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PRIME GAMMA-NEAR-RINGS WITH (σ, τ)-DERIVATIONS

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Abstract: Let N be a 2 torsion free prime Γ-near-ring with center $Z(N)$ and let d be a nontrivial derivation on N such that $d(N) \subseteq Z(N)$. Then we prove that N is commutative. Also we prove that if d be a nonzero (σ,τ)-derivation on N such that $d(N)$ commutes with an element $a$of$N$ then ether d is trivial or a is in $Z(N)$. Finally if d_1 be a nonzero (σ,τ)-derivation and d_2 be a nonzero derivation on N such that $d_1\tau = \tau d_1$, $d_1\sigma = \sigma d_1$, $d_2\tau = \tau d_2$, $d_2\sigma = \sigma d_2$ with $d_1(N)\Gamma\sigma(d_2(N)) = \tau(d_2(N))\Gamma d_1(N)$ then N is a commutative Γ-ring.

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1. Introduction

Bell and Mason [3] initiated the study of derivations in near-rings. The derivation in Γ-near-rings was first introduced by Cho and Jun [6]. They studied some

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basic properties of prime Γ-near-rings. Ozturk and Jun in [14] discussed prime Γ-near-rings with derivations. They obtained commutativity properties on prime Γ-near-rings. Later Asci [1] studied Γ-near-rings with (σ,τ)-derivations and obtained some commutativity results. The commutativity properties of prime Γ-near-rings with derivations also have been investigated by Uckun, Ozturk and Jun [18]. Golbasi [10] obtained analogue of Posner's Theorem from [17]. The commutativity conditions of prime Γ- near-rings with generalized derivations have been studied in [7] (also see [8]). The paper [9] has dealt with generalized derivations in semiprime gamma-rings.

In this paper we consider the prime Γ-near-rings with derivations to subject of commutativity conditions. The characterizations of prime Γ-near-rings with composition of two derivations are obtained. We also investigate conditions for prime Γ-near-ring to be commutative.

Organization of the paper is as follows. Sections 2 and 3 contain brief review and definitions which will be used further. Some of these results have been presented by others researchers (for example, see [1]). Main results are in Sections 3 and 4. The results of Section 3 concern derivations in Γ-rings (Theorem 3.6), whereas Section 4 deals with (σ,τ)-derivations in prime Γ-near-rings (Theorems 4.2 – 4.7).

2. Preliminaries

A Γ-near-ring is a triple $(N, +, \Gamma)$, where:

(i) $(N, +)$ is a group (not necessarily abelian),

(ii) Γ is a non-empty set of binary operations on N such that for each $\alpha \in \Gamma$, $(N, +, \alpha)$ is a left near-ring.

(iii) $a\alpha(b\beta c) = (a\alpha b)\beta c$, for all a, b, $c \in$N and α, $\beta \in \Gamma$.

Throughout this paper, N stands for a zero-symmetric left Γ-near-ring with multiplicative center $Z(N)$. Recall that a Γ-near-ring N is prime if $a\Gamma N\Gamma b = \{ 0 \}$ implies that $a = 0$ or $b = 0$. Let σ and τ be two Γ-near-ring automorphisms of N. For a, $b \in N$ and $\alpha \in \Gamma$, $[a, b]_\alpha$,

$[a, b]_\alpha^{(\sigma,\tau)}$ and (a, b) will denote the commutator $a\alpha b - b\alpha a$, $a\alpha\sigma(b) - \tau(b)\alpha a$ and $a + b - a - b$ respectively. An additive mapping d: $N \to N$ is called a derivation if $d(a\alpha b) = a\alpha d(b) + d(a)\alpha b$ holds for all a, $b \in N$, $\alpha \in \Gamma$. An additive mapping d: $N \to N$ is called a (σ, τ)-derivation if $d(a\alpha b) = \sigma(a)\alpha d(b) + d(a)\alpha\tau(b)$ holds for all a, $b \in N$, $\alpha \in \Gamma$. In particular $[a, b]_\alpha^{(1,1)} = [a, b]_\alpha$.

