PRIME GAMMA-NEAR-RINGS WITH \((\sigma, \tau)\)-DERIVATIONS

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Abstract: Let \(N\) be a 2 torsion free prime \(\Gamma\)-near-ring with center \(Z(N)\) and let \(d\) be a nontrivial derivation on \(N\) such that \(d(N) \subseteq Z(N)\). Then we prove that \(N\) is commutative. Also we prove that if \(d\) be a nonzero \((\sigma, \tau)\)-derivation on \(N\) such that \(d(N)\) commutes with an element \(a\) of \(N\) then either \(d\) is trivial or \(a\) is in \(Z(N)\). Finally if \(d_1\) be a nonzero \((\sigma, \tau)\)-derivation and \(d_2\) be a nonzero derivation on \(N\) such that \(d_1\tau = \tau d_1\), \(d_1\sigma = \sigma d_1\), \(d_2\tau = \tau d_2\), \(d_2\sigma = \sigma d_2\) with \(d_1(N)\Gamma \sigma(d_2(N)) = \tau(d_2(N))\Gamma d_1(N)\) then \(N\) is a commutative \(\Gamma\)-ring.

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1. Introduction

Bell and Mason [3] initiated the study of derivations in near-rings. The derivation in \(\Gamma\)-near-rings was first introduced by Cho and Jun [6]. They studied some
basic properties of prime Γ-near-rings. Ozturk and Jun in [14] discussed prime
Γ-near-rings with derivations. They obtained commutativity properties on
prime Γ-near-rings. Later Asci [1] studied Γ-near-rings with \((\sigma,\tau)\)-derivations
and obtained some commutativity results. The commutativity properties of
prime Γ-near-rings with derivations also have been investigated by Uckun, Ozturk and Jun [18]. Golbasi [10] obtained analogue of Posner’s Theorem from
[17]. The commutativity conditions of prime Γ-near-rings with generalized
derivations have been studied in [7] (also see [8]). The paper [9] has dealt with
generalized derivations in semiprime gamma-rings.

In this paper we consider the prime Γ-near-rings with derivations to subject
of commutativity conditions. The characterizations of prime Γ-near-rings with
composition of two derivations are obtained. We also investigate conditions for
prime Γ-near-ring to be commutative.

Organization of the paper is as follows. Sections 2 and 3 contain brief
review and definitions which will be used further. Some of these results have
been presented by others researchers (for example, see [1]). Main results are
in Sections 3 and 4. The results of Section 3 concern derivations in Γ-rings
(Theorem 3.6), whereas Section 4 deals with \((\sigma,\tau)\)-derivations in prime Γ-near-
rings (Theorems 4.2 – 4.7).

2. Preliminaries

A Γ-near-ring is a triple \((N, +, \Gamma)\), where:

(i) \((N, +)\) is a group (not necessarily abelian),

(ii) \(\Gamma\) is a non-empty set of binary operations on \(N\) such that for each \(\alpha \in \Gamma\),
     \((N, +, \alpha)\) is a left near-ring.

(iii) \(a\alpha(b\beta c) = (a\alpha b)\beta c\), for all \(a, b, c \in N\) and \(\alpha, \beta \in \Gamma\).

Throughout this paper, \(N\) stands for a zero-symmetric left Γ-near-ring with
multiplicative center \(Z(N)\). Recall that a Γ-near-ring \(N\) is prime if \(a\Gamma N\Gamma b = \{\)
\(0\} \) implies that \(a = 0\) or \(b = 0\). Let \(\sigma\) and \(\tau\) be two Γ-near-ring automorphisms
of \(N\). For \(a, b \in N\) and \(\alpha \in \Gamma\), \([a, b]_\alpha\),

\([a, b]^{(\sigma,\tau)}_\alpha\) and \((a, b)\) will denote the commutator \(a\alpha b - b\alpha a, a\alpha\sigma(b) - \tau(b)\alpha a\)
and \(a + b - a - b\) respectively. An additive mapping \(d: N \to N\) is called
a derivation if \(d(a\alpha b) = a\alpha d(b) + d(a)\alpha b\) holds for all \(a, b \in N, \alpha \in \Gamma\). An
additive mapping \(d: N \to N\) is called a \((\sigma, \tau)\)-derivation if \(d(a\alpha b) = \sigma(a)\alpha d(b) +
d(a)\alpha\tau(b)\) holds for all \(a, b \in N, \alpha \in \Gamma\). In particular \([a, b]^{(1,1)}_\alpha = [a, b]_\alpha\).
3. Properties of $\Gamma$ Rings with Derivations

**Lemma 3.1.** Let $d$ be an arbitrary additive endomorphism of $N$. Then
\[ d(a\alpha b) = a\alpha d(b) + d(a)\alpha b, \] for all $a, b \in N, \alpha \in \Gamma$
if and only if
\[ d(a\alpha b) = d(a)\alpha b + a\alpha d(b), \] for all $a, b \in N, \alpha \in \Gamma$.

