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GAUSS' METHOD OF CONSTRUCTING
A REGULAR HEPTADECAGON

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Abstract: This article explains how to construct a regular heptadecagon according to the theory of cyclotonic equations which was discovered by Gauss in 1796. The author also shows how to construct any root or fraction.

1. Introduction

We know about historically famous theories in mathematics, but there are many cases where we don't know the proofs. Fermat's last theorem, Galois' theorem, Gödel's incompleteness theorem..., once you start, there's nowhere to stop. Until recently I did not know the proof supporting Gauss' method for constructing a regular heptadecagon - a polygon with 17 sides. The construction method for an arbitrary regular n-sided polygon may be explained as follows, according to the 'Dictionary of Mathematics' (*Iwanami Shoten* Publishing). The neces-

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sary and sufficient conditions for a regular n-sided polygon to be possible to construct were established by Gauss. Given the prime factorization of n ,

$$n = 2^\lambda P_1 \cdots P_k \quad (\lambda \geq 0) \tag{1}$$

all of $P_1, \cdots, P_k$ must be distinct primes taking the form $2^h + 1$ (Fermat numbers). Inserting values for λ and h into this formula, the following values may be obtained for n.

$$n = 3, 4, 5, 6, 8, 10, 12, 15, 17, \cdots$$

When I reproduced the following quote from a certain document, "It is well-known that Gauss obtained a geometrical method for constructing a regular heptadecagon", I received the query, "By the way, how do you construct the regular heptadecagon?" I had copied verbatim from the dictionary, and didn't know the answer. The construction methods for regular triangles and regular pentagons were established by Euclid, the ancient Greek. However, considering whether we in our generation have completely mastered even the construction of regular pentagons, it is doubtful. For the regular pentagon, there are two construction methods. One begins by establishing a single edge, while the other contacts a circle internally. Neither requires a protractor, and both may be drawn using only a pair of compasses and a ruler. Since it's not the main topic of this chapter the explanation is abbreviated, but it utilizes the facts that

$$\cos\frac{\pi}{5} = \frac{1 + \sqrt{5}}{4} \tag{2}$$

and

$$\sin\frac{\pi}{5} = \frac{\sqrt{10 - 2\sqrt{5}}}{4}. \tag{3}$$

I am not an algebra specialist, so in the explanation that follows I'd like for the reader to understand the construction method for the regular heptadecagon attempted by Gauss as far as I have investigated it.

2. From Gauss' Diary

Chapter 1 of Takagi Teiji's 'History of Modern Mathematics' reports that when on the 30th March 1796, the 19-year-old Gauss opened his eyes and arose from his bed, a method for the outstanding problem of constructing a regular heptadecagon occurred to him and was thus recorded in his diary. An outline of the method goes as follows.[3]

If it is only required to prove the possibility of constructing a regular heptadecagon, the solution is clear and simple. Taking

$$360° = 17\phi,$$

if the value of $\cos\phi$ may be expressed as a square root then it is possible to construct the figure. $\cos\phi$ represents the x coordinate of a point on the circumference of a unit circle. Gauss demonstrated the computational process. Let's build on the explanation by looking at the method.

Firstly, Gauss made the following definitions.

$$\begin{aligned}
\cos\phi + \cos 4\phi &= a, \\
\cos 2\phi + \cos 8\phi &= b, \\
\cos 3\phi + \cos 5\phi &= c, \\
\cos 6\phi + \cos 7\phi &= d.
\end{aligned}$$

The thing to pay attention to here is the substitution of the parameters a, b, c, d for the values of $\cos\phi$ to $\cos 8\phi$. Perhaps this kind of substitution is only applied to this problem. If these are combined at random, then in total there are $_8C_2 \times _6C_2 \times _4C_2 = 2520$ possibilities. Do you think Gauss might have investigated every case? He did not. We will look at this issue in detail later, and it is closely related to the theory of cyclotomic equations.

Next, setting

$$a + b = e,$$

$$c + d = f,$$

then as is widely known,

$$e + f = -\frac{1}{2}. \tag{1}$$

In order to understand Equation (1) it's sufficient to remember a problem which often comes up in university entrance examinations.

