COlomposed Pencils on a Smooth Curve with  
a Singular Model in a Quadric Surface  

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Abstract: Let $C$ be the normalization of an integral curve of type $(a, a')$ on $\mathbb{P}^1 \times \mathbb{P}^1$. We give conditions on $\text{Sing}(Y)$ and $y$ for the non-existence of a pencil on $C$ partially composed with the $g_1^a$ or the $g_1^{a'}$ obtained in $C$ from the projections $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$.

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1. Introduction

Let $Q \subset \mathbb{P}^3$ be a smooth quadric surface. Let $\pi: Q \to \mathbb{P}^1$ and $\pi': Q \to \mathbb{P}^1$ be the two projections. We have $\text{Pic}(Q) \cong \mathbb{N}^2$. We have $h^0(\mathcal{O}_Q(a, b)) = (a + 1)(b + 1)$ and $h^1(Q, \mathcal{O}_Q(a, b)) = 0$ if $a \geq -1$ and $b \geq -1$. A curve $|\mathcal{O}_Q(a, b)|$ is said to have type $(a, b)$. The lines of $Q$ are the curve with either type $(1, 0)$ or type $(0, 1)$. We use the convention that the fibers of $\pi$ have type $(0, 1)$, while the fibers of $\pi'$ have type $(1, 0)$. Fix an integral $Y \in |\mathcal{O}_Q(a, a')|$. Let $w : C \to Y$ be the normalization map. Let $m : C \to \mathbb{P}^1$ (resp. $m' : C \to \mathbb{P}^1$) be the composition of $w$ with $\pi|Y$ (resp. $\pi'|Y$). We have $\deg(m) = a$ and $\deg(m') = a'$. Set $R := m^*(_{\mathbb{P}^1}\mathcal{O}_1(1))$.
and \( R' := m'^*(\mathcal{O}_{\mathbb{P}^1}(1)) \). \( R \) and \( R' \) are spanned line bundles of degree \( a \) and \( a' \), respectively. Under mild assumptions on \( Y \) we have \( h^0(R) = h^0(R') = 2 \) (see Remark 1). In this note we consider the following classical question. Under which assumptions on \( Y \) there is no spanned \( M \in \text{Pic}(C) \), \( M \neq R \) and \( M \neq R' \), such that \( h^0(C,M) = 2 \) and the morphism \( \phi : C \to \mathbb{P}^1 \) induced by \(|M|\) is composed either with \( m \) or with \( m' \), i.e. one of the two maps \((m,\phi)\) or \((m',\phi)\) from \( C \) into \( \mathbb{P}^1 \times \mathbb{P}^1 \) is not birational onto its image? Assume that \( M \) exists and let \( D \) be the normalization of either \((m,\phi)\) or \((m',\phi)\). Set \( y := \deg(M) \).

Theorem 1. Assume the existence of a spanned \( M \in \text{Pic}^y(C) \), \( M \neq R \) (resp. \( M \neq R' \)) such that \( \phi \) is composed with \( m \) (resp. \( m' \)), i.e. the map \((m,\phi)\) (resp. \((m',\phi)\)) has degree \( b \geq 2 \) onto its image. We have \( b|y \) and \( b < y \). We have \( b|a \) and \( b < a \) (resp. \( b|a' \) and \( b < a' \)). Since \( M \notin \{R,R'\} \), \( C \) has genus \( g \geq 2 \) and \( D \) has positive genus. In this note we prove the following result.

1. \( h^1(Q,\mathcal{J}(a - 2, a' - 2 - y/b)) = 0 \) (resp. \( h^1(Q,\mathcal{J}(a - 2 - y/b, a' - 2)) = 0 \));

2. \( Y \) has only ordinary nodes and ordinary cusps as singularities, \( \text{Sing}(Y) \) is formed by general points of \( Q \) and \( \sharp(\text{Sing}(Y)) \leq (a - 1)(a' - 1 - y/b) \) (resp. \( \sharp(\text{Sing}(Y)) \leq (a - 1 - y/b)(a' - 1) \));

3. assume \( a' \geq 2 + y/b \) (resp. \( a \geq 2 + y/b \)); set \( v := \max\{a - 2, a' - 2 - y/b\} \) (resp. \( v := \max\{a - 2 - y/b, a' - 2\} \) and \( u := \min\{a - 2 - y/b, a' - 2\} \)). Set \( \alpha := [u/3] \). \( Y \) has only ordinary nodes and ordinary cusps as singularities, no two of the points of \( \text{Sing}(Y) \) are contained in a line of \( Q \), at most \( u + v \) of the points of \( Z \) are contained in a curve of type \((1,1)\) and at most \( 3u + 1 \) of the points of \( Z \) are contained in a curve of type \((2,1)\) or \((1,2)\) and \( \sharp(\text{Sing}(Y)) \leq v - u + 10\alpha - 1 \);

4. \( Y \) has only ordinary nodes and ordinary cusps as singularities, \( a' \geq 2 + y/b \) (resp. \( a \geq 2 + y/b \)) and \( \sharp(\text{Sing}(Y)) \leq \min\{a - 1, a' - 1 - y/b\} \) ((resp. \( \sharp(\text{Sing}(Y)) \leq \min\{a - 1 - y/b, a' - 1\} \)).