3. Properties of Γ Rings with Derivations

Lemma 3.1. *Let d be an arbitrary additive endomorphism of N. Then*

$$d(a\alpha b) = a\alpha d(b) + d(a)\alpha b, \text{ for all } a,\, b \in N,\, \alpha \in \Gamma$$

if and only if

$$d(a\alpha b) = d(a)\alpha b + a\alpha d(b), \text{ for all } a,\, b \in N,\, \alpha \in \Gamma.$$

Proof. Suppose $d(a\alpha b) = a\alpha d(b) + d(a)\alpha b$ for all $a,\, b \in N,\, \alpha \in \Gamma$. Since

$$a\alpha(b + b) = a\alpha b + a\alpha b,$$

$$d(a\alpha(b + b)) = a\alpha d(b + b) + d(a)\alpha(b + b) = a\alpha d(b) + a\alpha d(b) + d(a)\alpha b + d(a)\alpha b$$

and

$$d(a\alpha b + a\alpha b) = d(a\alpha b) + d(a\alpha b) = a\alpha d(b) + d(a)\alpha b + a\alpha d(b) + d(a)\alpha b.$$

Then, we obtain

$$a\alpha d(b) + d(a)\alpha b = d(a)\alpha b + a\alpha d(b),$$

so $d(a\alpha b) = d(a)\alpha b + a\alpha d(b)$, for all $a,\, b \in N,\, \alpha \in \Gamma$.

The converse part is proved similarly.

Lemma 3.2. *Suppose that d is a derivation on N. Then N satisfies the following right distributive laws:*

(i) $(a\alpha d(b) + d(a)\alpha b)\beta c = a\alpha d(b)\beta c + d(a)\alpha b\beta c$,

(ii) $(d(a)\alpha b + a\alpha d(b))\beta c = d(a)\alpha b\beta c + a\alpha d(b)\beta c$ for every $a,\, b,\, c \in N$ and $\alpha,\, \beta \in \Gamma$.

Proof. (i) Consider

$$d((a\alpha b)\beta c) = a\alpha b\beta d(c) + d(a\alpha b)\beta c = a\alpha b\beta d(c) + (a\alpha d(b) + d(a)\alpha b)\beta c,$$

for all $a,\, b,\, c \in N$ and $\alpha,\, \beta \in \Gamma$, and

$$\begin{aligned} d(a\alpha(b\beta c)) = a\alpha d(b\beta c) + d(a)\alpha b\beta c &= a\alpha(b\beta d(c) + d(b)\beta c) + d(a)\alpha b)\beta c \\ &= a\alpha b\beta d(c) + a\alpha d(b)\beta c + d(a)\alpha b\beta c \end{aligned}$$

for all $a,\, b,\, c \in N$ and $\alpha,\, \beta \in \Gamma$.

From the above two relations we obtain

$$(a\alpha d(b) + d(a)\alpha b)\beta c = a\alpha d(b)\beta c + d(a)\alpha b\beta c,$$

for all a, b, $c \in N$ and α, $\beta \in \Gamma$.

(ii) Due to Lemma 3.1 we have,

$$d((a\alpha b)\beta c) = d(a\alpha b)\beta c + a\alpha b\beta d(c) = (d(a)\alpha b + a\alpha d(b))\beta c + a\alpha b\beta d(c),$$

for a, b, $c \in N, \alpha,\beta \in \Gamma$, and

$$\begin{aligned} d(a\alpha(b\beta c)) = d(a)\alpha b\beta c + a\alpha d(b\beta c) &= d(a)\alpha b\beta c + a\alpha(d(b)\beta c + b\alpha d(c)) \\ &= d(a)\alpha b\beta c + a\alpha d(b)\beta c + a\alpha b\beta d(c), \end{aligned}$$

for every a, b, $c \in N$ and α, $\beta \in \Gamma$.

These two relations imply that

$$(d(a)\alpha b + a\alpha d(b)\beta c = d(a)\alpha b\beta c + a\alpha d(b)\beta c,$$

for all a, b, $c \in N$ and α, $\beta \in \Gamma$.