**Proof.** Suppose $d(a\alpha b) = a\alpha d(b) + d(a)\alpha b$ for all $a, b \in N, \alpha \in \Gamma$. Since
\[ a\alpha(b + b) = a\alpha b + a\alpha b, \]
d$(a\alpha(b + b)) = a\alpha d(b + b) + d(a)\alpha(b + b) = a\alpha d(b) + a\alpha d(b) + d(a)\alpha b + d(a)\alpha b$
d and
\[ d(a\alpha b + a\alpha b) = d(a\alpha b) + d(a\alpha b) = a\alpha d(b) + d(a)\alpha b + a\alpha d(b) + d(a)\alpha b. \]
Then, we obtain
\[ a\alpha d(b) + d(a)\alpha b = d(a)\alpha b + a\alpha d(b), \]
so $d(a\alpha b) = d(a)\alpha b + a\alpha d(b)$, for all $a, b \in N, \alpha \in \Gamma$.
The converse part is proved similarly.

**Lemma 3.2.** Suppose that $d$ is a derivation on $N$. Then $N$ satisfies the following right distributive laws:

(i) $(a\alpha d(b) + a\alpha d(b))\beta c = a\alpha d(b)\beta c + d(a)\alpha b\beta c$,

(ii) $(d(a)\alpha b + a\alpha d(b))\beta c = d(a)\alpha b\beta c + a\alpha d(b)\beta c$ for every $a, b, c \in N$ and $\alpha, \beta \in \Gamma$.

**Proof.** (i) Consider
\[ d((a\alpha b)\beta c) = a\alpha b\beta d(c) + d(a\alpha b)\beta c = a\alpha b\beta d(c) + (a\alpha d(b) + d(a)\alpha b)\beta c, \]
for all $a, b, c \in N$ and $\alpha, \beta \in \Gamma$, and
\[ d(a\alpha(b\beta c)) = a\alpha d(b\beta c) + d(a)\alpha b\beta c = a\alpha(b\beta d(c) + d(b)\beta c) + d(a)\alpha b\beta c \]
\[ = a\alpha b\beta d(c) + a\alpha d(b)\beta c + d(a)\alpha b\beta c \]
for all $a, b, c \in N$ and $\alpha, \beta \in \Gamma$. 

From the above two relations we obtain

\[(a\alpha d(b) + d(a)\alpha b)\beta c = a\alpha d(b)\beta c + d(a)\alpha b\beta c,\]

for all \(a, b, c \in N\) and \(\alpha, \beta \in \Gamma\).

(ii) Due to Lemma 3.1 we have,

\[d((a\alpha b)\beta c) = d(a\alpha b)\beta c + a\alpha b\beta d(c) = (d(a)\alpha b + a\alpha d(b))\beta c + a\alpha b\beta d(c),\]

for \(a, b, c \in N, \alpha, \beta \in \Gamma\), and

\[d(a\alpha(b\beta c)) = d(a)\alpha b\beta c + a\alpha d(b)\beta c = d(a)\alpha b\beta c + a\alpha (d(b)\beta c + b\alpha d(c)) = d(a)\alpha b\beta c + a\alpha d(b)\beta c + a\alpha b\beta d(c),\]

for every \(a, b, c \in N\) and \(\alpha, \beta \in \Gamma\).

These two relations imply that

\[(d(a)\alpha b + a\alpha d(b))\beta c = d(a)\alpha b\beta c + a\alpha d(b)\beta c,\]

for all \(a, b, c \in N\) and \(\alpha, \beta \in \Gamma\).

**Lemma 3.3.** Let \(d\) be a derivation on a \(\Gamma\)-near-ring \(N\) and let \(u\) be not a left zero divisor in \(N\). If \([u, d(u)]_\alpha = 0\) for every \(\alpha \in \Gamma\), then \((a, u)\) is a constant for every \(a \in N\).

