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GENERALIZED INEQUALITIES FOR CONVEX MAPPINGS ON THE CO-ORDINATES

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Abstract: In this article, some another generalized Hermite-Hadamard type integral inequalities for s-convex mappings on the co-ordinates on the rectangles from the plain are obtained.

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1. Introduction

The following definition is well-known in the literature:

Definition 1. A mapping $f : I \subset R^+ \to R$ is said to be s-convex in the second sense on an interval I in R if the following inequality

$$f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y)$$

holds for all $x, y \in I$ and $\lambda \in [0,1]$, and for some fixed $s \in (0,1]$.

The inequality is in reversed order if f is an s-concave mapping in the second sense on an interval I in R.

Obviously one can see that if we choose $s = 1$, the above definition reduces to ordinary concept of convexity.

Many important inequalities have been established for the class of s-convex

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mapping in the second sense but the most famous is the Hermite-Hadamard inequality. In [8], Dragomir and Fitzpatrick established the following double inequality which was stated as:

Theorem 1.1. *Let $f : I \subset [0, b^*] \to R$ be an s-convex mapping in the second sense defined on an interval I in R, where $s \in (0,1)$ and $a, b \in I$ with $a < b$ and $b^* > 0$. Then the following inequality holds:*

$$2^{s-1} f(\frac{a+b}{2}) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{s+1}. \tag{1}$$

The inequalities are in reversed order if f is an s-concave mapping in the second sense on an interval I in R.

The inequalities (1) have become an important cornerstone in mathematical analysis and optimization and many uses of these inequalities have been discovered in variety of settings. Due to the rich geometrical significance of Hermite-Hadamard's inequality, there is growing literature providing its new proofs, extensions, refinements and generalizations, see for example [5, 7, 9] and the references therein.

Definition 2. Consider the bidimensional interval $\Delta = [a, b] \times [c, d]$ inR^2 with $a < b$ and $c < d$. The mapping $f : \Delta \to R$ is convex on Δ if the following inequality

$$f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \leq \lambda f(x,y) + (1-\lambda) f(z,w).$$

holds, for all $(x, y), (z, w) \in \Delta$ with $\lambda \in [0, 1]$.

Definition 3. Consider the bidimensional interval $\Delta = [a, b] \times [c, d]$ in R^2 with $a < b$ and $c < d$. A mapping $f : \Delta \to R$ is said to be co-ordinated convex on Δ if the partial mappings $f_y : u \in [a, b] \to f_y(u, y) \in R$ and $f_x : v \in [c, d] \to f_x(x, v) \in R$ are convex, for all $x \in (a, b)$ and $y \in (c, d)$.

In [7, 11], Latif and Dragomir proved the following Hermite-Hadamard type inequality for co-ordinated convex mappings on the rectangle from the plane:

Theorem 1.2. *If $f : \Delta \subseteq R^2 \to R$ is a co-ordinated convex partial differentiable mapping on a bidimensional interval $\Delta = [a, b] \times [c, d]$ with $a < b$ and $c < d$, then the following inequalities hold:*

$$\begin{aligned} & f(\frac{a+b}{2}, \frac{c+d}{2}) \\ \leq & \frac{1}{2}\Big[\frac{1}{b-a}\int_a^b f(x, \frac{c+d}{2})dx + \frac{1}{d-c}\int_c^d f(\frac{a+b}{2}, y)dy\Big] \end{aligned}$$

$$\leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)dydx$$
$$\leq \frac{1}{4}\Big[\frac{1}{b-a}\int_a^b f(x,c)dx + \frac{1}{b-a}\int_a^b f(x,d)dx$$
$$+ \frac{1}{d-c}\int_c^d f(a,y)dx + \frac{1}{d-c}\int_c^d f(b,y)dy\Big]$$
$$\leq \frac{1}{4}\big\{f(a,c)+f(a,d)+f(b,c)+f(b,d)\big\}.$$

Theorem 1.3. *Let $f : \Delta = [a,b]\times[c,d] \to R$ be convex on the co-ordinates on Δ. Then one has the inequalities:*

$$f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)dydx$$
$$\leq \frac{1}{4}\left[f(a,c)+f(b,c)+f(a,d)+f(b,d)\right]. \tag{2}$$

In [14], Özdemir and et. al established the following inequalities:

Theorem 1.4. *Let $f : \Delta = [a,b] \times [c,d] \subseteq R^2 \to R$ be a partial differentiable mapping on Δ with $a < b$, $c < d$ and $r, t \in [0,1]$. If $\left|\frac{\partial^2 f}{\partial t \partial r}\right|$ is convex on the co-ordinates on Δ, then the following inequality holds:*

$$\Big|\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)dydx + f(\frac{a+b}{2}, \frac{c+d}{2})$$
$$- \frac{1}{d-c}\int_c^d f(\frac{a+b}{2}, y)dy - \frac{1}{b-a}\int_a^b f(x, \frac{c+d}{2})dx\Big| (\equiv^{put} \Big|C\Big|)$$
$$\leq \frac{(b-a)(d-c)}{64}$$
$$\times \Big\{\Big|\frac{\partial^2 f(a,c)}{\partial t \partial r}\Big| + \Big|\frac{\partial^2 f(a,d)}{\partial t \partial r}\Big| + \Big|\frac{\partial^2 f(b,c)}{\partial t \partial r}\Big| + \Big|\frac{\partial^2 f(b,d)}{\partial t \partial r}\Big|\Big\}.$$

Theorem 1.5. *Let $f : \Delta = [a,b] \times [c,d] \subseteq R^2 \to R$ be a partial differentiable mapping on Δ with $a < b$, $c < d$ and $r, t \in [0,1]$. If $\left|\frac{\partial^2 f}{\partial t \partial r}\right|^q$ is convex on the co-ordinates on Δ, for $q \geq 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality holds:*

$$\Big|C\Big| \leq \frac{(b-a)(d-c)}{4^{2+\frac{1}{q}}}$$
$$\times \Big\{\Big|\frac{\partial^2 f}{\partial t \partial r}(a,c)\Big|^q + \Big|\frac{\partial^2 f}{\partial t \partial r}(a,d)\Big|^q + \Big|\frac{\partial^2 f}{\partial t \partial r}(b,c)\Big|^q + \Big|\frac{\partial^2 f}{\partial t \partial r}(b,d)\Big|^q\Big\}^{\frac{1}{q}},$$

where C is defined as in Theorem 1.3.

Theorem 1.6. *Let $f : \Delta = [a, b] \times [c, d] \subseteq R^2 \to R$ be a partial differentiable mapping on Δ with $a < b$, $c < d$ and $r, t \in [0, 1]$. If $\left|\frac{\partial^2 f}{\partial t \partial r}\right|^q$ is convex on the co-ordinates on Δ, for $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality holds:*

$$\left|C\right| \leq \frac{(b-a)(d-c)}{4^{1+\frac{1}{q}}(p+1)^{\frac{2}{p}}} \times \left\{\left|\frac{\partial^2 f}{\partial t \partial r}(a,c)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial r}(a,d)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial r}(b,c)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial r}(b,d)\right|^q\right\}^{\frac{1}{q}},$$

where C is defined as in Theorem 1.3.

Definition 4. Consider the bidimensional interval $\Delta = [a, b] \times [c, d]$ in $[0, \infty)^2$ with $a < b$ and $c < d$. The mapping $f : \Delta \to R$ is s-convex on Δ if the following inequality

$$f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \leq \lambda^s f(x, y) + (1-\lambda)^s f(z, w)$$

holds, for all $(x, y), (z, w) \in \Delta$ and $\lambda \in [0, 1]$ with some fixed $s \in (0, 1]$.

Definition 5. Let us consider the bidimensional interval $\Delta = [a, b] \times [c, d]$ in R^2 with $a < b$ and $c < d$. A mapping $f : \Delta \to R$ is said to be co-ordinated s-convex in the second sense if the partial mappings $f_y : u \in [a, b] \to f_y(u, y) \in R$ and $f_x : v \in [c, d] \to f_x(x, v) \in R$ are s-convex in the second sense, for all $x \in (a, b)$, $y \in (c, d)$ and $s \in (0, 1]$.

A formal definition for co-ordinated s-convex mappings in the second sense may be stated as follows;

Definition 6. A mapping $f : \Delta \to R$ will be called co-ordinated s-convex in the second sense on a bidimensional interval Δ if the following inequality

$$\begin{aligned} &f(tx + (1-t)z, ry + (1-r)w) \\ &\leq t^s r^s f(x, y) + (1-t)^s r^s f(z, y) \\ &\quad + t^s (1-r)^s f(x, w) + (1-t)^s (1-r)^s f(z, w) \end{aligned} \tag{3}$$

holds for all $r, t \in [0, 1]$ and $(x, y), (z, w) \in \Delta$ with some fixed $s \in (0, 1]$.

Obviously one can see that if we choose $s = 1$, the above formal definition reduces to ordinary concept of co-ordinary s-convexity.

In [7, 11], Latif and Dragomir proved the Hermite-Hadamard type inequality for co-ordinated convex mappings in the second sense on the rectangle from the plane.