Lemma 3.3. *Let d be a derivation on a Γ-near-ring N and let u be not a left zero divisor in N. If $[u, d(u)]_\alpha = 0$ for every $\alpha \in \Gamma$, then (a, u) is a constant for every $a \in N$.*

Proof. From $u\alpha(u + a) = u\alpha u + u\alpha a$, for all $x \in N$, $\alpha \in \Gamma$, we get

$$\begin{aligned} d(u\alpha(u + a)) =& u\alpha d(u + a) + d(u)\alpha(u + a) \\ =& u\alpha d(u) + u\alpha d(a) + d(u)\alpha u + d(u)\alpha a \\ =& u\alpha d(u) + u\alpha d(a) + u\alpha d(u) + a\alpha d(u) \end{aligned}$$

and

$$d(u\alpha u + u\alpha a) = d(u\alpha u) + d(u\alpha a) = u\alpha d(u) + d(u)\alpha u + u\alpha d(a) + d(u)\alpha a$$

which gives

$$u\alpha d(a) + d(u)\alpha u - d(u)\alpha u - u\alpha d(a) = 0,$$

for all $\alpha \in \Gamma$. Since $d(u)\alpha u = u\alpha d(u)$, $\alpha \in \Gamma$.

This equation can be written as

$$u\alpha(d(a) + d(u) - d(a) - d(u)) = u\alpha d(a + u - a - u) = u\alpha d((a, u)) = 0.$$

Thus $d((a, u)) = 0$. Hence (a, u) is constant. This completes the proof.

Lemma 3.4. *Let d be a (σ, τ)-derivation on a N and $a \in N$. Then*

$$(\tau(a)\alpha d(b) + d(a)\alpha\sigma(b))\beta\sigma(c) = \tau(a)\alpha d(b)\beta\sigma(c) + d(a)\alpha\sigma(b)\beta\sigma(c),$$

for $a, b, c \in N, \alpha, \beta \in \Gamma$.

Proof. Consider

$$\begin{aligned} d((a\alpha b)\beta c) &= \tau(a\alpha b)\beta d(c) + d(a\alpha b)\beta\sigma(c) \\ &= \tau(a)\alpha\tau(b)\beta d(c) + (\tau(a)\alpha d(b) + d(a)\alpha\sigma(b))\beta\sigma(c), \end{aligned}$$

for all $a, b, c \in$N, $\alpha, \beta \in \Gamma$, and

$$\begin{aligned} d(a\alpha(b\beta c)) &= \tau(a)\alpha d(b\beta c) + d(a)\alpha\sigma(b\beta c) \\ &= \tau(a)\alpha\tau(b)\beta d(c) + \tau(a)\alpha d(b)\beta\sigma(c) + d(a)\alpha\sigma(b)\beta\sigma(a) \end{aligned}$$

for all $a, b, c \in N, \alpha, \beta \in \Gamma$.

Since $d((a\alpha b)\beta c) = d(a\alpha(b\beta c))$, we have

$$(\tau(a)\alpha d(b) + d(a)\alpha\sigma(b))\beta\sigma(c) = \tau(a)\alpha d(b)\beta\sigma(c) + d(a)\alpha\sigma(b)\beta\sigma(c),$$

for all a, $b, c \in N, \alpha, \beta \in \Gamma$.

This completes the proof.

Lemma 3.5. *Suppose that N is a prime Γ-near-ring.*

(i) any nonzero element of the center of N is not zero divisor.

(ii) If there exist a nonzero element of $Z(N)$ such that $a + a \in Z(N)$, then $(N, +)$ is commutative.

(iii) If d $\neq$ 0 be a derivation on N. Then $a\Gamma d(N) = \{0\}$ implies $a = 0$, and $d(N)\Gamma a = \{0\}$ implies $a = 0$.

(iv) Let N be 2-torsion free and d be a derivation on N with $d^2 = 0$. Then $d = 0$.

Proof. (i) If $a \in Z(N)- \{ 0 \}$ and $a\alpha b = 0$, $b \in N$, $\alpha \in \Gamma$, then $a\alpha c\beta b = 0$, $b, c \in N$, $\alpha \in \Gamma$. Therefore we have, $a\Gamma N\Gamma b = 0$. Since N is prime and $a \neq 0$ then $b = 0$.

(ii) Let a$\in Z(N)- \{ 0 \}$ be an element such that $a + a \in Z(N)$ and let b, $c \in N$, $\alpha \in \Gamma$. By the distributivity of N we get $(b + c)\alpha(a + a) = b\alpha(a + a)+ c\alpha(a+a) = b\alpha a + b\alpha a + c\alpha a + c\alpha a = a\alpha b + a\alpha b + a\alpha c + a\alpha c = a\alpha(b+ b + c + c)$.