**Proof.** From \(u\alpha(u + a) = u\alpha u + u\alpha a\), for all \(x \in N, \alpha \in \Gamma\), we get

\[d(u\alpha(u + a)) = u\alpha d(u + a) + d(u)\alpha(u + a) = u\alpha d(u) + u\alpha d(a) + d(u)\alpha u + d(u)\alpha a = u\alpha d(u) + u\alpha d(a) + u\alpha d(u) + a\alpha d(u)\]

and

\[d(u\alpha u + u\alpha a) = d(u\alpha u) + d(u\alpha a) = u\alpha d(u) + d(u)\alpha u + u\alpha d(a) + d(u)\alpha a\]

which gives

\[u\alpha d(a) + d(u)\alpha u - d(u)\alpha u - u\alpha d(a) = 0,\]

for all \(\alpha \in \Gamma\). Since \(d(u)\alpha u = u\alpha d(u), \alpha \in \Gamma\).

This equation can be written as

\[u\alpha (d(a) + d(u) - d(a) - d(u)) = u\alpha d(a + u - a - u) = u\alpha d((a, u)) = 0.\]
Thus $d((a, u)) = 0$. Hence $(a, u)$ is constant. This completes the proof.

**Lemma 3.4.** Let $d$ be a $(\sigma, \tau)$-derivation on a $N$ and $a \in N$. Then

$$(\tau(a)\sigma d(b) + d(a)\sigma(b))\tau(c) = \tau(a)\sigma d(b)\tau(c) + d(a)\sigma(b)\tau(c),$$

for $a, b, c \in N, \alpha, \beta \in \Gamma$.

**Proof.** Consider

$$d((a\alpha b)\beta c) = \tau(a\alpha b)\beta d(c) + d(a\alpha b)\beta\tau(c)$$

$$= \tau(a)\alpha\tau(b)\beta d(c) + (\tau(a)\sigma d(b) + d(a)\sigma(b))\beta\tau(c),$$

for all $a, b, c \in N, \alpha, \beta \in \Gamma$, and

$$d(a\alpha(b\beta c)) = \tau(a)\alpha d(b\beta c) + d(a)\alpha\sigma(b\beta c)$$

$$= \tau(a)\alpha\tau(b)\beta d(c) + (\tau(a)\sigma d(b) + d(a)\sigma(b))\beta\tau(c) + d(a)\alpha\sigma(b)\beta\tau(c),$$

for all $a, b, c \in N, \alpha, \beta \in \Gamma$.

Since $d((a\alpha b)\beta c) = d(a\alpha(b\beta c))$, we have

$$(\tau(a)\alpha d(b) + d(a)\alpha\sigma(b))\beta\tau(c) = \tau(a)\alpha d(b)\beta\tau(c) + d(a)\alpha\sigma(b)\beta\tau(c),$$

for all $a, b, c \in N, \alpha, \beta \in \Gamma$.

This completes the proof.

**Lemma 3.5.** Suppose that $N$ is a prime $\Gamma$-near-ring.

(i) any nonzero element of the center of $N$ is not zero divisor.

(ii) If there exist a nonzero element of $Z(N)$ such that $a + a \in Z(N)$, then $(N, +)$ is commutative.

(iii) If $d \neq 0$ be a derivation on $N$. Then $a\Gamma d(N) = \{0\}$ implies $a = 0$, and $d(N)\Gamma a = \{0\}$ implies $a = 0$.

(iv) Let $N$ be 2-torsion free and $d$ be a derivation on $N$ with $d^2 = 0$. Then $d = 0$.

**Proof.** (i) If $a \in Z(N) - \{0\}$ and $a\alpha b = 0$, $b \in N, \alpha \in \Gamma$, then $a\alpha c\beta b = 0$, $b, c \in N, \alpha \in \Gamma$. Therefore we have, $a\Gamma N\Gamma b = 0$. Since $N$ is prime and $a \neq 0$ then $b = 0$.

(ii) Let $a \in Z(N) - \{0\}$ be an element such that $a + a \in Z(N)$ and let $b, c \in N, \alpha \in \Gamma$. By the distributivity of $N$ we get $(b + c)\alpha(a + a) = b\alpha(a + a) + c\alpha(a + a) = b\alpha a + b\alpha a + c\alpha a + c\alpha a = a\alpha b + a\alpha b + a\alpha c + a\alpha c = a\alpha(b + b + c + c)$. 