Theory. *For natural number n, setting*

$$S_n = \cos\phi + \cos 2\phi + \cdots + \cos n\phi,$$

implies that

$$2S_n \sin\frac{\phi}{2} = \sin\frac{2n+1}{2}\phi - \sin\frac{\phi}{2}.$$

The proof of this theory involves the application of the product $\rightarrow$ sum formula for $2\cos k\phi \sin \dfrac{\phi}{2}$ $(k = 1, \cdots, n)$, so that the intermediate term vanishes and the formula is simplified. Then, substituting $n = 8, \phi = \dfrac{2\pi}{17}$ into

$$S_n = (\sin \frac{2n+1}{2}\phi - \sin \frac{\phi}{2})/2\sin \frac{\phi}{2}$$

yields $\sin \dfrac{2n+1}{2}\phi = 0$. Therefore, $S_8 = -\dfrac{1}{2}$.

The products formed by each pair among a, b, c, d may now be obtained. By means of a simple calculation, and taking note of the fact that $\cos n\phi = \cos(17 - n)\phi$ the results are

$$\begin{aligned}
2ab\ &= e + f = -\frac{1}{2}, \\
2ac\ &= 2a + b + d, \\
2ad\ &= b + c + 2d, \\
2bc\ &= a + 2c + d, \\
2bd\ &= a + 2b + c, \\
2cd\ &= e + f = -\frac{1}{2}.
\end{aligned}$$

Looking at one of the equations above, for example, $2ab$, it is obtained as follows

$$2ab = 2(\cos\phi + \cos 4\phi)(\cos 2\phi + \cos 8\phi)$$
$$= 2\cos\phi\cos 2\phi + 2\cos\phi\cos 8\phi + 2\cos 4\phi\cos 2\phi + 2\cos 4\phi\cos 8\phi$$
$$= (\cos 3\phi + \cos\phi) + (\underline{\cos 9\phi} + \cos 7\phi) + (\cos 6\phi + \cos 2\phi) + (\underline{\cos 12\phi} + \cos 4\phi).$$

Here the substitution $\cos 9\phi = \cos 8\phi, \cos 12\phi = \cos 5\phi$ is made, and the result rearranged, yielding

$$= (\cos\phi + \cos 4\phi) + (\cos 2\phi + \cos 8\phi) + (\cos 3\phi + \cos 5\phi) + (\cos 6\phi + \cos 7\phi)$$
$$= a + b + c + d$$
$$= e + f$$
$$= -\frac{1}{2}$$

ergo,

$$2ac + 2ad + 2bc + 2bd = 4a + 4b + 4c + 4d,$$

i.e.,

$$2ef = -2,$$

or alternatively,

$$ef = -1. \tag{2}$$

The following is a solution method utilizing the relationship between the solutions and factors of 2nd order equations. In particular, from (1) and (2), the equations for e and f are the roots of

$$x^2 + \frac{1}{2}x - 1 = 0.$$

One is thus $-\frac{1}{4} + \sqrt{\frac{17}{16}}$, while the other is $-\frac{1}{4} - \sqrt{\frac{17}{16}}$. A glance is sufficient to reveal from their values that the first is e, and the second is f.

Now, the following equation has roots a and b.

$$x^2 - ex - \frac{1}{4} = 0.$$

The values of the roots are

$$\frac{1}{2}e \pm \sqrt{\frac{1}{4} + \frac{1}{4}e^2} = -\frac{1}{8} + \frac{1}{8}\sqrt{17} \pm \frac{1}{8}\sqrt{34 - 2\sqrt{17}}.$$

It is clear here that a is the upper (positive) sign, and b is the lower (negative) sign. The reason is that, trivially,

$$a - b = (\cos\phi - \cos 2\phi) + (\cos 4\phi - \cos 8\phi)$$

so in exactly the same way,

$$c = -\frac{1}{8} - \frac{1}{8}\sqrt{17} + \frac{1}{8}\sqrt{34 + 2\sqrt{17}},$$

and

$$d = -\frac{1}{8} - \frac{1}{8}\sqrt{17} - \frac{1}{8}\sqrt{34 + 2\sqrt{17}}$$

Now finally, $\cos\phi$ and $\cos 4\phi$ are clearly the roots of the following 2nd order equation (because the product $\cos\phi \cdot \cos 4\phi = \frac{1}{2}c$).