In [1, 2, 3, 4, 5], Alomari and Darus proved the following Hermite-Hadamard type inequality for co-ordinated s-convex mappings in the second sense on the rectangle from the plane:

Theorem 1.7. *If $f : \Delta \subseteq R^2 \to R$ is a co-ordinated s-convex partial differentiable mapping in the second sense on the co-ordinates on a bidimensional interval $\Delta = [a,b] \times [c,d]$ with $a < b$ and $c < d$, then the following inequalities*

$$\begin{aligned}
&4^{s-1} f(\frac{a+b}{2}, \frac{c+d}{2}) \\
\le & 2^{s-2}\Big[\frac{1}{b-a}\int_a^b f(x, \frac{c+d}{2})dx + \frac{1}{d-c}\int_c^d f(\frac{a+b}{2}, y)dy\Big] \\
\le & \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx \\
\le & \frac{1}{2(s+1)}\Big[\frac{1}{b-a}\int_a^b f(x,c)dx + \frac{1}{b-a}\int_a^b f(x,d)dx \\
&\quad + \frac{1}{d-c}\int_c^d f(a,y)dx + \frac{1}{d-c}\int_c^d f(b,y)dy\Big] \\
\le & \frac{\big\{f(a,c)+f(a,d)+f(b,c)+f(b,d)\big\}}{(s+1)^2},
\end{aligned}$$

hold for $x \in [a,b], y \in [c,d]$ and for some fixed $s \in (0,1]$.

For further Hermite-Hadamard-type and Ostrowski-type inequalities on co-ordinated convex mappings, co-ordinated s-convex mappings, (α, m)-convex mapping, (s,r)-convex mappings and h-convex mappings, see [1, 2, 3, 4, 5, 6, 7, 10, 11, 14, 15, 16, 17, 19].

The following lemmas are necessary and play important roles in establishing our main results[12, 18]:

Lemma 1. *Let $f : \Delta = [a,b] \times [c,d] \subseteq R^2 \to R$ be a partial differentiable mapping on Δ with with $a < b$, $c < d$ and $r, t \in [0,1]$. If $\frac{\partial^2 f}{\partial t \partial r} \in L(\Delta)$, then the following identity*

$$\begin{aligned}
&I(a,b,x;c,d,y;\lambda) \\
&\equiv^{put} \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,r)dtdr \\
&+ \frac{\lambda^2}{4}\Big\{f(a,c)+f(a,d)+f(b,c)+f(b,d)\Big\} \\
&+ \frac{(1-\lambda)^2}{4}\Big\{f(x,y)+f(x,c+d-y)
\end{aligned}$$

$$
\begin{aligned}
&\quad + f(a+b-x,c+d-y) + f(a+b-x,y)\Big\} \\
&+ \frac{\lambda(1-\lambda)}{4}\Big\{f(a,y) + f(x,c) + f(x,d) + f(b,y) \\
&\quad + f(a,c+d-y) + f(a+b-x,c) \\
&\quad + f(a+b-x,d) + f(b,c+d-y)\Big\} \\
&- \frac{1-\lambda}{2}\Big\{\frac{1}{d-c}\int_c^d \{f(x,r) + f(a+b-x,r)\}dr \\
&\quad + \frac{1}{b-a}\int_a^b \{f(t,c+d-y) + f(t,y)\}dt\Big\} \\
&- \frac{\lambda}{2}\Big\{\frac{1}{d-c}\int_c^d \{f(a,r) + f(b,r)\}dr \\
&\quad + \frac{1}{b-a}\int_a^b \{f(t,c) + f(t,d)\}dt\Big\} \\
&= \frac{1}{(b-a)(d-c)}\int_c^d\int_a^b P(x,t)Q(y,r)\frac{\partial^2 f(t,r)}{\partial t\partial r}dtdr \qquad (4)
\end{aligned}
$$

holds, for all $\lambda \in [0,1]$*, where* $P(\cdot,\cdot) : [a,b] \times [a,b] \to R$ *defined by*

$$
P(x,t) = \begin{cases} t - \left(a + \lambda\frac{b-a}{2}\right) & t \in [a,x] \\ t - \frac{a+b}{2} & t \in (x, a+b-x] \\ t - \left(b - \lambda\frac{b-a}{2}\right) & t \in (a+b-x, b], \end{cases}
$$

and, $Q(\cdot,\cdot) : [c,d] \times [c,d] \to R$ *defined by*

$$
Q(y,r) = \begin{cases} r - \left(c + \lambda\frac{d-c}{2}\right) & r \in [c,y] \\ r - \frac{c+d}{2} & r \in (y, c+d-y] \\ r - \left(d - \lambda\frac{d-c}{2}\right) & r \in (c+d-y, d], \end{cases}
$$

for all $x \in [a + \lambda\frac{b-a}{2}, \frac{a+b}{2}]$ *and* $y \in [c + \lambda\frac{d-c}{2}, \frac{c+d}{2}]$.