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Next, since \( a + a \in Z(N) \) then
\[(b+c)\alpha(a+a) = (b+c)\alpha a + (b+c)\alpha a = a\alpha(b+ c) + a\alpha(b+ c) = a\alpha(b+ c + b + c).
\]
Taking into account the last two we have, \( b + b + c + c = b + c + b + c \) and so \( b + c = c + b \). Hence \( (N, +) \) is commutative.

(iii) Let \( a\Gamma d(N) = \{ 0 \} \), \( b, c \in N \) and \( \alpha, \beta \in \Gamma \). Then \( 0 = a\alpha d(b\beta c) = a\alpha b\beta d(c) + a\alpha d(b)\beta c = a\alpha b\beta d(c) \). Thus \( a\Gamma\mathcal{N}d(N) = \{ 0 \} \), by using the primeness of \( N \) and \( d(N) \neq 0 \) we get \( a = 0 \).

The part (iii) can be proved similarly.

(iv) Note that for any \( a, b \in N, \alpha \in \Gamma \), \( 0 = d^2(a\alpha b) = d(a\alpha d(b) + d(a)\alpha b) = a\alpha d^2(b) + d(a)\alpha d(b) + d(a)d(\alpha b) + d^2(a)\alpha b \). Therefore we have \( 2d(a)\alpha d(b) = 0 \), for all \( a, b \in N, \alpha \in \Gamma \). Since \( N \) is 2-torsion free we get \( d(a)\alpha d(b) = 0 \), for each \( a \in N, \alpha \in \Gamma \). By using (iii) we obtain \( d = 0 \).

**Theorem 3.6.** Let \( N \) be a prime \( \Gamma \)-near-ring and let \( d \) be a nontrivial derivations on \( N \) such that \( d(N) \subseteq Z(N) \). Then \( (N, +) \) is commutative. Moreover if \( N \) is 2-torsion free, then \( N \) is a commutative \( \Gamma \)-ring.

**Proof.** Let \( a \) be a non-constant and \( c \) be a constant elements of \( N \). Then \( d(a\alpha c) = a\alpha d(c) + d(a)\alpha c = d(a)\alpha c \in Z(N), \alpha \in \Gamma \). Since \( d(a) \in Z(N) - \{ 0 \} \), it follows easily that \( c \in Z(N) \). Note that if \( c \) is a constant so \( c + c \) is a constant. It follows from lemma 3.5(ii) that \( (N, +) \) is commutative, provided that there exists a nonzero constant. Let us provide the last.

Suppose, that 0 is the only constant. Since \( d \) is obviously commuting, it follows from the lemma 3.3 that all \( u \) which are not zero divisors belong to by \( Z(N) \). In particular, if \( a \neq 0 \), \( d(a) \in Z(N) \). But then for all \( b \in N \), \( d(b) + d(a) - d(b) - d(a) = d((b, a)) = 0 \), hence \( (b, a) = 0 \).

Now, we assume that \( N \) is 2-torsion free. By Lemma 3.1,
\[(a\alpha d(b) + d(a)\alpha b)\beta c = a\alpha d(b)\beta c + d(a)\alpha b\beta c\]
for all \( a, b, c \in N, \alpha, \beta \in \Gamma \), and using the fact that \( d(a\alpha b) \in Z, \alpha \in \Gamma \), we get \( c\alpha d(b) + c\alpha d(\alpha b)\beta c = a\alpha d(b)\beta c + a\alpha d(b)\beta c, \alpha, \beta \in \Gamma \). Since \( (N, +) \) is commutative and \( d(N) \subseteq Z(N) \), we obtain \( d(b)\alpha[c, a]_\beta = d(a)\alpha[b, c]_\beta \) for all \( a, b, c \in N, \alpha, \beta \in \Gamma \).

Suppose now that \( N \) is not commutative. Choosing \( b, c \in N \), with \( [b, c]_\beta \neq 0, \beta \in \Gamma \), and letting \( u = d(a) \), we get \( d^2(a)\alpha[b, c]_\beta = 0 \), for all \( a \in N, \alpha, \beta \in \Gamma \), and since the central elements \( d^2(a) \) cannot be a nonzero divisor of zero, we conclude that \( d^2(a) = 0 \) for all \( a \in N \). By lemma 3.5(iv), this is not true for nontrivial \( d \).
4. \( \Gamma \)-Near-Rings with \((\sigma, \tau)\)-Derivations

**Lemma 4.1.** Suppose that \( N \) is a prime \( \Gamma \)-near-ring with a nonzero \((\sigma, \tau)\)-derivation \( d \) and \( a \in N \). If at least one of the following (i) \( d(N)\Gamma\sigma(a) \) or (ii) \( a\Gamma d(N) \) is zero then \( a = 0 \).

