YET ANOTHER X-RANK
CHARACTERIZATION OF RATIONAL NORMAL CURVES

E. Ballico
Department of Mathematics
University of Trento
38 123 Povo (Trento) - Via Sommarive, 14, ITALY

Abstract: Fix positive integers $s, k_i, 1 \leq i \leq s$, such that $k_1 \geq 2$ and $2k < n$, where $k := k_1 + \cdots + k_s$. Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate curve. For any $P \in \mathbb{P}^n$ the $X$-rank $r_X(P)$ of $P$ is the minimal cardinality of a set $S \subset X$ such that $P \in \langle S \rangle$. We prove that $X$ is not a rational normal curve if and only if the following condition holds: fix $s$ general points $P_1, \ldots, P_s \in X_{reg}$ and set $Z := \sum_{i=1}^s k_i P_i$; then there is some $P \in \langle Z \rangle$ such that $P \notin \langle Z' \rangle$ for any $Z' \subsetneq Z$ and $r_X(P) \leq n + 1 - k$.

Moreover, if $X$ is not a rational normal curve and we fix a finite set $E \subset X$, then we may find a set $S \subset X \setminus E$ with $\sharp(S) \leq n + 1 - k$ and $P \in \langle S \rangle$.

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1. Introduction

Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate variety defined over an algebraically closed field $\mathbb{K}$. For any $P \in \mathbb{P}^n$ the $X$-rank $r_X(P)$ of $P$ is the minimal cardinality of a finite set $S \subset X$ such that $P \in \langle S \rangle$, where $\langle \cdot \rangle$ denote the linear span. In characteristic zero we have $r_X(P) \leq n + 1 - \dim(X)$ for any $X$ and any $P$ (see [5], Proposition 4.1). In positive characteristic this is true, except
for at most one $P$ (see [1]). In this short note we prove the following result.

**Theorem 1.** Fix positive integers $s, k_i$, $1 \leq i \leq s$, such that $k_1 \geq 2$ and $2k < n$, where $k := k_1 + \cdots + k_s$. Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate curve. $X$ is not a rational normal curve if and only if the following condition ♠ holds:

Condition ♠: Fix $s$ general points $P_1, \ldots, P_s \in X_{\text{reg}}$ and set $Z := \sum_{i=1}^{s} k_i P_i$. Then there is some $P \in \langle Z \rangle$ such that $P \notin \langle Z' \rangle$ for any $Z' \subset Z$ and $r_X(P) \leq n + 1 - k$.

Moreover, if $X$ is not a rational normal curve and we fix a finite set $E \subset X$, then we may find a set $S \subset X \setminus E$ with $\sharp(S) \leq n + 1 - k$ and $P \in \langle S \rangle$.

In the case $k_i = 1$ for all $i$ we prove the following result.

**Theorem 2.** Fix integers $k \geq 2$ and $n > 2k$. Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate curve. Fix a general $Z \subset X$ such that $\sharp(Z) = s$. $X$ is not a rational normal curve if and only if there is some $P \in \langle Z \rangle$ such that $P \notin \langle Z' \rangle$ for any $Z' \subset Z$ and $W \supseteq Z$ for every set $W \subset X$ such that $\sharp(W) \leq n + 1 - k$, $W \cap Z = \emptyset$ and $P \in \langle W \rangle$. Moreover, if $X$ is not a rational normal curve for any fixed finite set $E \subset X$ we may find $W$ as above with $W \cap E = \emptyset$.

Instead of $W \cap Z = \emptyset$ we may assume $W \not\subset Z$.

**Question 1.** For general $Z$ is it possible to take as $P$ a general element of $\langle Z \rangle$? Or, even, every $P \in \langle Z \rangle$ such that $P \notin \langle Z' \rangle$ for any $Z' \subset Z$?

## 2. The Proofs

**Lemma 1.** Let $C \subset \mathbb{P}^r$ be an integral and non-degenerate curve. Fix a general $P \in C$ and a finite set $E \subset C$ with $P \notin E$. There is a finite set $S \subset C \setminus E$ such that $P \in \langle S \rangle$, $P \notin S$ and $\sharp(S) \leq r$ if and only if $C$ is not a rational normal curve.