In this article, motivated by notion given in Theorem 1.3, Theorem 1.4, Theorem 1.5 and Liu and Park [12], the author give some new lemma which give another generalized results, and using this lemma, some new inequalities that give estimate of the difference between the middle and the rightmost terms in Theorem 1.1 for differentiable co-ordinated convex and s-convex mappings on rectangle from the plain R^2 are obtained.

2. Inequalities for Co-Ordinated Convexity

Theorem 2.1. *Let $f : \Delta = [a,b] \times [c,d] \to R$ be a partial differentiable mapping on a bidimensional interval Δ in R^2 with $a < b$, $c < d$ and $\frac{\partial^2 f}{\partial t \partial r} \in L(\Delta)$ for $t, r \in [0,1]$. If $\left|\frac{\partial^2 f}{\partial t \partial r}\right|$ is a convex mapping on the co-ordinates on Δ, then the following inequality*

$$\begin{aligned}
&\Big|I(a,b,x;c,d,y;\lambda)\Big| \\
\leq &\frac{R(a,b;x;\lambda)R(c,d;y;\lambda)}{(b-a)^2(d-c)^2} \\
&\times \Big\{\Big|\frac{\partial^2 f}{\partial t \partial r}(a,c)\Big| + \Big|\frac{\partial^2 f}{\partial t \partial r}(a,d)\Big| + \Big|\frac{\partial^2 f}{\partial t \partial r}(b,c)\Big| + \Big|\frac{\partial^2 f}{\partial t \partial r}(b,d)\Big|\Big\}
\end{aligned} \tag{5}$$

holds, for all $\lambda \in [0,1]$ and $x \in [a + \lambda\frac{b-a}{2}, \frac{a+b}{2}]$, $y \in [c + \lambda\frac{d-c}{2}, \frac{c+d}{2}]$, where

$$\begin{aligned}
&R(\alpha,\beta;\omega;\lambda) \\
=&\frac{\beta-\alpha}{8}\Big[(1+2\lambda^2)\beta^2 + (5-4\lambda+2\lambda^2)\alpha^2 - 4(1+\lambda)\beta\omega \\
&+ 8\omega^2 + 2\alpha\{(1+2\lambda-2\lambda^2)\beta + 2(\lambda-3)\omega\}\Big].
\end{aligned}$$

Proof. From Lemma 1(4), we can write

$$\Big|I(a,b,x;c,d,y;\lambda)\Big|(b-a)(d-c) \leq \int_a^b \int_c^d \Big|P(x,t)Q(y,r)\Big|\Big|\frac{\partial^2 f(t,r)}{\partial t \partial r}\Big| dr dt, \tag{6}$$

Since $\left|\frac{\partial^2 f}{\partial t \partial r}\right|$ is convex on the co-ordinates on Δ, we get

$$\begin{aligned}
\Big|\frac{\partial^2}{\partial t \partial r} f(t,r)\Big| \leq &\frac{(b-t)(d-r)}{(b-a)(d-c)}\Big|\frac{\partial^2 f(a,c)}{\partial t \partial r}\Big| + \frac{(b-t)(r-c)}{(b-a)(d-c)}\Big|\frac{\partial^2 f(a,d)}{\partial t \partial r}\Big| \\
&+ \frac{(t-a)(d-r)}{(b-a)(d-c)}\Big|\frac{\partial^2 f(b,c)}{\partial t \partial r}\Big| + \frac{(t-a)(r-c)}{(b-a)(d-c)}\Big|\frac{\partial^2 f(b,d)}{\partial t \partial r}\Big|.
\end{aligned} \tag{7}$$

By substituting (7) in (6), we have

$$\begin{aligned}
&\Big|I(a,b,x;c,d,y;\lambda)\Big|(b-a)(d-c) \\
\leq &\frac{1}{(b-a)^2(d-c)^2} \\
&\times \Big[\Big\{\int_a^b \Big|P(x,t)\Big|(b-t)dt\Big\}\Big\{\int_c^d \Big|Q(y,r)\Big|(d-r)dr\Big\}\Big|\frac{\partial^2 f(a,c)}{\partial t \partial r}\Big|
\end{aligned}$$