**Proof.** (i) Consider

\[
0 = d(b\beta c)\alpha\sigma(a) = (\tau(b)\beta d(c) + d(b)\beta\sigma(c))\alpha\sigma(a) = \tau(b)\beta d(c)\alpha\sigma(a) + d(b)\beta\sigma(c)\alpha\sigma(a),
\]

for all \( a, b, c \in N, \alpha, \beta \in \Gamma \).

Since \( d(c)\alpha\sigma(a) = 0 \), we have \( d(b)\beta\sigma(c)\alpha\sigma(a) = 0 \), i.e., \( d(b)\Gamma\Gamma\sigma\sigma(a) = 0 \). (Bearing in mind that \( \sigma \) is an automorphism). Since \( d \neq 0 \) and \( N \) is prime, we get \( \sigma(a) = 0 \) which gives \( a = 0 \).

The proof of (ii) is similar.

**Theorem 4.2.** Suppose that \( d \) is a nonzero \((\sigma, \tau)\)-derivation of a prime \( \Gamma \)-near-ring \( N \) and \( a \in N, \alpha \in \Gamma \). Let \([d(N), a]^{(\sigma, \tau)} = 0\), then \( d(a) = 0 \) or \( a \in Z(N) \).

**Proof.** Note that for all \( b \in N \) and \( \alpha \in \Gamma, a\alpha b \in N \). Therefore, from \([d(N), a]^{(\sigma, \tau)} = 0\) we have, \( d(a\alpha b)\beta\sigma(a) = \tau(a)\beta d(a\alpha b) \) for \( \beta \in \Gamma \).

Hence

\[
(\tau(a)\alpha d(b) + d(a)\alpha\sigma(b))\beta\sigma(a) = \tau(a)\beta(\tau(a)\alpha d(b) + d(a)\alpha\sigma(b)).
\]

Then by Lemma 3.4, we have

\[
\tau(a)\alpha d(b)\beta\sigma(a) + d(a)\alpha\sigma(b)\beta\sigma(a) = \tau(a)\beta\tau(a)\alpha d(b) + \tau(a)\beta d(a)\alpha\sigma(b).
\]

Due to the condition of the theorem, we have

\[
\tau(a)\alpha\tau(a)\beta d(b) + d(a)\alpha\sigma(b)\beta\sigma(a) = \tau(a)\beta\tau(a)\alpha d(b) + d(a)\alpha\sigma(a)\beta\sigma(b),
\]

that is

\[
d(a)\alpha\sigma([b, a]_{\beta}) = 0 \tag{1}
\]

for all \( b \in N, \alpha, \beta \in \Gamma \).

Substituting \( b\delta c, (c \in N, \delta \in \Gamma) \) for \( a \) in (1), we have \( d(a)\alpha\sigma(b)\beta\sigma([c, a]_{\alpha}) = 0 \) for all \( b, c \in N, \alpha, \beta, \delta \in \Gamma \). Since \( \sigma \) is automorphism we get \( d(a) = 0 \) or \( a \in Z(N) \). This completes the proof.
Lemma 4.3. Suppose that \( N \) is a prime \( \Gamma \)-near-ring and \( d_1 \) is a nonzero \((\sigma, \tau)\)-derivation and \( d_2 \) is a derivation of \( N \) such that

\[
d_1(a)\Gamma \sigma(d_2(b)) + \tau(d_2(a))\Gamma d_1(b) = 0
\]

for all \( a, b \in N \). Then \((N, +)\) is commutative.

Proof. Consider \( d_1(a)\Gamma \sigma(d_2(b)) + \tau(d_2(a))\Gamma d_1(b) = 0 \) for all \( \alpha \in \Gamma \). Replacing \( b \) by \( u + v \) we have,

\[
0 = d_1(a)\alpha \sigma(d_2(u + v)) + \tau(d_2(a))\alpha d_1(u + v)
\]

\[
= d_1(a)\alpha \sigma(d_2(u)) + d_1(a)\alpha \sigma(d_2(v)) + \tau(d_2(a))\alpha d_1(u) + \tau(d_2(a))\alpha d_1(v),
\]

for \( u, v \in N, \alpha \in \Gamma \).