Next, since $a + a \in Z(N)$ then

$$(b+c)\alpha(a+a) = (b+c)\alpha a+(b+c)\alpha a = a\alpha(b+c)+a\alpha(b+c) = a\alpha(b+c+b+c).$$

Taking into account the last two we have, $b + b + c + c = b + c + b + c$ and so $b + c = c + b$. Hence $(N, +)$ is commutative.

(iii) Let $a\Gamma d(N) = \{ 0 \}$, b, $c \in$N and α, $\beta \in \Gamma$. Then $0 = a\alpha d(b\beta c) = a\alpha b\beta d(c) + a\alpha d(b)\beta c = a\alpha b\beta d(c)$. Thus $a\Gamma N\Gamma d(N) = \{ 0 \}$, by using the primeness of N and $d(N) \neq 0$ we get $a = 0$.

The part (iii) can be proved similarly.

(iv) Note that for any a, $b \in N$, $\alpha \in \Gamma$, $0 = d^2(a\alpha b) = d(a\alpha d(b)+d(a)\alpha b) = a\alpha d^2(b) + d(a)\alpha d(b) + d(a)\alpha d(b)+ d^2(a)\alpha b$. Therefore we have $2d(a)\alpha d(b) = 0$, for all a, $b \in N$, $\alpha \in \Gamma$. Since N is 2-torsion free we get $d(a)\alpha d(b) = 0$, for each $a \in N$, $\alpha \in \Gamma$. By using (iii) we obtain $d = 0$.

Theorem 3.6. *Let N be a prime Γ-near-ring and let d be a nontrivial derivations on N such that $d(N) \subseteq Z(N)$. Then $(N, +)$ is commutative. Moreover if N is 2-torsion free, then $N$is a commutative Γ-ring.*

Proof. Let a be a non-constant and c be a constant elements of N. Then $d(a\alpha c) = a\alpha d(c) + d(a)\alpha c = d(a)\alpha c \in Z(N)$, $\alpha \in \Gamma$. Since $d(a) \in Z(N)- \{ 0 \}$, it follows easily that $c \in Z(N)$. Note that if c is a constant so $c + c$ is a constant. It follows from lemma 3.5(ii) that $(N, +)$ is commutative, provided that there exists a nonzero constant. Let us provide the last.

Suppose, that 0 is the only constant. Since d is obviously commuting, it follows from the lemma 3.3 that all u which are not zero divisors belong to by $Z(N)$. In particular, if $a \neq 0$, $d(a) \in Z(N)$. But then for all $b \in N$, $d(b) + d(a) - d(b) - d(a) = d((b, a)) = 0$, hence $(b, a) = 0$.

Now, we assume that N is 2-torsion free. By Lemma 3.1,

$$(a\alpha d(b) + d(a)\alpha b)\beta c = a\alpha d(b)\beta c + d(a)\alpha b\beta c$$

for all a, b, $c \in N$, α, $\beta \in \Gamma$, and using the fact that $d(a\alpha b) \in Z$, $\alpha \in \Gamma$, we get $c\alpha a\beta d(b) + c\alpha d(a)\beta b = a\alpha d(b)\beta c + a\alpha d(b)\beta c$, α, $\beta \in \Gamma$. Since $(N, +)$ is commutative and $d(N) \subseteq Z(N)$, we obtain $d(b)\alpha[c, a]_\beta = d(a)\alpha[b, c]_\beta$ for all a, b, $c \in N$, α, $\beta \in \Gamma$.

Suppose now that N is not commutative. Choosing b, $c \in N$, with $[b, c]_\beta \neq 0$, $\beta \in \Gamma$, and letting $u = d(a)$, we get $d^2(a)\alpha[b, c]_\beta = 0$, for all $a \in N$, α, $\beta \in \Gamma$, and since the central elements $d^2(a)$ cannot be a nonzero divisor of zero, we conclude that $d^2(a) = 0$ for all $a \in N$. By lemma 3.5(iv), this is not true for nontrivial d.