We work over an algebraically closed field \( \mathbb{K} \).
2. The Proof

Let $Z \subset Q$ be a zero-dimensional scheme. Let $\Delta_Z$ be the union of all lines $L \subset Q$ such that $L \cap Z = \emptyset$. Notice that $\Delta_Z$ is a finite union of lines. This is a fundamental difference between $Q$ and $\mathbb{P}^2$.

**Lemma 1.** Fix $(x, v) \in \mathbb{N}^2$ and the ideal sheaf $\mathcal{J}$ of a zero-dimensional scheme $Z$ such that $h^1(\mathcal{J}(u, v)) = 0$. Fix a set $B \subset Q$ such that $B \cap \Delta_Z = \emptyset$ and there is an integer $b > 0$ such that for each $I \in |\mathcal{O}_Q(0, 1)|$ either $I \cap B = \emptyset$ or $\mathfrak{g}(I \cap B) = b$. Set $y := \mathfrak{g}(B)$. Assume $b \leq u + 1$. Then $Z \cap B = \emptyset$, $b|y$ and $h^1(\mathcal{I}_{\mathcal{J} \cup B}(u, v + y/b)) = 0$.

**Proof.** Since $Z \subseteq \Delta_Z$ and $B \cap \Delta_Z = \emptyset$, we have $B \cap Z = \emptyset$. By assumption there is a curve $F \in |\mathcal{O}_Q(0, y/b)|$, $F$ union of $y/b$ distinct lines such that $B \subset F$ and $\mathfrak{g}(F \cap I) = b$ for each connected component $I$ of $F$. Since $B \cap \Delta_Z = \emptyset$, we have $Z \cap F = \emptyset$. Hence deg $((Z \cup B) \cap I) = b$ for each component $I$ of $F$. Since $b \leq u + 1$, we get $h^1(F, \mathcal{I}_{Z \cup B}(u, v + y/b)) = 0$. Since $Z = (Z \cup B) \setminus (Z \cup B) \cap F$.

Use the exact sequence

$$0 \rightarrow \mathcal{I}_Z(u, v) \rightarrow \mathcal{I}_{Z \cup B}(u, v + y/b) \rightarrow \mathcal{I}_{(Z \cup B) \cap F, F}(u, v + y/b) \rightarrow 0$$

and the assumption $h^1(\mathcal{J}(u, v)) = 0$. $\square$

**Remark 1.** The adjunction formula gives that $h^0(R) = 2$ (resp. $h^0(R') = 2$) if and only if $h^1(\mathcal{J}(a - 2, a' - 3)) = 0$ (resp. $h^1(\mathcal{J}(a - 3, a' - 2)) = 0$). Just the existence of a reduced curve $Y \in |\mathcal{O}_Q(a, a')|$ implies $h^1(\mathcal{J}(a - 2, a' - 2)) = 0$.

**Proof of Theorem 1.** We first assume that $|M|$ is composed with the $g^1_a$ on $C$ induced by $\pi$. Fix a general $A \in |M|$ and set $B := w(A) \subset Y$. Since $|M|$ is a complete linear system, even in positive characteristic we see that $\phi$ is not composed with a Frobenius. Hence $\phi$ is separable. Since $\phi$ is separable and $A$ is general in $|M|$, $A$ is formed by $y$ distinct points. Since $|M|$ is spanned and $A$ is general, we have $A \cap w^{-1}(\text{Sing}(Y)) = \emptyset$. Hence $B \subset Y$, $B \cap Z = \emptyset$ and $\mathfrak{g}(B) = y$. Since $|M|$ has no base point, $\Delta_Z$ is a finite union of lines and $A$ is general, we have $B \cap \Delta_Z = \emptyset$. Fix $O \in A$. Since $M$ has no base points, we have $h^0(C, \mathcal{O}_C(A \setminus \{O\})) = h^0(C, \mathcal{O}_C(A)) - 1$, i.e. $h^0(C, \omega_C(-A)) = h^0(C, \omega_C(-A))(O)$ (Riemann-Roch and Serre duality). The adjunction formula gives $\omega_Y \cong \mathcal{O}_Y(a - 2, a' - 2)$. Since $h^i(\mathcal{O}_Q(-2, -2)) = 0$, $i = 0, 1$, we get that $|\omega_C|$ is induced by the linear system $|\mathcal{J}(a - 2, a' - 2)|$ on $Q$. Since $A \cap w^{-1}(\text{Sing}(Y)) = \emptyset$, we get $h^1(C, \omega_C(-A)) = h^1(\mathcal{I}_{Z \cup B}(a - 2, a' - 2))$. Hence $h^1(\mathcal{I}_{Z \cup B}(a - 2, a' - 2)) > 0$ since $h^1(\mathcal{I}_{Z \cup B}(a - 2, a' - 2)) > 0$, lemma 1 gives a contradiction.
Now assume that $|R|$ is composed with the $g_a^1$, induced by $\pi'$. We conclude as above taking $a'$ instead of $a$.

Hence we proved Theorem 1 under the assumption (1). We only need to prove that in the remaining cases the assumption of (1) is satisfied. If $Y$ has only ordinary nodes and ordinary cusps, then $Z$ is the set Sing($Y$) with its reduced structure. In case (2) uses the definition of “general set”. In case (3) use [1], lemma 2. The proof that (4) implies (1) is an easy exercise.

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References