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AXIOMATIC DIFFERENTIAL GEOMETRY II-4 – ITS DEVELOPMENTS – CHAPTER 4: THE FRÖLICHER-NIJENHUIS ALGEBRA

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Abstract: In our previous paper (Axiomatic Differential Geometry II-3) we have discussed the general Jacobi identity, from which the Jacobi identity of vector fields follows readily. In this paper we derive Jacobi-like identities of tangent-vector-valued forms from the general Jacobi identity.

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1. Introduction

In our previous paper [20] we presented the general Jacobi identity, from which the Jacobi identity of vector fields followed readily. The principal objective in this paper is to show that the Jacobi-like identity of tangent-vector-valued forms follows no less readily from the general Jacobi identity.

In orthodox differential geometry, establishing the Jacobi identity of vector fields on a smooth manifold is a trifling exercise, because we are able to identify vector fields with derivations. Similarly, since we are capable of identifying

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tangent-vector-valued forms with derivations of a certain kind over the algebra of differential form, the Jacobi-like identity of tangent-vector-valued forms is essentially no less difficult, though somewhat cumbersome, which Frölicher and Nijenhuis did in the 1950's.

Within our general framework of axiomatic differential geometry, such luxury is no longer permitted, so that the significance of the general Jacobi identity could not be exaggerated. Generally speaking, tangent-vector-valued forms are defined to be mappings subject to three conditions (the details will be seen in Section 4), namely, the first condition being what might be called the Dirac condition after Dirac distributions, the second condition being multi-linearity, and the third condition being anti-symmetry. Our general approach enables us to discern three levels in which the Jacobi-like identity holds, namely, tangent-vector-valued forms without multi-linearity or anti-symmetry, those without multi-linearity, and those without anti-symmetry. The case of tangent-vector-valued forms without multi-linearity or anti-symmetry is most fundamental. By taking Jacobi-like identities of tangent-vector-valued forms of the above three kinds at once, we get the very Jacobi-like identity that Frölicher and Nijenhuis provided more than half a century ago.

This paper is organized as follows. After some preliminaries in Section 2, we review the general Jacobi identity in Section 3. Section 4 is devoted to two distinct viewpoints towards tangent-vector-valued forms, just as we gave two viewpoints towards vector fields in . Section 5 is concerned with Jacobi-like identities of tangent-vector-valued forms. We are working within the axiomatics of differential geometry in [19], namely, a DG-category

$$(\mathcal{K}, \mathbb{R}, \mathbf{T}, \alpha)$$

2. Preliminaries

2.1. Weil Algebras and Infinitesimal Objects

The notion of a *Weil algebra* was introduced by Weil himself in [22]. We denote by $\mathbf{W}$ the category of Weil algebras. Roughly speaking, each Weil algebra corresponds to an infinitesimal object in the shade. By way of example, the Weil algebra $\mathbb{R}[X]/(X^2)$ (=the quotient ring of the polynomial ring $\mathbb{R}[X]$ of an indeterminate X over $\mathbb{R}$ modulo the ideal (X^2) generated by X^2) corresponds to the infinitesimal object of first-order nilpotent infinitesimals, while the Weil algebra $\mathbb{R}[X]/(X^3)$ corresponds to the infinitesimal object of second-order nilpotent infinitesimals. Although an infinitesimal object is undoubtedly imaginary in the real world, as has harassed both mathematicians and philosophers of the 17th and the 18th centuries (because mathematicians at that time preferred to talk infinitesimal objects as if they were real entities), each Weil algebra yields its corresponding *Weil functor* on the category of smooth manifolds of some kind to itself, which is no doubt a real entity. By way of example, the Weil algebra $\mathbb{R}[X]/(X^2)$ yields the tangent bundle functor as its corresponding Weil functor. Intuitively speaking, the Weil functor corresponding to a Weil algebra stands for the exponentiation by the infinitesimal object corresponding to the Weil algebra at issue. For Weil functors on the category of finite-dimensional smooth manifolds, the reader is referred to §35 of [3], while the reader can find a readable treatment of Weil functors on the category of smooth manifolds modelled on convenient vector spaces in §31 of [4].