4. Γ-Near-Rings with (σ, τ)-Derivations

Lemma 4.1. *Suppose that N is a prime Γ-near-ring with a nonzero (σ,τ)-derivation d and $a \in N$. If at least one of the following (i) $d(N)\Gamma\sigma(a)$ or (ii) $a\Gamma d(N)$ is zero then $a = 0$.*

Proof. (i) Consider

$$\begin{aligned} 0 =& d(b\beta c)\alpha\sigma(a) = (\tau(b)\beta d(c) + d(b)\beta\sigma(c))\alpha\sigma(a) \\ =& \tau(b)\beta d(c)\alpha\sigma(a) + d(b)\beta\sigma(c)\alpha\sigma(a), \end{aligned}$$

for all a, b, $c \in N$, α, $\beta \in \Gamma$.

Since $d(c)\alpha\sigma(a) = 0$, we have $d(b)\beta\sigma(c)\alpha\sigma(a) = 0$, i.e., $d(b)\Gamma N\Gamma\sigma(a) = 0$. (Bearing in mind that σ is an automorphism). Since $d \neq 0$ and N is prime, we get $\sigma(a) = 0$ which gives $a = 0$.

The proof of (ii) is similar.

Theorem 4.2. *Suppose that d is a nonzero (σ, τ)-derivation of a prime Γ-near-ring N and $a \in N$, $\alpha \in \Gamma$. Let $[d(N), a]_\alpha^{(\sigma,\tau)} = 0$, then $d(a) = 0$ or $a \in Z(N)$.*

Proof. Note that for all $b \in N$ and $\alpha \in \Gamma$, $a\alpha b \in N$. Therefore, from $[d(N), a]_\alpha^{(\sigma,\tau)} = 0$ we have, $d(a\alpha b)\beta\sigma(a) = \tau(a)\beta d(a\alpha b)$ for $\beta \in \Gamma$.

Hence

$$(\tau(a)\alpha \underline{d}(b) + d(a)\alpha\sigma(b))\beta\sigma(a) = \tau(a)\beta(\tau(a)\alpha d(b) + d(a)\alpha\sigma(b)).$$

Then by Lemma 3.4, we have

$$\tau(a)\alpha d(b)\beta\sigma(a) + d(a)\alpha\sigma(b)\beta\sigma(a) = \tau(a)\beta\tau(a)\alpha d(b) + \tau(a)\beta d(a)\alpha\sigma(b).$$

Due to the condition of the theorem, we have

$$\tau(a)\alpha\tau(a)\beta d(b) + d(a)\alpha\sigma(b)\beta\sigma(a) = \tau(a)\beta\tau(a)\alpha d(b) + d(a)\alpha\sigma(a)\beta\sigma(b),$$

that is

$$d(a)\alpha\sigma([b, a]_\beta) = 0 \tag{1}$$

for all $b \in N$, α, $\beta \in \Gamma$.

Substituting $b\delta c$, ($c \in N$, $\delta \in \Gamma$) for a in (1), we have $d(a)\alpha\sigma(b)\beta\sigma([c, a]_\alpha) = 0$ for all b, $c \in N$, α, β, $\delta \in \Gamma$. Since σ is automorphism we get $d(a) = 0$ or $a \in Z(N)$. This completes the proof.

Lemma 4.3. *Suppose that N is a prime Γ-near-ring and d_1 is a nonzero (σ,τ)-derivation and d_2 is a derivation of N such that*

$$d_1(a)\Gamma\sigma(d_2(b)) + \tau(d_2(a))\Gamma d_1(b) = 0$$

for all $a, b \in N$. Then $(N, +)$ is commutative.

Proof. Consider $d_1(a)\Gamma\sigma(d_2(b)) + \tau(d_2(a))\Gamma d_1(b) = 0$ for all $\alpha \in \Gamma$. Replacing b by $u+v$ we have,

$$\begin{aligned} 0 &= d_1(a)\alpha\sigma(d_2(u+v)) + \tau(d_2(a))\alpha d_1(u+v) \\ &= d_1(a)\alpha\sigma(d_2(u)) + d_1(a)\alpha\sigma(d_2(v)) + \tau(d_2(a))\alpha d_1(u) + \tau(d_2(a))\alpha d_1(v), \end{aligned}$$

for $u, v \in N$, $\alpha \in \Gamma$.