$$x^2 - ax + \frac{1}{2}c = 0$$

Consequently,

$$\cos\phi = +\frac{1}{2}a + \sqrt{\frac{1}{4}a^2 - \frac{1}{2}c}\,,$$

and

$$\cos 4\phi = +\frac{1}{2}a - \sqrt{\frac{1}{4}a^2 - \frac{1}{2}c}\,.$$

Rearranging however yields,

$$2a^2 = 2 + b + 2c$$

so,

$$\cos\phi = \frac{1}{2}a + \sqrt{\frac{1}{4} + \frac{1}{8}b - \frac{1}{4}c} = -\frac{1}{16} + \frac{1}{16}\sqrt{17} + \frac{1}{16}\sqrt{34 - 2\sqrt{17}}$$
$$+\frac{1}{8}\sqrt{17 + 3\sqrt{17} - \sqrt{34 - 2\sqrt{17}} - 2\sqrt{34 + 2\sqrt{17}}}$$

which is the value Gauss obtained for $\cos\phi$.

3. Gauss' Theory of Cyclotomic Equations

We have seen how the value of $\cos\phi$ needed for the construction of a regular heptadecagon can be obtained, but this calculation was just a confirmation. So far we have not even touched upon the essential question of why it turns out as it does, *i.e.*, the reason for the substitution of the four parameters a, b, c, d for the terms from $\cos\phi$ to $\cos 8\phi$ must be stated.

The reason is not written in Gauss' diary. In order to find out, one must enlist the help of Kurata Reijirou's 'Gauss' Theory of Cyclotonic Equations',[2] or 'Gauss' Theory of Numbers' translated by Takase Masahito.[1] Think about the following equation.

$$x^n - 1 = 0 \tag{4}$$

It goes without saying that the roots of this equation are the nth roots of 1, and as is widely known,

$$e^{\frac{2\pi k i}{n}} \quad (k = 0, 1, 2, \cdots, n-1).$$

This has a relationship with Euler's equation.

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{5}$$

Among the n roots, those which first equal 1 when raised to the power n are known as the primitive nth roots of unity. $e^{\frac{2\pi i}{n}}$ is a primitive nth root of

unity. This is a point with angle $\dfrac{2\pi}{n}$ on a unit circle in the complex plain, *i.e.,* it expresses an nth equal part of a complete circumference. $e^{\frac{2\pi k i}{n}}$ expresses the point on the circumference with k-times the angle.

Through the intermediate agency of the complex plain, Equation (4) is tied to th n-sided regular polygon figures originating in ancient Greece. This was Gauss' underlying perspective, and the first mathematician to introduce the complex plain was Gauss.

Now, Equation (4) is

$$x^n - 1 = (x - 1)(x^{n-1} + x^{n-2} + \cdots + x + 1),$$

so excluding 1, all the roots of Equation (4) are roots of the following function.

$$F(x) = x^{n-1} + x^{n-2} + \cdots + x + 1 \tag{6}$$

This is known as a cyclotomic equation, or alternatively, as a circle-partitioning equation.

In $x^n - 1 = 0$, if a primitive nth root of unity, denoted ω , is already known, then the solutions of this equation are

$$1, \omega, \omega^2, \cdots, \omega^{n-1}.$$

For example, thinking about $x^3 - 1 = 0$,

$$x^3 - 1 = (x - 1)(x^2 + x + 1) = 0,$$

so the 3rd root is 1, $\dfrac{-1 \pm \sqrt{3}i}{2}$. Taking $\dfrac{-1 + \sqrt{3}i}{2} = \omega$, we have $\dfrac{-1 - \sqrt{3}i}{2} = \omega^2$, and it can be seen that the roots are $1, \omega$ and ω^2. In this case, $3\theta = 2\pi$ so

$$\omega = \cos\theta + i\sin\theta = e^{i\theta}$$
$$\omega^2 = \cos 2\theta + i\sin 2\theta = e^{i2\theta}$$
$$\omega^3 = \cos 3\theta + i\sin 3\theta = e^{i3\theta} = 1$$

and it can be seen that the 3 roots cycle.