$$+\Big\{\int_a^b \Big|P(x,t)\Big|(b-t)dt\Big\}\Big\{\int_c^d \Big|Q(y,r)\Big|(r-c)dr\Big\}\Big|\frac{\partial^2 f(a,d)}{\partial t\partial r}\Big|$$
$$+\Big\{\int_a^b \Big|P(x,t)\Big|(t-a)dt\Big\}\Big\{\int_c^d \Big|Q(y,r)\Big|(d-r)dr\Big\}\Big|\frac{\partial^2 f(b,c)}{\partial t\partial r}\Big|$$
$$+\Big\{\int_a^b \Big|P(x,t)\Big|(t-a)dt\Big\}\Big\{\int_c^d \Big|Q(y,r)\Big|(r-c)dr\Big\}\Big|\frac{\partial^2 f(b,d)}{\partial t\partial r}\Big|\Big]$$

$$=\frac{R(a,b;x;\lambda)R(c,d;y;\lambda)}{(b-a)^2(d-c)^2}$$
$$\times\Big\{\Big|\frac{\partial^2 f(a,c)}{\partial t\partial r}\Big|+\Big|\frac{\partial^2 f(a,d)}{\partial t\partial r}\Big|+\Big|\frac{\partial^2 f(b,c)}{\partial t\partial r}\Big|+\Big|\frac{\partial^2 f(b,d)}{\partial t\partial r}\Big|\Big\},$$

where we have used the facts that

$$\int_a^b \big|P(x,t)\big|(b-t)dt=\int_a^b \big|P(x,t)\big|(t-a)dt=R(a,b;x;\lambda),$$
$$\int_c^d \big|Q(y,r)\big|(d-r)dr=\int_c^d \big|Q(y,r)\big|(r-c)dr=R(c,d;y;\lambda).$$

Corollary 2.1. *If we choose $x=\frac{a+b}{2}, y=\frac{c+d}{2}$ and $\lambda=0$ in Theorem 2.1, we obtain Theorem 1.4.*

Theorem 2.2. *Let $f:\Delta=[a,b]\times[c,d]\to R$ be a partial differentiable mapping on a bidimensional interval Δ in R^2 with $a<b$, $c<d$ and $\frac{\partial^2 f}{\partial t\partial r}\in L(\Delta)$ for $t,r\in[0,1]$. If $\left|\frac{\partial^2 f}{\partial t\partial r}\right|^q$ is a convex mapping on the co-ordinates on Δ for $q\geq 1$ with $\frac{1}{p}+\frac{1}{q}=1$, then the following inequality*

$$\Big|I(a,b,x;c,d,y;\lambda)\Big|$$
$$\leq\Big\{R_1(a,b;x;\lambda)R_1(c,d;y;\lambda)\Big\}^{\frac{1}{p}}\Big\{\frac{R^{\frac{1}{q}}(a,b;x;\lambda)R^{\frac{1}{q}}(c,d;y;\lambda)}{(b-a)^{1+\frac{1}{q}}(d-c)^{1+\frac{1}{q}}}\Big\} \tag{8}$$
$$\times\Big\{\Big|\frac{\partial^2 f(a,c)}{\partial t\partial r}\Big|^q+\Big|\frac{\partial^2 f(a,d)}{\partial t\partial r}\Big|^q+\Big|\frac{\partial^2 f(b,c)}{\partial t\partial r}\Big|^q+\Big|\frac{\partial^2 f(b,d)}{\partial t\partial r}\Big|^q\Big\}^{\frac{1}{q}},$$

holds, for all $\lambda\in[0,1]$ and $x\in[a+\lambda\frac{b-a}{2},\frac{a+b}{2}], y\in[c+\lambda\frac{d-c}{2},\frac{c+d}{2}]$, where

$$R_1(\alpha,\beta;\omega;\lambda)$$
$$=\frac{1}{4}\Big\{(1+2\lambda^2)\beta^2+(5-4\lambda+2\lambda^2)\alpha^2+2\alpha\beta(1+2\lambda-2\lambda^2)$$

$$+8\omega^2-4\beta(1+\lambda)\omega+4\alpha(\lambda-3)\omega\Big\},$$

and $R(\alpha,\beta;\omega;\lambda)$ is defined as in Theorem 2.1.

Proof. From Lemma 1(4) and applying the well-known power mean inequality for double integrals, we can write

$$\begin{aligned}
&\Big|I(a,b,x;c,d,y;\lambda)\Big|(b-a)(d-c)\\
\le&\int_a^b\int_c^d\Big|P(x,t)Q(y,r)\Big|\Big|\frac{\partial^2 f(t,r)}{\partial t\partial r}\Big|drdt\\
\le&\Big\{\int_a^b\Big|P(x,t)\Big|dt\Big\}^{\frac{1}{p}}\Big\{\int_c^d\Big|Q(y,r)\Big|dr\Big\}^{\frac{1}{p}}\\
&\times\Big\{\int_a^b\int_c^d\Big|P(x,t)Q(y,r)\Big|\Big|\frac{\partial^2 f(t,r)}{\partial t\partial r}\Big|^q drdt\Big\}^{\frac{1}{q}}.
\end{aligned}\tag{9}$$