Using the condition of the lemma, we get

\[
0 = d_1(a)\alpha \sigma(d_2(u)) + d_1(a)\alpha \sigma(d_2(v)) - d_1(a)\alpha \sigma(d_2(u)) - d_1(a)\alpha \sigma(d_2(v)).
\]

As result we get,

\[
d_1(a)\alpha \sigma(d_2(u, v)) = 0
\]

for all \( a, u, v \in N, \alpha \in \Gamma \).

By Lemma 4.1(i), we obtain that \( d_2(u, v) = 0 \) for all \( u, v \in N \). Note that for any \( w \in N \), we have \( d_2(\omega u, \omega v) = 0 \). Using (2), we obtain that \( d_2(\omega \alpha(u, v)) = 0 \). This yields \( d_2(\omega \alpha(u, v)) = 0 \), for all \( w, u, v \in N, \alpha \in \Gamma \). By Lemma 3.5(iii), we get \( (u, v) = 0 \), for all \( u, v \in N \).

Theorem 4.4. Let \( N \) be a prime \( \Gamma \)-near-ring with \( 2N \neq \{0\} \), \( d_1 \) be a \((\sigma, \tau)\)-derivation and \( d_2 \) be a derivation such that \( d_1 \tau = \tau d_1, d_1 \sigma = \sigma d_1, d_2 \tau = \tau d_2 \) and \( d_2 \sigma = \sigma d_2 \). If \( d_1(a)\Gamma \sigma(d_2(b)) + \tau(d_2(a))\Gamma d_1(b) = 0 \) for all \( a, b \in N \), then either \( d_1 = 0 \) or \( d_2 = 0 \).

Proof. Let \( d_1 \neq 0 \) and \( d_2 \neq 0 \). According to Lemma 4.3 we know that \((N, +)\) is commutative. Now, for all \( u, v \in N \) we take \( a = u \beta v \) in \( d_1(a)\Gamma \sigma(d_2(b)) + \tau(d_2(a))\Gamma d_1(b) = 0 \) to obtain

\[
0 = d_1(u \beta v)\alpha \sigma(d_2(b)) + \tau(d_2(u \beta v))\alpha d_1(b)
\]

\[
= (\tau(u)\beta d_1(v) + d_1(u)\beta \sigma(v))\alpha \sigma(d_2(b)) + \tau(u) \beta d_2(v) + d_2(u) \beta v) \alpha d_1(b).
\]

Using the left distributive law, we get,

\[
0 = \tau(u) \beta d_1(v)\alpha \sigma(d_2(b)) + d_1(u) \beta \sigma(v)\alpha \sigma(d_2(b)) + \tau(u) \beta \tau(d_2(v)) \alpha \sigma d_1(b)
\]
for all \( u, v, b \in N, \alpha, \beta \in \Gamma \).

Hence,

\[
0 = \tau(u)\beta(d_1(v)\alpha\sigma(d_2(b) + \tau(d_2(v))\alpha d_1(b)) + d_1(u)\beta\sigma(v)\alpha\sigma(d_2(b)) + d_2(\tau(u))\beta\sigma(v)\alpha d_1(b).
\]

Now due to the condition of the theorem, we obtain

\[
d_1(u)\beta\sigma(v)\alpha\sigma(d_2(b)) + d_2(\tau(u))\beta\sigma(v)\alpha d_1(b) = 0, \tag{3}
\]

for all \( b, u, v \in N, \alpha, \beta \in \Gamma \).

Replacing \( b \) by \( b\delta t \), \( t \in N, \delta \in \Gamma \), in (3) we have

\[
0 = d_1(u)\alpha\beta\sigma(v)\alpha\sigma(d_2(b\delta t)) + d_2(\tau(u))\beta\sigma(v)\alpha d_1(b\delta t)
= d_1(u)\beta\sigma(v)\alpha\sigma(d_2(t)) + d_1(u)\beta\sigma(v)\alpha\sigma(d_2(b))\delta\sigma(t)
+ d_2(\tau(u))\beta\sigma(v)\alpha\sigma(y)\delta d_1(t) + d_2(\tau(u))\beta\sigma(v)\alpha d_1(b)\delta\sigma(t)
= d_1(u)\beta\sigma(v)\alpha\sigma(d_2(t)) + d_2(\tau(u))\beta\sigma(v)\alpha\sigma(d_2(b))\delta\sigma(t)
+ d_1(u)\beta\sigma(v)\alpha\sigma(d_2(b))\delta\sigma(t) + d_2(\tau(u))\beta\sigma(v)\alpha d_1(b)\delta\sigma(t).
\]

Using (3) we again obtain,

\[
d_1(u)\beta\sigma(v)\beta\sigma(d_2(b))\beta\sigma(t) + d_2(\tau(u))\beta\sigma(v)\alpha d_1(b)\delta\sigma(t) = 0, \tag{4}
\]

for all \( u, v, b, t \in N, \alpha, \beta, \delta \in \Gamma \).