Using the condition of the lemma, we get

$$0 = d_1(a)\alpha\sigma(d_2(u)) + d_1(a)\alpha\sigma(d_2(v)) - d_1(a)\alpha\sigma(d_2(u)) - d_1(a)\alpha\sigma(d_2(v)).$$

As result we get,

$$d_1(a)\alpha\sigma(d_2(u,v)) = 0 \tag{2}$$

for all $a, u, v \in N$, $\alpha \in \Gamma$.

By Lemma 4.1(i), we obtain that $d_2(u, v) = 0$ for all $u, v \in N$. Note that for any $w \in N$, we have $d_2(w\alpha u, w\alpha v) = 0$. Using (2), we obtain that $d_2(w\alpha(u, v)) = 0$. This yields $d_2(w)\alpha(u, v) = 0$, for all $w, u, v \in N$, $\alpha \in \Gamma$. By Lemma 3.5(iii), we get $(u, v) = 0$, for all $u, v \in N$.

Theorem 4.4. *Let N be a prime Γ-near-ring with $2N \neq \{0\}$, d_1 be a (σ, τ)-derivation and d_2 be a derivation such that $d_1\tau = \tau d_1$, $d_1\sigma = \sigma d_1$, $d_2\tau = \tau d_2$ and $d_2\sigma = \sigma d_2$. If $d_1(a)\Gamma\sigma(d_2(b)) + \tau(d_2(a))\Gamma d_1(b) = 0$ for all $a, b \in N$, then either $d_1 = 0$ or $d_2 = 0$.*

Proof. Let $d_1 \neq 0$ and $d_2 \neq 0$. According to Lemma 4.3 we know that $(N, +)$ is commutative. Now, for all $u, v \in N$ we take $a = u\beta v$ in $d_1(a)\Gamma\sigma(d_2(b)) + \tau(d_2(a))\Gamma d_1(b) = 0$ to obtain

$$\begin{aligned} 0 &= d_1(u\beta v)\alpha\sigma(d_2(b)) + \tau(d_2(u\beta v))\alpha d_1(b) \\ &= (\tau(u)\beta d_1(v) + d_1(u)\beta\sigma(v))\alpha\sigma(d_2(b)) + \tau(u\beta d_2(v) + d_2(u)\beta v)\alpha d_1(b). \end{aligned}$$

Using the left distributive law, we get,

$$0 = \tau(u)\beta d_1(v)\alpha\sigma(d_2(b)) + d_1(u)\beta\sigma(v)\alpha\sigma(d_2(b)) + \tau(u)\beta\tau(d_2(v))\alpha\sigma d_1(b)$$

$$+ \tau(d_2(u))\beta\tau(v)\alpha d_1(b),$$

for all u, v, $b \in N$, α, $\beta \in \Gamma$.

Hence,

$$0 = \tau(u)\beta(d_1(v)\alpha\sigma(d_2(b) + \tau(d_2(v))\alpha d_1(b)) + d_1(u)\beta\sigma(v)\alpha\sigma(d_2(b)) \\ + d_2(\tau(u))\beta\tau(v)\alpha d_1(b).$$

Now due to the condition of the theorem, we obtain

$$d_1(u)\beta\sigma(v)\alpha\sigma(d_2(b)) + d_2(\tau(u))\beta\tau(v)\alpha d_1(b) = 0, \quad (3)$$

for all b, u, $v \in N$, α, $\beta \in \Gamma$.

Replacing b by $b\delta t$, $t \in N$, $\delta \in \Gamma$, in (3) we have

$$\begin{aligned} 0 =& d_1(u)\alpha\beta\sigma(v)\alpha\sigma(d_2(b\delta t)) + d_2(\tau(u))\beta\tau(v)\alpha d_1(b\delta t) \\ =& d_1(u)\beta\sigma(v)\alpha\sigma(b)\delta\sigma(d_2(t)) + d_1(u)\beta\sigma(v)\alpha\sigma(d_2(b))\delta\sigma(t) \\ &+ d_2(\tau(u))\beta\tau(v)\alpha\tau(y)\delta d_1(t) + d_2(\tau(u))\beta\tau(v)\alpha d_1(b)\delta\sigma(t) \\ =& d_1(u)\beta\sigma(v\beta b)\delta\sigma(d_2(t)) + d_2(\tau(u))\beta\tau(v\beta b)\delta d_1(t) \\ &+ d_1(u)\beta\sigma(v)\alpha\sigma(d_2(b))\delta\sigma(t) + d_2(\tau(u))\beta\tau(v)\beta d_1(b)\delta\tau(t). \end{aligned}$$