At this point I'd like to introduce the following theorem which utilizes the concept of a remainder.

Theorem. *When p is a prime number, the set of roots of the cyclotomic equation*

$$F(x) = x^{n-1} + x^{n-2} + \cdots + x + 1.$$

may be written as Ω, and apart from 1, all the elements $r \in \Omega$ (which satisfy $x^n - 1 = 0$) are complex numbers. Furthermore, for a positive or negative integer e which is not divisible by p, the following are satisfied.

(1) $r^p = 1, r^{2p} = r^{3p} = \cdots = 1, r^{ep} = 1$;

(2) for integers λ and μ

$$\lambda \equiv \mu \pmod{p} \Leftrightarrow r^\lambda = r^\mu;$$

(3) for $r \in \Omega$

$$\Omega = \{r^e, r^{2e}, \cdots, r^{e(p-1)}\}, \quad r^e + r^{2e} + \cdots + r^{e(p-1)} = -1.$$

Gauss defined a parameter with a value representing an f-step cycle.

Definition. For an odd prime p, and a primitive pth root of unity, denoted r, and taking $p - 1 = fe$, g as a primitive root of p, and λ as an arbitrary integer, the value of the f-step cycle, denoted (f, λ), is defined as follows

$$(f, \lambda) = [\lambda] + [\lambda h] + [\lambda h^2] + \cdots + [\lambda h^{f-1}]$$

(note that $h = g^e$).

At this point let's perform the calculation for a regular heptadecagon. Taking 3 as a primitive root of the prime number 17,

$$p = 17, p - 1 = 16 = 16 \times 1 = f \times e, \quad g = 3, \quad h = g^e = 3^1 = 3,$$

and the 16-step cycle is as follows.

$$\begin{aligned}(16, 1) = &[1] + [3] + [9] + [10] + [13] + [5] + [15] + [11] + [16] + [14] \\ &+[8] + [7] + [4] + [12] + [2] + [6]\end{aligned}$$

That is to say, when 3 is taken as the primitive root for $\{[1], \cdots, [16]\}$, the result is a cyclic group modulo 17. For example, the 4th term on the right-hand side of the equation above, $[10]$, can be obtained as

$$\lambda h^3 = 1 \times 3^3 = 27 = 10 \pmod{17}.$$

An explanation of primitive roots, modulo, and cyclic groups is deferred to books specialized in algebra, number theory, groups, and so on.

At any rate, primes and primitive roots have a truly ingenious relationship. I first heard of the term 'primitive root' when I studied subroutines for generating pseudorandom numbers on computers. As a consequence, computers

can be made to generate randomly ordered sequences of all the integers they are capable of expressing. For the problem in question, the significant point is that the 16 roots can be reordered according to the concept of remainders. This was how the 16-step cycle could be obtained. Gauss presented a theory decomposing an f-step cycle.

Theorem. When $p - 1 = abc$, the bc-step cycle (bc, λ) is the sum of b c-step cycles

$$(bc, \lambda) = (c, \lambda) + \left(c, \lambda g^a\right) + \left(c, \lambda g^{2a}\right) + \cdots + \left(c, \lambda g^{a(b-1)}\right)$$

Let's attempt a decomposition of the 16-step cycle into 2 8-step cycles as described on p27-28 of reference (Kurata, 1988).[2] From $p = 17, p - 1 = 16 = 1 \times 2 \times 8 = a \times b \times c$,

$$(16, 1) = (8, 1) + (8, 3).$$

Defining $(8, 1)$ and $(8, 3)$ we have the following equations.

$$
\begin{aligned}
(8, 1) &= [1] + [9] + [9^2] + [9^3] + [9^4] + [9^5] + [9^6] + [9^7] \\
&= [1] + [9] + [81] + [729] + [6561] + [59049] + [531441] + [4782969] \\
&= [1] + [9] + [13] + [15] + [16] + [8] + [4] + [2] \pmod{17} \\
&= [1] + [2] + [4] + [8] + [9] + [13] + [15] + [16] \\
(8, 3) &= [3] + [3 \cdot 9] + [3 \cdot 9^2] + [3 \cdot 9^3] + [3 \cdot 9^4] + [3 \cdot 9^5] + [3 \cdot 9^6] + [3 \cdot 9^7] \\
&= [3] + [27] + [243] + [2187] + [19683] + [177147] + [1594323] \\
&\quad + [14348907] \\
&= [3] + [10] + [5] + [11] + [14] + [7] + [12] + [6] \pmod{17} \\
&= [3] + [5] + [6] + [7] + [10] + [11] + [12] + [14]
\end{aligned}
$$