Substituting \( d_1(t) \) for \( t \) in (4), we get

\[
d_1(u)\alpha\sigma(v)\beta\sigma(d_2(b))\delta\sigma(d_1(t)) + d_2(\tau(u))\beta\sigma(v)\alpha d_1(b)\delta\sigma(d_1(t)) = 0 \tag{5}
\]

for all \( u, v, b, t \in N, \alpha, \beta, \delta \in \Gamma \).

Now if we take \( \tau(b) \) instead of \( y \) in (5), we have,

\[
d_1(u)\alpha\sigma(v)\beta\sigma(d_2(\tau(b)))\delta\sigma(d_1(t)) + d_2(\tau(u))\beta\sigma(v)\alpha d_1(\tau(b))\delta\sigma(d_1(t)) = 0, \tag{6}
\]

for all \( u, v, b, t \in N, \alpha, \beta, \delta \in \Gamma \).

Substituting \( v \) and \( b \) in (3) by \( v\gamma d_1(b) \) and \( \sigma(t) \) respectively, we obtain

\[
d_1(u)\alpha\sigma(v)\beta\sigma(d_1(y))\delta\sigma(d_2(\sigma(t))) + d_2(\tau(u))\beta\sigma(v)\delta\sigma(d_1(y))\gamma d_1(\sigma(t)) = 0, \tag{7}
\]

for all \( u, v, b, t \in N, \alpha, \beta \in \Gamma \).
Now, if we subtract (6) from (7) and make use $d_1\sigma = \sigma d_1$, $d_1\tau = \tau d_1$, then
\[ d_1(u)\alpha \sigma (v) \delta (d_1(\sigma(y)))\beta \sigma (d_2(\sigma(t))) - \sigma (d_2(\tau(y)))\beta \sigma (d_1(t)) = 0. \]

Therefore,
\[ d_1(N)\Gamma \sigma (d_1(b))\Gamma d_2(\sigma(t)) - d_2(\tau(b))\beta d_1(t) = 0, \tag{8} \]
for all $t, b \in N$.

Since $N$ is a prime $\Gamma$-near-ring and $d_1 \neq 0$ we conclude that,
\[ d_1(b)\alpha d_2(\sigma(t)) - d_2(\tau(b))\beta d_1(t) = 0. \]

Using $d_2\sigma = \sigma d_2$ and the hypothesis, we get
\[ d_1(b)\beta \sigma (d_2(t) + d_2(t)) = 0, \tag{9} \]
for all $b, t \in N$.

Due to Lemma 4.1(i) and $d_1 \neq 0$ we get, $d_2(t) + d_2(t) = 0$, for all $t \in N$. Hence $0 = d_2(s\alpha t) + d_2(s\alpha t) = d_2(s)\alpha (t + t)$ for all $s, t \in N, \alpha \in \Gamma$, and so $d_2(N)\alpha (t + t) = 0$, for all $t \in N$. If we apply Lemma 3.5(iii), we get $t + t = 0$ for all $t \in N$, which contradicts that $2N \neq \{0\}$.

This completes the proof.

**Theorem 4.5.** Let $N$ be a prime $\Gamma$-near-ring with $2N \neq \{0\}$, $d_1$ be a $(\sigma, \tau)$-derivation and $d_2$ be a derivation such that $d_1\tau = \tau d_1$, $d_1\sigma = \sigma d_1$, $d_2\tau = \tau d_2$ and $d_2\sigma = \sigma d_2$.(i) If $d_1(a)\Gamma \sigma (d_2(b)) + \tau (d_2(a))\Gamma d_1(b) = 0$ for all $a, b \in N$, then either $d_1 = 0$ or $d_2 = 0$.(ii) If $d_1d_2$ acts as a $(\sigma, \tau)$-derivation on $N$, then either $d_1 = 0$ or $d_2 = 0$.

