ON KROPINA CHANGE
OF TWO-DIMENSIONAL FINSLER SPACES

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Abstract: The purpose of the present paper is to obtained the relationship between the main scalars, geodesic and scalar curvature among two-dimensional Finsler spaces $F^2$ and a Finsler space $F^*2$ due to Kropina change.

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1. Introduction

Let $(M^n, L)$ be an n-dimensional Finsler space on a differentiable manifold $M^n$, equipped with the fundamental function $L(x, y)$. In 1971, Matsumoto [1] introduced the transformation of Finsler metric:

$$L'(x, y) = L(x, y) + \beta(x, y)$$

where, $\beta(x, y) = b_i(x)y^i$, $b_i(x)$ are components of a covariant vector which is a function of position alone. If $L(x, y)$ is a metric function of Riemannian...
space then $L'(x, y)$ reduces to the metric function of Randers space. Such a Finsler metric was first introduced by G. Randers [2] from the standpoint of general theory of relativity and applied to the theory of electron microscope by R. S. Ingarden [3], who first named it as Randers space. The geometrical property of this space has been studied by various authors [4, 5, 6, 7, 8]. In 1978, Numata [9] has studied the properties of $(M^n, L')$ which is obtained from Minkowskian space $(M^n, L)$ by the transformation (1). In 1984, Shibata [10] has studied the properties of Finsler space $(M^n, L^*)$ whose metric function $L^*(x, y)$ is obtained from $L(x, y)$ by the relation $L^*(x, y) = f(L, \beta)$ where $f$ is positively homogeneous of degree one in $L$ and $\beta$. This change of metric function is called a $\beta$-change. The change (1) is a particular case of $\beta$-change called Randers change.

Another particular $\beta$-change of Finsler metric function is a Kropina change of metric function given by

$$
L^*(x, y) = \frac{L^2(x, y)}{\beta(x, y)}
$$

If $L(x, y)$ reduces to the metric function of Riemannian space then $L^*(x, y)$ reduces to the metric function of Kropina space [11]. Due to this reason the transformation (2) has been called the Kropina change of Finsler metric.

2. The Finsler Space $(M^n, L^*)$

Let $(M^n, L)$ be a given Finsler space and let $b_i(x) y^i$ be a one form on $M^n$. We shall define on $M^n$ a function $L^*(x, y) 0$ by the equation,

$$
L^*(x, y) = \frac{L^2(x, y)}{\beta(x, y)}
$$

where, we put $\beta(x, y) = b_i(x) y^i$.

To find the metric tensor $g_{ij}^*$, the angular metric tensor $h_{ij}^*$, the Cartan tensor $C_{ijk}^*$, and the v-curvature tensor of $(M^n, L^*)$ [12], we use the following results,

$$
\dot{\partial}_l \beta = b_l, \quad \dot{\partial}_l L = l_l, \quad \dot{\partial}_l l_l = L^{-1} h_{ij}
$$

where, $\dot{\partial}_l$ stands for partial derivative with respect to $y^i$ and $h_{ij}$ are components of angular metric tensor of $(M^n, L)$ given by $h_{ij} = g_{ij} - l_l l_j = L \dot{\partial}_i \dot{\partial}_j L$. The successive differentiation of (3) with respect to $y^i$ and $y^j$ gives,

$$
l_{ij}^* = \frac{2L}{\beta} l_l + \frac{L^2}{\beta^2} b_i
$$
\[
h^*_ij = \frac{2L^2}{\beta^2} h_{ij} + \frac{2L^2}{\beta^2} l_i l_j + \frac{2L^3}{\beta^3} (l_i b_j + l_j b_i) + \frac{2L^4}{\beta^4} b_i b_j \tag{6}
\]

From (5) and (6) we get the following relation between metric tensors of \((M^n, L)\) and \((M^n, L^*)\),

\[
g^*_ij = \frac{2L^2}{\beta^2} g_{ij} + \frac{4L^2}{\beta^2} l_i l_j + \frac{4L^3}{\beta^3} (l_i b_j + l_j b_i) + \frac{3L^4}{\beta^4} b_i b_j \tag{7}
\]