Since $\left|\frac{\partial^2 f}{\partial t\partial r}\right|^q$ is convex on the co-ordinates on Δ, we get

$$\begin{aligned}
&\Big|\frac{\partial^2 f(t,r)}{\partial t\partial r}\Big|^q\\
\le&\frac{(b-t)(d-r)}{(b-a)(d-c)}\Big|\frac{\partial^2 f(a,c)}{\partial t\partial r}\Big|^q+\frac{(b-t)(r-c)}{(b-a)(d-c)}\Big|\frac{\partial^2 f(a,d)}{\partial t\partial r}\Big|^q\\
&+\frac{(t-a)(d-r)}{(b-a)(d-c)}\Big|\frac{\partial^2 f(b,c)}{\partial t\partial r}\Big|^q+\frac{(t-a)(r-c)}{(b-a)(d-c)}\Big|\frac{\partial^2 f(b,d)}{\partial t\partial r}\Big|^q.
\end{aligned}\tag{10}$$

Note that

$$\int_a^b\Big|P(x,t)\Big|dt=R_1(a,b;x;\lambda),\quad \int_c^d\Big|Q(y,r)\Big|ds=R_1(c,d;y;\lambda).\tag{11}$$

By the inequalities (10) and (11), also we have

$$\begin{aligned}
&\int_a^b\int_c^d\Big|P(x,t)Q(y,r)\Big|\Big|\frac{\partial^2 f(t,r)}{\partial t\partial r}\Big|^q drdt\\
\le&\frac{R(a,b;x;\lambda)R(c,d;y;\lambda)}{(b-a)(d-c)}\\
&\times\Big\{\Big|\frac{\partial^2 f(a,c)}{\partial t\partial r}\Big|^q+\Big|\frac{\partial^2 f(a,d)}{\partial t\partial r}\Big|^q+\Big|\frac{\partial^2 f(b,c)}{\partial t\partial r}\Big|^q+\Big|\frac{\partial^2 f(b,d)}{\partial t\partial r}\Big|^q\Big\}.
\end{aligned}\tag{12}$$

By substituting (11) and (12) in (9), we get the desired result (8).

Corollary 2.2. *If we choose $x = \frac{a+b}{2}, y = \frac{c+d}{2}$ and $\lambda = 0$ in Theorem 2.2, we obtain Theorem 1.5.*

Theorem 2.3. *Let $f : \Delta = [a,b] \times [c,d] \to R$ be a partial differentiable mapping on a bidimensional interval Δ in R^2 with $a < b, c < d$ and $\frac{\partial^2 f}{\partial t \partial r} \in L(\Delta)$ for $t, s \in [0,1]$. If $\left|\frac{\partial^2 f}{\partial t \partial r}\right|^q$ is a convex mapping on the co-ordinates on Δ, for $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality*

$$\begin{aligned}
&\Big|I(a,b,x;c,d,y;\lambda)\Big| \\
\le&\Big\{\frac{R_2(a,b;x;\lambda)R_2(a,b;x;\lambda)}{(b-a)(d-c)}\Big\}^{\frac{1}{p}}\Big\{\frac{1}{4^{\frac{1}{q}}}\Big\} \\
&\times\left\{\Big|\frac{\partial^2 f(a,c)}{\partial t \partial r}\Big|^q + \Big|\frac{\partial^2 f(a,d)}{\partial t \partial r}\Big|^q + \Big|\frac{\partial^2 f(b,c)}{\partial t \partial r}\Big|^q + \Big|\frac{\partial^2 f(b,d)}{\partial t \partial r}\Big|^q\right\}^{\frac{1}{q}}
\end{aligned} \tag{13}$$

holds, for all $\lambda \in [0,1]$ and $x \in [a + \lambda\frac{b-a}{2}, \frac{a+b}{2}], y \in [c + \lambda\frac{d-c}{2}, \frac{c+d}{2}]$, where

$$\begin{aligned}
R_2(\alpha,\beta;\omega;\lambda) =&\frac{2^{-p}}{1+p}\Big[(\beta-\alpha)^{1+p}\lambda^{1+p} + (\alpha+\beta-2\omega)^{1+p} \\
&+ \big\{2(\omega-\alpha) - (\beta-\alpha)\lambda\big\}^{1+p}\Big],
\end{aligned}$$

and $R(\alpha,\beta;\omega;\lambda)$ is defined as in Theorem 2.1.