Here, the 16-step cycle $(16, 1)$ is decomposed into 2 8-step cycles, $(8, 1)$ and $(8, 3)$. If the decomposition theory is applied repeatedly, the decomposition can be continued down to a final 1-step cycle. Figure 1 shows a breakdown of the whole decomposition process. The parameters a, b, c, d, e, f and values of $\cos \phi$ to $\cos 8\phi$ recorded in "Gauss' diary" are compiled in this figure. Can you understand the reason why Gauss chose the values

$$\cos \phi + \cos 4\phi = a, \quad \cos 3\phi + \cos 5\phi = c,$$

$$\cos 2\phi + \cos 8\phi = b, \quad \cos 6\phi + \cos 7\phi = d$$

$$a + b = e, \quad c + d = f$$

by means of this result?

For $p = 17$, the 16 roots besides 1 may be calculated as shown in Table 1, in a similar way. Figure 2 shows a diagram of a regular heptadecagon. By means of such a decomposition, it becomes possible to factorize the 2nd term on the right-hand side of the equation
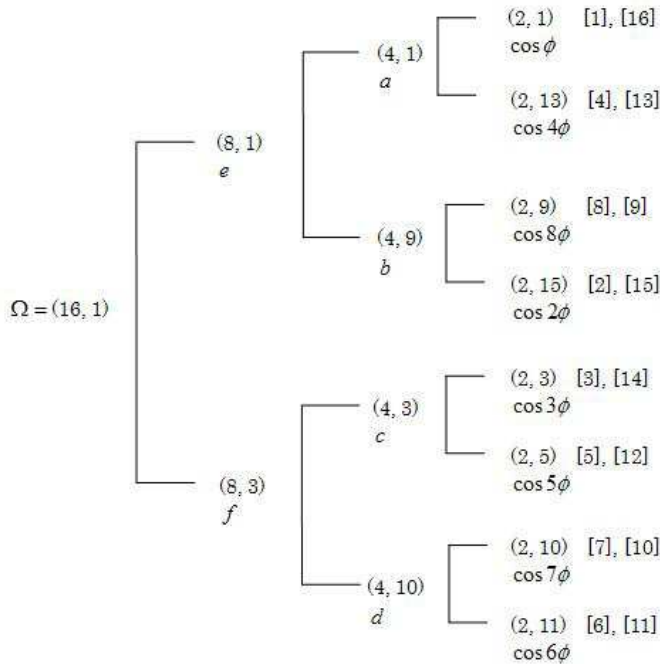

Figure 1: $x^{17} - 1 = (x - 1)(x^{16} + x^{15} + \cdots + x + 1)$.

Why it turns out like this, at present, I do not know. In order to find out it will be necessary to make further detailed study, and this is surely a place where the profound beauty of algebra lies. I have a feeling that I have experienced a glimpse of this beauty.
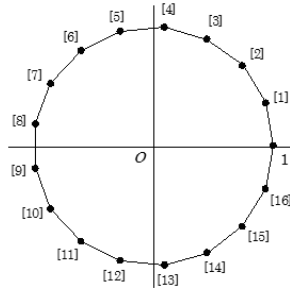
Figure 2: The decomposition process of the f-step cycle

$$
\begin{array}{ll}
[1], [16] & 0.9324722294 \pm 0.3612416662i \\
[2], [15] & 0.7390089172 \pm 0.6736956436i \\
[3], [14] & 0.4457383558 \pm 0.8951632914i \\
[4], [13] & 0.0922683595 \pm 0.9957341763i \\
[5], [12] & -0.2736629901 \pm 0.9618256432i \\
[6], [11] & -0.6026346364 \pm 0.7980172273i \\
[7], [10] & -0.8502171357 \pm 0.5264321629i \\
[8], [9] & -0.9829730997 \pm 0.1837495178i
\end{array}
$$

Table 1. The 16 roots

4. Constructing roots, and more...

The value of $\cos \phi$ was expressed using roots, so let's look at their construction process specifically. Construction problems in mathematics are not solved by calculating numerical values using a computer and drawing diagrammatic illustrations. The figure must be drawn using only a ruler and compasses, and given two points on a plane (points 0 and 1, *i.e.,* a line segment) as a unit length. The ruler is only used for drawing straight lines, and is considered incapable of measurement.