The contravariant component of the metric tensor of \((M^n, L^*)\) will be derived from (7) as follows,

\[
g^{*ij} = \frac{\beta^2}{2L^2} g^{ij} + \frac{\beta^2}{L^2} (1 - \frac{2\beta^2}{L^2} b^i b^j) l^i l^j - \frac{\beta^2}{2L^2 b^2} b^i b^j + \frac{\beta^3}{L^3 b^2} (l^i b^j + l^j b^i) \tag{8}
\]

where, we put \(b^2 = g^{ij} b_i b_j, b^i = g^{ij} b_j, l^i = g^{ij} l_j\). Differentiating (7) with respect to \(y^k\) and using (4), we get the following relation between the Cartan tensors of \((M^n, L)\) and \((M^n, L^*)\)

\[
C^*_{ijk} = \frac{2L^2}{\beta^2} C_{ijk} - \frac{2L^2}{\beta^2} (h_{ij} d_k + h_{jk} d_i + h_{ki} d_j) - \frac{6L^4}{\beta^5} d_i d_j d_k \tag{9}
\]

where, \(d_i = b_i - \frac{\beta}{L} l_i\). It is to be noted that,

\[
d_i l^i = 0, \quad d_i b^i = b^2 - \frac{\beta^2}{L^2}, \quad h_{ij} l^j = 0, \quad h_{ij} d^i = h_{ij} b^i = d_i \tag{10}
\]

where, \(d^i = g^{ij} d_j = b^i - \frac{\beta}{L} l^i\).

To find \(C^*_{jk} = g^{*in} C^*_{ijn}\), we use (8), (9) and (10), we have,

\[
C^*_{jk} = C^*_{ik} - \beta^{-1} (h^1_{ik} d_k + h^1_k d_i + h^1_{jk}) - 3L^2 \beta^{-3} d_j d_k d^i - b^{-2} C_{jke^i} + (3L^2 \beta^{-3} - \beta^{-1} b^{-2}) d_j d_k e^i + (\beta^{-1} - \beta L^{-2} b^{-2}) h_{jke} e^i \tag{11}
\]

where, \(e^i = b^i - 2\beta L^{-1} l^i = d^i - \beta L^{-1} l^i\) and \(C_{jk} = C_{ijk} b^i\).

Throughout this paper we use the symbol ‘\(^\prime\)’ to denote the contraction with \(b^i\).

**Proposition 1.** Let \(F^{*n} = (M^n, L^*)\) be an \(n\)-dimensional Finsler space obtained from the Kropina change of the Finsler space \(F^n = (M^n, L)\), then their normalized supporting element \(l^*_i\), angular metric tensor \(h^*_ij\), fundamental metric tensor \(g^*_ij\) and \((h)hv\)-torsion tensor \(C^*_ijk\) is given by (5), (6), (7) and (9) respectively.
3. Kropina Change of Main Scalar

The \((h)hv\)-torsion tensor for a two-dimensional Finsler space \(F^2\) is given by,

\[
LC_{ijk} = I_{im}m_jm_k
\]  

(12)

where, \(I = C_{222}\) is the main scalar \([13]\) in \(F^2\).

Similarly, the \((h)hv\)-torsion tensor for a two-dimensional Finsler space \(F^{*2}\) is given by,

\[
L^*C^*_{ijk} = I^*_{im}m^*_jm^*_k
\]  

(13)

where, \(I^*\) is the main scalar \([13]\) in \(F^{*2}\), and \(m^*_i\) is unit vector orthogonal to \(l^*_i\) in two-dimensional Finsler space.

Putting, \(i = k\) in equation in (11), we get

\[
C^*_i = C_i - \beta^{-1}(n + 1)d_i - b^{-2}C_{..i}
\]  

(14)

The normalized torsion vector \(m^i = \frac{C^i}{C}, \) in \(F^2\) and \(m^*i = \frac{C^*i}{C^*}\) in \(F^{*2}\) is the length of \(C^i\) and \(C^*i\) respectively.