**Proof.** Consider
\[ d_2d_1(a\sigma b) = \tau (a)\alpha d_2d_1(b) + d_2d_1(a)\alpha \sigma (b), \]
and
\[ d_2d_1(a\sigma b) = d_2(\tau (a))\alpha d_1(b) + \tau (a)\alpha d_2d_1(b) + d_2d_1(a)\alpha \sigma (b) + d_1(a)\alpha d_2(\sigma(b)). \]

These two expressions give $\tau (d_2(a))\alpha d_1(b) + d_1(a)\alpha \sigma (d_2(b)) = 0$, for all $a, b \in N$. Now due to Theorem 4.4, we get $d_1 = 0$ or $d_2 = 0$.

**Theorem 4.6.** Let $N$ be a prime $\Gamma$-near-ring with a nonzero $(\sigma, \tau)$-derivation $d$ such that $d(a\sigma b) = d(b\alpha a)$ for all $a, b \in N, \alpha \in \Gamma$, then $N$ is a commutative $\Gamma$-near-ring.
Proof. Let us consider
\[\tau(a)\alpha d(b) + d(a)\alpha \sigma(b) = \tau(b)\alpha d(a) + d(b)\alpha \sigma(a),\] (10)
for all \(a, b \in N,\ \alpha \in \Gamma.\)

We substitute \(b\beta a\) for \(a\) in (10) and obtain
\[\tau(b)\beta \tau(a)\alpha d(b) + d(b)\beta a\alpha \sigma(b) = \tau(b)\beta d(b\alpha a) + d(b)\alpha \tau(b)\alpha \sigma(a).\]

By the partial distributive law for \(a, b \in N,\ \alpha, \beta \in \Gamma\) and using \(d(b\alpha a) = d(a\alpha b)\) we get
\[\tau(b)\beta \tau(a)\alpha d(b) + \tau(b)\beta d(a)\alpha \sigma(b) + d(b)\alpha \sigma(a)\beta \sigma(b)\]
\[= \tau(b)\beta \tau(a)\alpha d(b) + \tau(b)\beta d(a)\alpha \sigma(b) + d(b)\alpha \sigma(b)\beta \sigma(a).\]

Therefore,
\[d(b)\alpha \sigma(a)\beta \sigma(b) = d(b)\alpha \sigma(b)\beta \sigma(a),\] (11)
for all \(a, b \in N,\ \alpha, \beta \in \Gamma.\)

Replacing \(a\) by \(a\delta c,\ c \in N\) in the last equality we get
\[d(b)\beta \sigma(a)\beta \sigma(c)\beta \sigma(b) = d(b)\alpha \sigma(b)\alpha \sigma(a)\alpha \sigma(c) = d(b)\alpha \sigma(a)\beta \sigma(b)\alpha \sigma(c)\]
and so \(d(b)\Gamma \sigma([c, b]_\alpha) = 0,\ \text{for all} a, c \in N,\ \alpha, \beta \in \Gamma.\)

By the primeness of \(N,\) we obtain \(d(b) = 0\) or \(b \in Z(N).\) Since \(d\) is a nonzero \((\sigma, \tau)\)-derivation on \(N,\) we have \(b \in Z(N).\) So, \(N\) is a commutative \(\Gamma\)-near-ring.

Theorem 4.7. Let \(N\) be a prime 2-torsion free \(\Gamma\)-near-ring and \(d_1\) be a nonzero \((\sigma, \tau)\)-derivation, \(d_2\) a nonzero derivation on \(N\) such that \(d_1\tau = \tau d_1, d_1\sigma = \sigma d_1, d_2\tau = \tau d_2\) and \(d_2\sigma = \sigma d_2.\) If \(d_1(a)\Gamma \sigma(d_2(b)) = \tau(d_2(a))\Gamma d_1(b)\) for all \(a, b \in N,\) then \(N\) is a commutative \(\Gamma\)-ring.

Proof. Consider \([d_1(a), d_2(b)]^{(\sigma, \tau)}_{\alpha} = 0,\ \text{for all} a, b \in N,\ \alpha \in \Gamma.\)

Applying Theorem 3.6, we obtain that \(d_1d_2(N) = 0\) or \(d_2(N) \subset Z(N).\) Since \(d_1\) and \(d_2\) are nonzero derivations, we have \(d_2(N) \subset Z(N).\) Now according to Theorem 4.2 we get the commutativity of \(N.\)

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