Proof. From Lemma 1(4) and applying the well-known power mean inequality for double integrals, we can write

$$\begin{aligned}
&\Big|I(a,b,x;c,d,y;\lambda)\Big|(b-a)(d-c) \\
\le&\int_a^b\int_c^d \Big|P(x,t)Q(y,r)\Big|\Big|\frac{\partial^2 f(t,r)}{\partial t \partial r}\Big|drdt \\
\le&\Big\{\int_a^b \Big|P(x,t)\Big|^p dt \int_c^d \Big|Q(y,r)\Big|^p dr\Big\}^{\frac{1}{p}}\Big\{\int_a^b\int_c^d \Big|\frac{\partial^2 f(t,r)}{\partial t \partial r}\Big|^q drdt\Big\}^{\frac{1}{q}}.
\end{aligned} \tag{14}$$

Since $\left|\frac{\partial^2 f}{\partial t \partial r}\right|^q$ is convex on the co-ordinates on Δ, by the inequality (10) we get

$$\begin{aligned}
&\int_a^b\int_c^d \Big|\frac{\partial^2 f(t,r)}{\partial t \partial r}\Big|^q drdt\Big\{\frac{4}{(b-a)(d-c)}\Big\} \\
\le&\Big\{\Big|\frac{\partial^2 f(a,c)}{\partial t \partial r}\Big|^q + \Big|\frac{\partial^2 f(a,d)}{\partial t \partial r}\Big|^q + \Big|\frac{\partial^2 f(b,c)}{\partial t \partial r}\Big|^q + \Big|\frac{\partial^2 f(b,d)}{\partial t \partial r}\Big|^q\Big\}.
\end{aligned} \tag{15}$$

Note that

$$\int_a^b \left|P(x,t)\right|^p dt = R_2(a,b;x;\lambda),$$
$$\int_c^d \left|Q(y,s)\right|^p ds = R_2(c,d;y;\lambda). \tag{16}$$

By substituting (15) and (16) in (14), we get the following inequalities:

$$\begin{aligned}
&\Big|I(a,b,x;c,d,y;\lambda)\Big|(b-a)(d-c)\\
\leq &\int_a^b\int_c^d \Big|P(x,t)Q(y,r)\Big|\Big|\frac{\partial^2 f(t,r)}{\partial t\partial r}\Big|drdt\\
\leq &\Big\{R_2(a,b;x;\lambda)R_2(a,b;x;\lambda)\Big\}^{\frac{1}{p}}\Big\{\frac{(b-a)(d-c)}{4}\Big\}^{\frac{1}{q}}\\
&\times\Big\{\Big|\frac{\partial^2 f(a,c)}{\partial t\partial r}\Big|^q+\Big|\frac{\partial^2 f(a,d)}{\partial t\partial r}\Big|^q+\Big|\frac{\partial^2 f(b,c)}{\partial t\partial r}\Big|^q+\Big|\frac{\partial^2 f(b,d)}{\partial t\partial r}\Big|^q\Big\}^{\frac{1}{q}}.
\end{aligned}$$

By the simple calculation, we get the desired result (13).

Corollary 2.3. *If we choose $x=\frac{a+b}{2}, y=\frac{c+d}{2}$ and $\lambda=0$ in Theorem 2.3, we obtain Theorem 1.6.*

3. Inequalities for Co-Ordinated s-Convexity

Theorem 3.1. *Let $f:\Delta=[a,b]\times[c,d]\subset[0,\infty)^2\to R$ be a partial differentiable mapping on a bidimensional interval Δ in R^2 with $a<b$, $c<d$ and $\frac{\partial^2 f}{\partial t\partial r}\in L(\Delta)$ for t,r in $[0,1]$. If $\left|\frac{\partial^2 f}{\partial t\partial r}\right|^q$, $q>1$, is an s-convex mapping on the co-ordinates on Δ, then the following inequality*

$$\begin{aligned}
&\Big|I(a,b,x;c,d,y;\lambda)\Big|(b-a)^{2-\frac{1}{q}}(d-c)^{2-\frac{1}{q}}\\
\leq &\frac{1}{(1+s)^{\frac{2}{q}}}\Big\{R_2(a,b;x;\lambda)R_2(a,b;x;\lambda)\Big\}^{\frac{1}{p}}\\
&\times\Big\{\Big|\frac{\partial^2 f(a,c)}{\partial t\partial r}\Big|^q+\Big|\frac{\partial^2 f(a,d)}{\partial t\partial r}\Big|^q\Big|\frac{\partial^2 f(b,c)}{\partial t\partial r}\Big|^q+\Big|\frac{\partial^2 f(b,d)}{\partial t\partial r}\Big|^q\Big\}^{\frac{1}{q}},
\end{aligned}\tag{17}$$

holds, for all $\lambda\in[0,1]$ and $x\in[a+\lambda\frac{b-a}{2},\frac{a+b}{2}]$, $y\in[c+\lambda\frac{d-c}{2},\frac{c+d}{2}]$, where $R_2(\alpha,\beta;\omega;\lambda)$ is defined as in Theorem 2.3.