The equation (14) can also be written

\[
m^*_i = \lambda m_i + \mu d_i + \phi C_{..i}
\]  

(15)

where, \(\lambda = \frac{C}{C^*}, \ \mu = \frac{-(n+1)\beta^{-1}}{C^*}, \ \text{and} \ \phi = \frac{-b^{-2}}{C^*}.\) since,

\[
C^{*2} = g^{*ij}C^*_iC^*_j = \frac{\beta^2}{2L^2}C^2 - \frac{\beta^2}{2L^2b^2}B^2 - \beta^{-1}(n + 1)[\frac{\beta^2}{L^2}D - \frac{\beta^2}{L^2b^2}B(\frac{b^2}{L^2})] - \frac{n + 1}{2L^2b^2}(b^2 - \frac{\beta^2}{L^2})^2 + \beta^{-1}(n + 1)b^{-2}[\frac{\beta^2}{L^2}H - \frac{\beta^2}{L^2b^2}(b^2 - \frac{\beta^2}{L^2})C_{..i}] + \frac{b^{-4}\beta^2}{2L^2}J - \frac{\beta^2}{L^2b^2}b^{-2}C^2
\]  

(16)

where, \(B = b\delta C_i, \ \ D = g^{ij}C_jd_j = g^{ij}C_jd_i, \ \ E = g^{ij}C_{..j} = g^{ij}C_{..i}C_j, \ \ F = g^{ij}d_id_j, \ \ H = g^{ij}d_iC_{..j} = g^{ij}C_{..i}d_j, \ \ J = g^{ij}C_{..i}C_{..j}\) are an scalars.

The contravariant component of \(l^{*i}\) and \(m^{*i}\) is given by

\[
l^{*i} = \frac{g^{*ij}l^j}{L} = \frac{\beta}{L}l^i
\]  

(17)
where, \( l^i_i = 1 \).
Again,
\[
m^*i = g^{*ij}m^*_j = Mm^i + Nb^i + Pdi + Qd^i + SC^i
\]
(18)
where, \( M = \frac{\lambda\beta^2}{2L^2}, \ N = -(\frac{\lambda\beta^2}{2L^2}K + \frac{\mu\beta^2}{L^2}b^i + \frac{\phi\beta^2}{L^2}C_{...}), \ P = \frac{\lambda\beta^2}{L^2}b^i, \ b^im_i = K \) is an certain scalar.

**Proposition 2.** Let \( F^* = (M^*, L^*) \) be an n-dimensional Finsler space obtained from the Kropina change of the Finsler space \( F^n = (M^n, L) \), then contravariant and covariant components of the Berwald frame \((l, m)\) in two-dimensional Finsler space is given by (17), (18), (5) and (15) respectively.

**Proposition 3.** Let \( F^* = (M^*, L^*) \) be an n-dimensional Finsler space obtained from the Kropina change of the Finsler space \( F^n = (M^n, L) \), then the relationship between the length of the components \( C_i \) and \( C_i^* \) is given by (16).

Since, the (h)hv-torsion tensor given by (9) can be rewritten as in two-dimensional as follows:
\[
I^*m^*_im^*_j = \frac{2L^2}{\beta^2}Im_i m_j m_k - \frac{6L^2}{\beta^3}b_2 m_i m_j m_k + \frac{2L}{\beta^2}(m^i m_j l_k + \frac{6L^4}{\beta^3}b_2 m_i m_j m_k - \frac{6L^2}{\beta^3}b_2 (l_i m_j k + l_k m_j i + l_k m_i j) + \frac{6L^3}{\beta^4}b_2 (l_i m_j k + l_k m_j i + l_k m_i j) + \frac{6L^4}{\beta^5}b_2 l_i m_j k)
\]
(19)
where, \( h_{ij} = m_i m_j \) and \( b_i = b_1 l_i + b_2 m_i \), then \( b_i b^i = 0 \implies b_1 = 0 \). So, \( b_i = b_2 m_i \), \( b_1 \) and \( b_2 \) are certain scalars.

From equation (15) and (19), we have
\[
I^*(\lambda + \mu b_2 + \phi K^2)^3 m_i m_j m_k = \frac{2L^2}{\beta^2}I m_i m_j m_k - \frac{6L^2}{\beta^3}b_2 m_i m_j m_k + \frac{2L}{\beta^2}(m^i m_j l_k + \frac{6L^4}{\beta^3}b_2 m_i m_j m_k - \frac{6L^2}{\beta^3}b_2 (l_i m_j k + l_k m_j i + l_k m_i j) + \frac{6L^3}{\beta^4}b_2 (l_i m_j k + l_k m_j i + l_k m_i j) + \frac{6L^4}{\beta^5}b_2 l_i m_j k)
\]
(20)
Contracting (20) by \( m_i m_j m_k \), we have,
\[
(\lambda + \mu b_2 + \phi K^2)^3 I^* = \frac{2L^2}{\beta^2}I - \frac{6L^2}{\beta^3}b_2
\]
(21)
Theorem 1. Let $F^\ast n = (M^n, L^\ast)$ be an $n$-dimensional Finsler space obtained from the Kropina change of the Finsler space $F^n = (M^n, L)$, then the relationship between the main scalar $I^\ast$ and $I$ of the Finsler space is given by (21).

