha \left(\frac{1}{\sqrt{4}}\right)^r + (1 - \alpha)\left\langle ((A_k^* + A_j^*)(A_k + A_j))^{\frac{r}{1-r}} x, x \right\rangle\right)^{\frac{1}{r}}
\end{aligned}$$

$$\begin{aligned}
 &= \left\langle \sum_{k=1}^n A_k^* A_k x, x \right\rangle \\
 &+ \sum_{1 \leq j \neq k \leq n} \left(\left\langle \left(\alpha \left(\frac{1}{\sqrt{4}} \right)^r I + (1 - \alpha) ((A_k^* + A_j^*)(A_k + A_j))^{\frac{r}{1-r}} \right) x, x \right\rangle \right)^{\frac{1}{r}}.
 \end{aligned}$$

The proof is finished by taking the supremum over all $x \in H$ with $\|x\| = 1$. $\square$

Similarly to what we have done in the proof of Corollary 4, Theorem 5 implies the following.

Corollary 6. *Let $A \in B(H)$. Then*

$$\|A\|^2 \leq \frac{1}{2} \|A^* A + A A^*\| + 2 \left\| \left(\frac{1}{2} \right)^{4r+1} I + \frac{1}{2} (A^* A)^{2r} \right\|^{\frac{1}{r}}.$$

Following the same arguments used in the proof of Theorem 5, we achieve the following lemma.

Lemma 7. *Let $A_1, \dots, A_n \in B(H)$. Then for any $0 \leq \alpha < 1$,*

$$\begin{aligned}
 \left\| \sum_{k=1}^n A_k \right\|^2 &\leq \left\| \sum_{k=1}^n A_k^* A_k \right\| + \alpha n(n-1) \\
 &+ (1 - \alpha) \left\| \sum_{1 \leq j \neq k \leq n} \left(\left(\frac{A_k + A_j}{2} \right)^* \left(\frac{A_k + A_j}{2} \right) \right)^{\frac{1}{1-r}} \right\|.
 \end{aligned}$$

Proof. Let $x \in H$ be a unit vector. Lemma 2 and the approaches used in the proof of Theorem 5 give that

$$\begin{aligned}
 \left\| \sum_{k=1}^n A_k x \right\|^2 &\leq \sum_{k=1}^n \langle A_k^* A_k x, x \rangle + \sum_{1 \leq j \neq k \leq n} \left\| \left(\frac{A_k + A_j}{2} \right) x \right\|^2 \\
 &\leq \sum_{k=1}^n \langle A_k^* A_k x, x \rangle \\
 &+ (1 - \alpha) \sum_{1 \leq j \neq k \leq n} \left\langle \left(\frac{A_k + A_j}{2} \right)^* \left(\frac{A_k + A_j}{2} \right) x, x \right\rangle^{\frac{1}{1-r}} + \alpha n(n-1) \\
 &\leq \left\langle \sum_{k=1}^n A_k^* A_k x, x \right\rangle + \alpha n(n-1)
 \end{aligned}$$

$$+ (1 - \alpha) \left\langle \sum_{1 \leq j \neq k \leq n} \left(\left(\frac{A_k + A_j}{2} \right)^* \left(\frac{A_k + A_j}{2} \right) \right)^{\frac{1}{1-\alpha}} x, x \right\rangle.$$

Take the supremum over all unit vectors $x \in H$, we obtain our lemma. $\square$

Applying the above lemma with $\alpha = 0$, we get inequality (2).

Let us use Lemma 7 to generalize some known inequalities. In particular, we get the following.

Lemma 8. *Let $A_1, A_2, \dots, A_n \in B(H)$ and $0 \leq \alpha < 1$. Then*

$$\begin{aligned} \left\| \sum_{k=1}^n A_k \right\|^2 + (1 - \alpha) \sum_{k=1}^n \|A_k\|^{\frac{2}{1-\alpha}} \\ \leq \left\| \sum_{k=1}^n A_k^* A_k \right\| + \alpha n(n-1) + (1 - \alpha) \sum_{j,k=1}^n \left\| \frac{A_k + A_j}{2} \right\|^{\frac{2}{1-\alpha}}. \end{aligned}$$

Proof. Let $x \in H$ be a unit vector. Use the same manner we used in proving Lemma 7, we obtain

$$\left\| \sum_{k=1}^n A_k x \right\|^2 \leq \left\langle \sum_{k=1}^n A_k^* A_k x, x \right\rangle + \alpha n(n-1) + (1 - \alpha) \sum_{1 \leq j \neq k \leq n} \left\| \left(\frac{A_k + A_j}{2} \right) x \right\|^{\frac{2}{1-\alpha}}.$$

Take the supremum over all unit vectors $x \in H$, we conclude

$$\begin{aligned} \left\| \sum_{k=1}^n A_k \right\|^2 \leq \left\| \sum_{k=1}^n A_k^* A_k \right\| + \alpha n(n-1) \\ + (1 - \alpha) \sum_{j,k=1}^n \left\| \frac{A_k + A_j}{2} \right\|^{\frac{2}{1-\alpha}} - \sum_{k=1}^n \|A_k\|^{\frac{2}{1-\alpha}}. \quad \square \end{aligned}$$

Use Lemma 8, with $\alpha = 0$, we get [Theorem 2.3, [1]]. Also, use the same lemma with $\alpha = 0$ and $n = 2$, we deduce [Inequality (5), [7]].

We end our results with the following.

Theorem 9. *Let $A_1, \dots, A_n \in B(H)$. Then*

$$\left\| \sum_{k=1}^n A_k \right\|^2 \leq \left\| \sum_{k=1}^n A_k^* A_k \right\| + \frac{1}{2} \left\| I + \sum_{1 \leq j \neq k \leq n} A_j^* A_k \right\|^2 - \frac{1}{2}.$$

Proof. For a unit vector $x \in H$, we have that

$$\begin{aligned} \left\| \sum_{k=1}^n A_k x \right\|^2 &= \sum_{k=1}^n \langle A_k^* A_k x, x \rangle + \frac{1}{2} \left\| Ix + \sum_{1 \leq j \neq k \leq n} A_j^* A_k x \right\|^2 - \frac{\|x\|^2}{2} \\ &\quad - \frac{1}{2} \left\| \sum_{1 \leq j \neq k \leq n} A_j^* A_k \right\|^2 \\ &\leq \sum_{k=1}^n \langle A_k^* A_k x, x \rangle + \frac{1}{2} \left\| \left(I + \sum_{1 \leq j \neq k \leq n} A_j^* A_k \right) x \right\|^2 - \frac{1}{2}. \end{aligned}$$

By taking the supremum over all unit vectors $x \in H$, we complete the proof. $\square$

Using the cartesian decomposition of the operator $A \in B(H)$, Theorem 10 with $n = 2$ implies

$$\|A\|^2 \leq \frac{1}{2} \|A^* A + A A^*\| + \frac{1}{2} \left\| I + \frac{A^* A - A A^*}{2} \right\|^2 - \frac{1}{2}.$$

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