Using (3) we again obtain,

$$d_1(u)\beta\sigma(v)\beta\sigma(d_2(b))\beta\sigma(t) + d_2(\tau(u))\beta\tau(v)\alpha d_1(b)\delta\sigma(t) = 0, \quad (4)$$

for all u, v, b, $t \in N$, α, β, $\delta \in \Gamma$.

Substituting $d_1(t)$ for t in (4), we get

$$d_1(u)\alpha\sigma(v)\beta\sigma(d_2(b))\delta\sigma(d_1(t)) + d_2(\tau(u))\beta\tau(v)d_1(b)\delta\sigma(d_1(t)) = 0 \quad (5)$$

for all u, v, b, $t \in N$, α, β, $\delta \in \Gamma$.

Now if we take $\tau(b)$ instead of y in (5), we have,

$$d_1(u)\alpha\sigma(v)\beta\sigma(d_2(\tau(b)))\delta\sigma(d_1(t)) + d_2(\tau(u))\beta\tau(v)\alpha d_1(\tau(b))\delta\sigma(d_1(t)) = 0, \quad (6)$$

for all u, v, b, $t \in$N, $\alpha \in \Gamma$.

Substituting v and b in (3) by $v\gamma d_1(b)$ and $\sigma(t)$ respectively, we obtain,

$$d_1(u)\alpha\sigma(v)\beta\sigma(d_1(y))\delta\sigma(d_2(\sigma(t))) + d_2(\tau(u))\beta\tau(v)\delta\tau(d_1(y))\gamma d_1(\sigma(t)) = 0, \quad (7)$$

for all u, v, b, $t \in N$, α, $\beta \in \Gamma$.

Now, if we subtract (6) from (7) and make use $d_1\sigma = \sigma d_1$, $d_1\tau = \tau d_1$, then

$$d_1(u)\alpha\sigma(v)\delta(d_1(\sigma(y))\beta\sigma(d_2(\sigma(t)) - \sigma(d_2(\tau(y))\beta\sigma(d_1(t)) = 0.$$

Therefore,

$$d_1(N)\Gamma N\Gamma(\sigma(d_1(b)\Gamma d_2(\sigma(t)) - d_2(\tau(b))\beta d_1(t)) = 0, \tag{8}$$

for all t, $b \in N$.

Since N is a prime Γ-near-ring and $d_1 \neq 0$ we conclude that,

$$d_1(b)\alpha d_2(\sigma(t)) - d_2(\tau(b))\beta d_1(t) = 0.$$

Using $d_2\sigma = \sigma d_2$ and the hypothesis, we get

$$d_1(b)\beta\sigma(d_2(t) + d_2(t)) = 0, \tag{9}$$

for all b, $t \in N$.

Due to Lemma 4.1(i) and $d_1 \neq 0$ we get, $d_2(t) + d_2(t) = 0$, for all $t \in N$. Hence $0 = d_2(s\alpha t) + d_2(s\alpha t) = d_2(s)\alpha(t + t)$ for all s, $t \in N$, $\alpha \in \Gamma$, and so $d_2(N)\alpha(t + t) = 0$, for all $t \in N$. If we apply Lemma 3.5(iii), we get $t + t = 0$ for all $t \in N$, which contradicts that $2N \neq \{0\}$.

This completes the proof.