**Proof.** If $C$ is a rational normal, then no such set exists, because any $r + 1$ points of $C$ are linearly independent. Now assume that $C$ is a rational normal curve. Take a general hyperplane $H \subset \mathbb{P}^r$ containing $P$. For general $H$ we have $H \cap E = \emptyset$. Since $P$ is general in $C$, $H$ may be seen as a general hyperplane. Hence $C \cap H$ is a general hyperplane section of $C$. Hence $C \cap H$ is formed by $\deg(C) > n$ distinct points and $C \cap H$ spans $H$. If $C$ is not very strange in the sense of [6], then we may take as $S$ any $r$ points of $C \cap H \setminus \{P\}$. If $C$ is very strange, then we need to check that not all $r$-ples of point of $C \cap H$
spanning \( H \) contains \( P \). Fix \( P_1, P_2 \in C \cap H \setminus \{P\} \) such that \( P_1 \neq P_2 \). If \( r = 2 \) we take \( S = \{P_1, P_2\} \). Assume \( r > 2 \). For every integer \( t \in \{2, \ldots, r - 2\} \) all \( t \)-dimensional linear subspaces of \( H \) spanned by points of \( C \cap H \) contain the same number of points of \( C \cap H \). Hence there is \( P_3 \in C \cap H \setminus (\langle P_1, P_2 \rangle \cup \{P\}) \).

And so on if \( r > 3 \).

\[ \Box \]

**Proof of Theorems 1 and 2.** In characteristic zero the “if” part of Theorem 1 is the easy part of theorem of Sylvester (see [4], [3], [5]). In arbitrary characteristic it is just [2], Lemma 1, and the observation that if \( X \) is a rational normal curve, then every zero-dimensional scheme \( A \subset X \) with \( \deg(A) \leq n + 1 \) is linearly independent. Hence it is sufficient to prove the “only if” part. In the set-up of Theorem 2 take \( s := k \) and write \( Z = \{P_1, \ldots, P_s\} \). Fix a finite set \( E \subset X \). Let \( \ell : \mathbb{P}^n \setminus \{P_s\} \to \mathbb{P}^{n-1} \) denote the linear projection from \( P_s \). Since \( P_s \) is general in \( X \), a dimensional count gives that for a general \( Q \in X \) the line \( \langle Q, P_s \rangle \) meets \( X \) only at \( Q \) and \( P_s \) and that a general tangent line of \( X \) does not contain \( P_s \). Hence \( \ell|X \setminus \{P_s\} \) is birational onto its image. Let \( C \subset \mathbb{P}^{n-1} \) denote the closure of \( \ell(X \setminus \{P_s\}) \) in \( \mathbb{P}^{n-1} \). Since \( \deg(C) = \deg(X) - 1 \), \( X \) is a rational normal curve if and only if \( C \) is a rational normal curve. Assume that \( C \) is not a rational normal curve. Set \( Q_i := \ell(P_i), 1 \leq i \leq s - 1 \). Let \( Q_s \in C \) be the only point corresponding to the tangent line of \( X \) at \( P_s \). Set \( E' := \ell(E \setminus \{P_s\}) \cup \{Q_1, \ldots, Q_s\} \). Set \( B := \sum_{i=1}^{s-1} k_i Q_i + (k_s - 1) Q_s \) with the convention that \( 0 Q_s \) is the zero divisor \( C \). Since \( P_1, \ldots, P_s \) are general in \( X \), \( Q_1, \ldots, Q_s \) are general in \( C \). We use induction on \( k \). First assume \( k = 2 \). In this case \( B = Q_1 \). Lemma 1 with the set \( E' \) gives the existence of a set \( S' \subset C \setminus (E_1 \cup Z_{\text{red}}) \) such that \( \#(S') \leq n + 1 - k \) and \( Q_1 \in \langle S' \rangle \). Since \( Q_s \notin S \), there is a unique set \( S \subset X \) such that \( \ell(S) = S' \). By construction we have \( S \cap E = \emptyset \). Since \( Q_1 \in \langle S' \rangle \) and \( \ell \) is the linear projection from \( P_1 \), we have \( \langle Z \rangle \cap \langle S \rangle \). Since \( P_s \notin S \) and \( P_i \notin S \), the set \( \langle Z \rangle \cap \langle S \rangle \) is a unique point, \( P \), and \( P \notin \langle Z' \rangle \) for any \( Z' \subset Z \). Now assume \( k > 2 \). We apply the inductive assumption the integer \( k - 1 \). Take \( S' \subset C \setminus S' \) such that \( \#(S') \leq n + k - 1 \) and \( \langle S' \rangle \cap \langle B \rangle \) contains a point \( P' \) such that \( P' \notin \langle B' \rangle \) for any \( B' \subset B \). Then as in the case \( k = 2 \) we may take as \( S \) the only subset of \( X \) with \( \ell(S) = S' \). \[ \Box \]

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References