4. Kropina Change of Geodesic

Let us consider $s$ be the arc-length, then the equations of a geodesic [14] of $F^n = (M^n, L)$ is written in the well-known from,

$$\frac{d^2 x^i}{ds^2} + 2G^i(x, \frac{dx}{ds}) = 0 \tag{22}$$

where, functions $G^i(x, y)$ are given by

$$2G^i = g^{ir}(y^j \partial_r \partial_j F - \partial_r F), \quad F = \frac{L^2}{2}$$

Now, suppose $s^\ast$ be the arc-length in the Finsler space $F^\ast n = (M^n, L^\ast)$, then the equations of a geodesic can be written as,

$$\frac{d^2 x^i}{ds^\ast^2} + 2G^\ast i(x, \frac{dx}{ds^\ast}) = 0 \tag{23}$$

where, functions $G^\ast i(x, y)$ are given by

$$2G^\ast i = g^{\ast ir}(y^j \partial_r \partial_j F^\ast - \partial_r F^\ast), \quad F^\ast = \frac{L^\ast 2}{2}$$

Since, $ds^\ast = L^\ast(x, dx)$, this is also be written as,

$$ds^\ast = \frac{L^2(x,y)}{\beta(x,y)} = \frac{ds^2}{b_i dx^i}$$

Since, $ds= L(x, dx)$

Thus we have,

$$\frac{dx^i}{ds} = \frac{dx^i}{ds^\ast} \frac{2ds}{\beta} - \frac{ds^2}{\beta^2 b_i dx^i} \tag{24}$$

Differentiating (24) with respect to $ds$, we have,

$$\frac{d^2 x^i}{ds^2} = \frac{d^2 x^i}{ds^\ast^2} \frac{2ds}{\beta} - \frac{ds^2}{\beta^2 b_i dx^i} + \frac{dx^i}{ds^\ast} \frac{2ds}{\beta} - \frac{2ds}{\beta^2 b_i dx^i} - \frac{(ds^2 b_i dx^i)^2}{(ds)^2}$$

$$- \frac{ds}{\beta} (\frac{2ds}{\beta} - \frac{2(ds)^2}{\beta^3} b_i dx^i) \tag{25}$$
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Since, \(2G_{*i} = g^*_{ir}(y^j \partial_r \partial_j \frac{L^2}{2} - \partial_r \frac{L^2}{2})\)

Then,

\[
G_{*i} = \frac{2L^2}{\beta^2} G_i + y^j \left[ \frac{2L^2}{\beta} \left( \frac{l_i}{L^2 b^2} \right) \partial_j L - \frac{L^4}{\beta^3} \partial_i \partial_j \beta - \right]
\]

\[
\frac{2L^3}{\beta^2} l_i \partial_j \beta + \frac{2L^4}{\beta^4} b_i \partial_j \beta + \frac{2L^3}{\beta^3} l_i \partial_j \beta - \frac{2L^3}{\beta^3} b_i \partial_j L + \frac{L^4}{\beta^3} \partial_j \beta
\]