Theorem 4.5. *Let N be a prime Γ-near-ring with $2N \neq \{ 0 \}$, d_1 be a (σ, τ)-derivation and d_2 be a derivation such that $d_1\tau = \tau d_1$, $d_1\sigma = \sigma d_1$, $d_2\tau = \tau d_2$ and $d_2\sigma = \sigma d_2$.(i) If $d_1(a)\Gamma\sigma(d_2(b)) + \tau(d_2(a))\Gamma d_1(b) = 0$ for all a, $b \in N$, then either $d_1 = 0$ or $d_2 = 0$.(ii) If d_1d_2 acts as a (σ, τ)-derivation on N, then either $d_1 = 0$ or $d_2 = 0$.*

Proof. Consider

$$d_2d_1(a\alpha b) = \tau(a)\alpha d_2d_1(b) + d_2d_1(a)\alpha\sigma(b),$$

and

$$d_2d_1(a\alpha b) = d_2(\tau(a))\alpha d_1(b) + \tau(a)\alpha d_2d_1(b) + d_2d_1(a)\alpha\sigma(b) + d_1(a)\alpha d_2(\sigma(b)).$$

These two expressions give $\tau(d_2(a))\alpha d_1(b) + d_1(a)\alpha\sigma(d_2(b)) = 0$, for all a, $b \in N$. Now due to Theorem 4.4, we get $d_1 = 0$ or $d_2 = 0$.

Theorem 4.6. *Let N be a prime Γ-near-ring with a nonzero (σ, τ)-derivation d such that $d(a\alpha b) = d(b\alpha a)$ for all $a, b \in N$, $\alpha \in \Gamma$, then N is a commutative Γ-near-ring.*

Proof. Let us consider

$$\tau(a)\alpha d(b) + d(a)\alpha\sigma(b) = \tau(b)\alpha d(a) + d(b)\alpha\sigma(a), \tag{10}$$

for all $a, b \in N$, $\alpha \in \Gamma$.

We substitute $b\beta a$ for a in (10) and obtain

$$\tau(b)\beta\tau(a)\alpha d(b) + d(b\beta a)\alpha\sigma(b) = \tau(b)\beta d(b\alpha a) + d(b)\alpha\tau(b)\alpha\sigma(a).$$

By the partial distributive law for $a, b \in N$, α, $\beta \in \Gamma$ and using $d(b\alpha a) = d(a\alpha b)$ we get

$$\begin{aligned}\tau(b)\beta\tau(a)\alpha d(b) + \tau(b)\beta d(a)\alpha\sigma(b) + d(b)\alpha\sigma(a)\beta\sigma(b)\\ = \tau(b)\beta\tau(a)\alpha d(b) + \tau(b)\beta d(a)\alpha\sigma(b) + d(b)\alpha\sigma(b)\beta\sigma(a).\end{aligned}$$

Therefore,

$$d(b)\alpha\sigma(a)\beta\sigma(b) = d(b)\alpha\sigma(b)\beta\sigma(a), \tag{11}$$

for all $a, b \in N$, α, $\beta \in \Gamma$.

Replacing a by $a\delta c$, $c \in N$ in the last equality we get

$$d(b)\beta\sigma(a)\beta\sigma(c)\beta\sigma(b) = d(b)\alpha\sigma(b)\alpha\sigma(a)\alpha\sigma(c) = d(b)\alpha\sigma(a)\beta\sigma(b)\alpha\sigma(c)$$

and so $d(b)\Gamma N\Gamma\sigma([c, b]_\alpha) = 0$, for all a, $c \in N$, α, $\beta \in \Gamma$. By the primeness of N, we obtain $d(b) = 0$ or $b \in Z(N)$. Since d is a nonzero (σ, τ)-derivation on N, we have $b \in Z(N)$. So, N is a commutative Γ-near-ring.

Theorem 4.7. *Let N be a prime 2-torsion free Γ-near-ring and d_1 be a nonzero (σ, τ)-derivation, d_2 a nonzero derivation on N such that $d_1\tau = \tau d_1, d_1\sigma = \sigma d_1$, $d_2\tau = \tau d_2$ and $d_2\sigma = \sigma d_2$. If $d_1(a)\Gamma\sigma(d_2(b)) = \tau(d_2(a))\Gamma d_1(b)$ for all $a, b \in N$, then N is a commutative Γ-ring.*

Proof. Consider $[d_1(a), d_2(b)]_\alpha^{(\sigma,\tau)} = 0$, for all $a, b \in N$, $\alpha \in \Gamma$. Applying Theorem 3.6, we obtain that $d_1d_2(N) = 0$ or $d_2(N) \subset Z(N)$. Since d_1 and d_2 are nonzero derivations, we have $d_2(N) \subset Z(N)$. Now according to Theorem 4.2 we get the commutativity of N.

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