Proof. From Lemma 1(4), we can write

$$
\begin{aligned}
&\Big|I(a,b,x;c,d,y;\lambda)\Big|(b-a)(d-c) \\
\le&\int_a^b\int_c^d\Big|P(x,t)Q(y,r)\Big|\Big|\frac{\partial^2 f(t,r)}{\partial t\partial r}\Big|drdt \\
\le&\Big\{\int_a^b\int_c^d\Big|P(x,t)Q(y,r)\Big|^p drdt\Big\}^{\frac{1}{p}}\Big\{\int_a^b\int_c^d\Big|\frac{\partial^2 f(t,r)}{\partial t\partial r}\Big|^q drdt\Big\}^{\frac{1}{q}}.
\end{aligned}
\tag{18}
$$

Since $\left|\frac{\partial^2 f}{\partial t\partial r}\right|^q$ is s-convex on the co-ordinates on Δ, we get

$$
\begin{aligned}
&\Big|\frac{\partial^2}{\partial t\partial r}f(t,r)\Big|^q \\
\le&\Big(\frac{b-t}{b-a}\Big)^s\Big(\frac{d-r}{d-c}\Big)^s\Big|\frac{\partial^2 f(a,c)}{\partial t\partial r}\Big|^q+\Big(\frac{b-t}{b-a}\Big)^s\Big(\frac{r-c}{d-c}\Big)^s\Big|\frac{\partial^2 f(a,d)}{\partial t\partial r}\Big|^q \\
&+\Big(\frac{t-a}{b-a}\Big)^s\Big(\frac{d-r}{d-c}\Big)^s\Big|\frac{\partial^2 f(b,c)}{\partial t\partial r}\Big|^q+\Big(\frac{t-a}{b-a}\Big)^s\Big(\frac{r-c}{d-c}\Big)^s\Big|\frac{\partial^2 f(b,d)}{\partial t\partial r}\Big|^q.
\end{aligned}
\tag{19}
$$

By using the simple calculations we have

$$
\int_a^b\int_c^d\Big|P(x,t)Q(y,r)\Big|^p drdt=R_2(a,b;x;\lambda)R_2(a,b;x;\lambda),
\tag{20}
$$

and, by using (19) also we get

$$
\begin{aligned}
&\int_a^b\int_c^d\Big|\frac{\partial^2 f(t,r)}{\partial t\partial r}\Big|^q drdt\Big\{(b-a)(d-c)\Big\}^{\frac{s}{q}} \\
\le&\Big[\Big\{\int_a^b(b-t)^s dt\Big\}\Big\{\int_c^d(d-r)^s dr\Big\}\Big|\frac{\partial^2 f(a,c)}{\partial t\partial r}\Big|^q \\
&+\Big\{\int_a^b(b-t)^s dt\Big\}\Big\{\int_c^d(r-c)^s dr\Big\}\Big|\frac{\partial^2 f(a,d)}{\partial t\partial r}\Big|^q \\
&+\Big\{\int_a^b(t-a)^s dt\Big\}\Big\{\int_c^d(d-r)^s dr\Big\}\Big|\frac{\partial^2 f(b,c)}{\partial t\partial r}\Big|^q \\
&+\Big\{\int_a^b(t-a)^s dt\Big\}\Big\{\int_c^d(r-c)^s dr\Big\}\Big|\frac{\partial^2 f(b,d)}{\partial t\partial r}\Big|^q\Big]^{\frac{1}{q}} \\
\le&\Big\{\frac{(b-a)^{1+s}(d-c)^{1+s}}{(1+s)^2}\Big\}\Big\{\Big|\frac{\partial^2 f(a,c)}{\partial t\partial r}\Big|^q \\
&+\Big|\frac{\partial^2 f(a,d)}{\partial t\partial r}\Big|^q+\Big|\frac{\partial^2 f(b,c)}{\partial t\partial r}\Big|^q+\Big|\frac{\partial^2 f(b,d)}{\partial t\partial r}\Big|^q\Big\}^{\frac{1}{q}},
\end{aligned}
\tag{21}
$$

where we have used the facts that

$$\int_a^b (b-t)^s dt = \int_a^b (t-a)^s dt = \frac{(b-a)^{1+s}}{1+s}$$

and

$$\int_c^d (d-r)^s dr = \int_c^d (r-c)^s dr = \frac{(d-c)^{1+s}}{1+s}.$$

By substituting (20) and (21) in (18), we get the desired result (17).

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