Synthetic differential geometry (usually abbreviated to SDG), which is a kind of differential geometry with a cornucopia of nilpotent infinitesimals, was forced to invent its models, in which nilpotent infinitesimals were visible. For a standard textbook on SDG, the reader is referred to [5], while he or she is referred to [2] for the model theory of SDG constructed vigorously by Dubuc [**?**] and others. Although we do not get involved in SDG herein, we will exploit locutions in terms of infinitesimal objects so as to make the paper highly readable. Thus we prefer to write $\mathcal{W}_D$ and $\mathcal{W}_{D_2}$ in place of $\mathbb{R}[X]/(X^2)$ and $\mathbb{R}[X]/(X^3)$ respectively, where D stands for the infinitesimal object of first-order nilpotent infinitesimals, and D_2 stands for the infinitesimal object of second-order nilpotent infinitesimals. To Newton and Leibniz, D stood for

$$\{d \in \mathbb{R} \mid d^2 = 0\}$$

while D_2 stood for

$$\{d \in \mathbb{R} \mid d^3 = 0\}$$

We will write $\mathcal{W}_{d\in D_2 \mapsto d^2 \in D}$ for the homomorphim of Weil algebras $\mathbb{R}[X]/(X^2) \to \mathbb{R}[X]/(X^3)$ induced by the homomorphism $X \to X^2$ of the polynomial ring $\mathbb{R}[X]$ to itself. Such locutions are justifiable, because the category $\mathbf{W}$ of Weil algebras in the real world and the category of infinitesimal objects in the shade are dual to each other in a sense. Thus we have a contravariant functor $\mathcal{W}$ from the category of infinitesimal objects in the shade to the category of Weil algebras in the real world. Its inverse contravariant functor from the category of Weil algebras in the real world to the category of Weil algebras in the real world is denoted by $\mathcal{D}$. By way of example, $\mathcal{D}_{\mathbb{R}[X]/(X^2)}$ and $\mathcal{D}_{\mathbb{R}[X]/(X^3)}$ stand for D and D_2 respectively. To familiarize himself or herself with such locutions, the reader is strongly encouraged to read the first two chapters of [5], even if he or she is not interested in SDG at all.

We need to fix notation and terminology for simplicial objects, which form an important subclass of infinitesimal objects. *Simplicial objects* are infinitesimal objects of the form

$$D^n\{\mathsf{p}\}$$
$$= \{(d_1, ..., d_n) \in D^n \mid d_{i_1}...d_{i_k} = 0 \ (\forall (i_1, ..., i_k) \in \mathsf{p})\}$$

where p is a finite set of finite sequences $(i_1, ..., i_k)$ of natural numbers between 1 and n, including the endpoints, with $i_1 < ... < i_k$. If p is empty, $D^n\{\mathsf{p}\}$ is D^n itself. If p consists of all the binary sequences, then $D^n\{\mathsf{p}\}$ represents $D(n)$ in the standard terminology of SDG. Given two simplicial objects $D^m\{\mathsf{p}\}$ and $D^n\{\mathsf{q}\}$, we define a simplicial object $D^m\{\mathsf{p}\} \oplus D^n\{\mathsf{q}\}$ to be

$$D^{m+n}\{\mathsf{p} \oplus \mathsf{q}\}$$

where

$$\mathsf{p} \oplus \mathsf{q}$$
$$= \mathsf{p} \cup \{(j_1 + m, ..., j_k + m) \mid (j_1, ..., j_k) \in \mathsf{q}\}$$
$$\cup \{(i, j + m) \mid 1 \leq i \leq m, \ 1 \leq j \leq n\}$$

Since the operation $\oplus$ is associative, we can combine any finite number of simplicial objects by $\oplus$ without bothering about how to insert parentheses. Given morphisms of simplicial objects $\Phi_i : D^{m_i}\{\mathsf{p}_i\} \to D^m\{\mathsf{p}\}$ $(1 \leq i \leq n)$, there exists a unique morphism of simplicial objects $\Phi : D^{m_1}\{\mathsf{p}_1\} \oplus ... \oplus D^{m_n}\{\mathsf{p}_n\} \to D^m\{\mathsf{p}\}$ whose restriction to $D^{m_i}\{\mathsf{p}_i\}$ coincides with Φ_i for each i. We denote this Φ by $\Phi_1 \oplus ... \oplus \Phi_n$. We write $D(n)$ for $\{(d, ..., d) \in D^n \mid d_i d_j = 0$ for any $i \neq j\}$.