Now, we have

\[
G^*_{*i} = g^*_{*ir} G^*_{r} = G^i + \frac{2L^2}{\beta^2} G_r \left[ \frac{\beta^3}{L^3 b^2} (l^i b^r + l^r b^i) \right] + \left[ \frac{\beta^2}{L^2 b^2} L^2 \partial^r \frac{L^2}{b^2} \right] - \frac{2L^2}{\beta^2} b^i b^r + \frac{\beta^2}{L^3 b^2} \left( l^i b^r + l^r b^i \right) + \frac{\beta^2}{L^2} (1 - \frac{2\beta^2}{L^2 b^2}) L^2 \partial^r \frac{L^2}{b^2} \right] + \frac{\beta^3}{L^2 b^2} \left( l^i b^r + l^r b^i \right)
\]

\[
\frac{\beta^2}{\beta^3} \left( \frac{L^2}{\beta^2} \right) \partial_i \partial_j L - \frac{L^4}{\beta^3} \partial_i \partial_j \beta - \frac{2L^3}{\beta^2} l_i \partial_j \beta + \frac{2L^3}{\beta^2} b_i \partial_j \beta + \frac{2L^3}{\beta^3} l_i \partial_j \beta - \frac{2L^3}{\beta^3} b_i \partial_j L + \frac{L^4}{\beta^3} \partial_j \beta
\]

Proposition 4. Let \(F^*n = (M^n, L^*)\) be an n-dimensional Finsler space obtained from the Kropina change of the Finsler space \(F^n = (M^n, L)\), then the relationship between the Berwald connection function \(G^*_{*i}\) and \(G^i\) is given by (27).

Proposition 5. Let \(F^*n = (M^n, L^*)\) be an n-dimensional Finsler space obtained from the Kropina change of the Finsler space \(F^n = (M^n, L)\), then the relationship between the arc-length \(s^*\) and \(s\) is given by (24).

Theorem 2. Let \(F^*n = (M^n, L^*)\) be an n-dimensional Finsler space obtained from the Kropina change of the Finsler space \(F^n = (M^n, L)\), then the equation of geodesic is given by (23) where \(\frac{d^2 x^i}{ds^2}\) and \(G^*_{*i}\) is given by (25) and (27) respectively.

5. Kropina Change of Scalar Curvature

The (v)h-torsion tensor \(R^i_{jk}\) in two-dimensional Finsler space may be written as,

\[
R^i_{jk} = LR^i (l_j m_k - l_k m_j)
\]
where, $R$ is the h-scalar curvature.

Again, the (v)h-torsion tensor $R^{*i}_{jk}$ in Finsler space $F^{*2}$ is,

$$R^{*i}_{jk} = L^i R^* m^* (l^*_j m^*_k - l^*_k m^*_j) \tag{29}$$

The equation (29) can also be written as,

$$\frac{R^{*i}_{jk}}{R^*} = L^i m^* (l^*_j m^*_k - l^*_k m^*_j)$$

In view of (3), (5), (15) and (17), we have,

$$\frac{R^{*i}_{jk}}{R^*} = 2 L^2 \lambda \beta^2 M M^i (l^*_j m^*_k - l^*_k m^*_j) + 2 L^3 \lambda \beta^2 (Nb^i + Pl^i + Qd^i + SC^i (l^*_j m^*_k - l^*_k m^*_j) + \frac{L^2}{\beta} (MM^i + Nb^i + Pl^i + Qd^i + SC^i [\frac{2L^2}{\beta} (l^*_j d_k - l^*_k d_j) + \frac{2L^2}{\beta} (l^*_j C..k - l^*_k C..j) - \frac{L^2}{\beta^2} (b^*_j m^*_k - b^*_k m^*_j) - \frac{L^2}{\beta^2} (b^*_j C..k - b^*_k C..j)])]$$

Using (28) in (30), we have,

$$\frac{R^{*i}_{jk}}{R^*} = 2 L^2 \lambda \beta^2 M R^i j k R^* + 2 L^3 \lambda \beta^2 (Nb^i + Pl^i + Qd^i + SC^i (l^*_j m^*_k - l^*_k m^*_j) + \frac{L^2}{\beta} (MM^i + Nb^i + Pl^i + Qd^i + SC^i [\frac{2L^2}{\beta} (l^*_j d_k - l^*_k d_j) + \frac{2L^2}{\beta} (l^*_j C..k - l^*_k C..j) - \frac{L^2}{\beta^2} (b^*_j m^*_k - b^*_k m^*_j) - \frac{L^2}{\beta^2} (b^*_j C..k - b^*_k C..j)])]$$

**Theorem 3.** Let $F^{*n} = (M^n, L^*)$ be an n-dimensional Finsler space obtained from the Kropina change of the Finsler space $F^n = (M^n, L)$, then the relationship between (v)h-torsion tensor and scalar curvature is given by (31).

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References