The problem is then how to draw the figure using root lengths. It is possible to construct the length of any root, in a manner similar to those shown in Figure 3. First, a unit square is drawn. By the triple angle theory, the diagonal edge has length $\sqrt{2}$. Secondly, a circular arc with radius $\sqrt{2}$ is drawn, and this length is transferred to the line extended from the base edge of the square. In this way, a rectangle with height 1, and width $\sqrt{2}$ can be constructed.

Once again applying the triple angle theory to this rectangle, the length of the diagonal is $\sqrt{3}$. By proceeding in this manner, the lengths $\sqrt{4}, \sqrt{5}, \sqrt{6}, \sqrt{7}$ and so on can be obtained. The construction of the length $\sqrt{17}$ can also be thus achieved.

Next, rather than the roots of integers, let's think about the construction of roots of arbitrary numerical values. Pages 50-51 of Kurata (1988) discuss this in more detail and contain an explanation of how to construct fractions (of the form $\dfrac{a}{b}$) and roots (of the form $\sqrt{a}$). See Figures 4 and 5. Even in the present age, when computers have advanced so far, the construction method for the regular heptadecagon discovered by Gauss around 200 years ago in 1796 still impresses anew. For me it simply reaffirmed Gauss' magnificent mental capacity and creative abilities.[?]
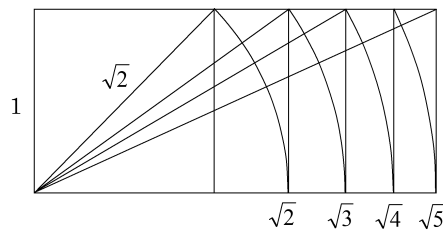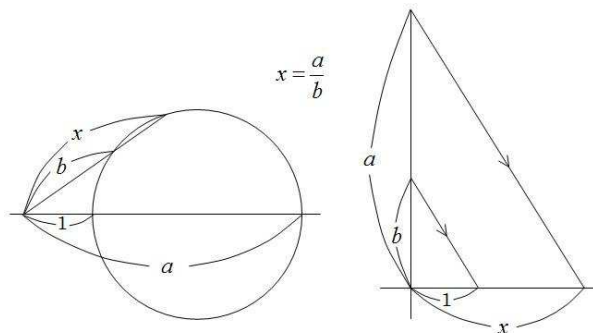

Figure 3: The length of a root

Figure 4: The construction of a fraction $(\dfrac{a}{b})$
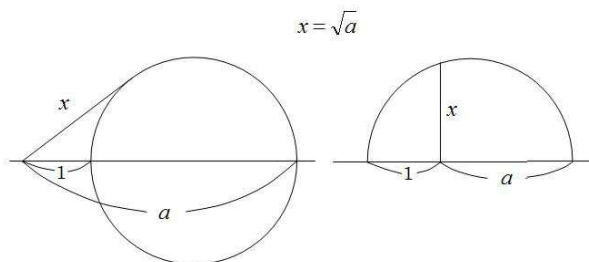
$$x = \sqrt{a}$$

Figure 5: The construction of a root ($\sqrt{a}$)

References

[1] C.F. Gauss, (trans. by M. Takase), *Gauss Seisuron* [Gauss' Number Theory, Disquisitiones Arithmeticae], Tokyo: Asakura, (1995).

[2] R. Kurata, *Gauss Enbun Hoteishikiron* [Gauss' Theory of Cyclotonic Equations], Nagoya: Kawai cultural education research laboratory, (1988).

[3] T. Takagi, *Kinsei Sugakushidan* [History of Modern Mathematics], 3rd edition, Tokyo: Kyoritsu, (1987).