2.2. Some Conventions

Notation 1. We use the following notations:

1. We write
$$[A \to B]$$
for the exponential B^A.

2. We denote a canonical injection $A \to B$ by i_A^B.

3. We denote a canonical projection $A \to B$ by π_B^A.

4. An object M is always assumed to be microlinear.

5. The evaluation morphism $[A \rightarrow B] \times A \rightarrow B$ is denoted by

$$\mathrm{ev}_{[A \rightarrow B] \times A}$$

6. We denote by $\mathbb{S}_p$ the set of permutations of $\{1, ..., p\}$. Given $\sigma \in \mathbb{S}_p$, its signature is denoted by ε_σ.

3. The General Jacobi Identity

Proposition 2. *The diagram*

$$\begin{array}{ccccc} & & \alpha_{\mathcal{W}_\varphi}(M) & & \\ & \mathbf{T}^{\mathcal{W}_{D^3\{(1,3),(2,3)\}}}M & \rightarrow & \underset{2}{\mathbf{T}^{\mathcal{W}_{D^2}}M} & \\ \alpha_{\mathcal{W}_\psi}(M) & \downarrow & & \downarrow & \alpha_{\mathcal{W}_{i^{D^2}_{D(2)}}}(M) \\ & \underset{1}{\mathbf{T}^{\mathcal{W}_{D^2}}M} & \rightarrow & \mathbf{T}^{\mathcal{W}_{D(2)}}M & \\ & & \alpha_{\mathcal{W}_{i^{D^2}_{D(2)}}}(M) & & \end{array}$$

is a pullback diagram, where the assumptive mapping

$$\varphi : D^2 \rightarrow D^3\{(1,3),(2,3)\}$$

is

$$(d_1, d_2) \in D^2 \mapsto (d_1, d_2, 0) \in D^3\{(1,3),(2,3)\}$$

while the assumptive mapping

$$\psi : D^2 \rightarrow D^3\{(1,3),(2,3)\}$$

is

$$(d_1, d_2) \in D^2 \mapsto (d_1, d_2, d_1 d_2) \in D^3\{(1,3),(2,3)\}$$

Remark 3. The numbers $1, 2$ under $\mathbf{T}^{\mathcal{W}_{D^2}}M$ are given simply so as for the reader to easily relate each occurrence of $\mathbf{T}^{\mathcal{W}_{D^2}}M$ in the above Proposition to its corresponding occurrence in the following Corollary.

Corollary 4. *We have*

$$\mathbf{T}^{\mathcal{W}_{D^3\{(1,3),(2,3)\}}}M = \underset{1}{\mathbf{T}^{\mathcal{W}_{D^2}}M} \times_{\mathbf{T}^{\mathcal{W}_{D(2)}}M} \underset{2}{\mathbf{T}^{\mathcal{W}_{D^2}}M}$$

Notation 5. We will write

$$\dot{\zeta}^{-}(M) : \mathbf{T}^{\mathcal{W}_{D^2}}M \times_{\mathbf{T}^{\mathcal{W}_{D(2)}}M} \mathbf{T}^{\mathcal{W}_{D^2}}M \to \mathbf{T}^{\mathcal{W}_{D}}M$$

for the morphism

$$\begin{aligned} &\mathbf{T}^{\mathcal{W}_{D^2}}M \times_{\mathbf{T}^{\mathcal{W}_{D(2)}}M} \mathbf{T}^{\mathcal{W}_{D^2}}M \\ &= \mathbf{T}^{\mathcal{W}_{D^3\{(1,3),(2,3)\}}}M \\ &\xrightarrow{\alpha_{\mathcal{W}_{d\in D\mapsto(0,0,d)\in D^3\{(1,3),(2,3)\}}}} \\ &\mathbf{T}^{\mathcal{W}_{D}}M \end{aligned}$$

Proposition 6. *The morphism*

$$\begin{aligned} &\underset{1}{\mathbf{T}^{\mathcal{W}_{D^2}}}M \times_{\mathbf{T}^{\mathcal{W}_{D(2)}}M} \underset{2}{\mathbf{T}^{\mathcal{W}_{D^2}}}M \\ &\xrightarrow{\dot{\zeta}^{-}(M)} \\ &\mathbf{T}^{\mathcal{W}_{D}}M \end{aligned}$$

and the morphism

$$\begin{aligned} &\underset{1}{\mathbf{T}^{\mathcal{W}_{D^2}}}M \times_{\mathbf{T}^{\mathcal{W}_{D(2)}}M} \underset{2}{\mathbf{T}^{\mathcal{W}_{D^2}}}M \\ &= \underset{2}{\mathbf{T}^{\mathcal{W}_{D^2}}}M \times_{\mathbf{T}^{\mathcal{W}_{D(2)}}M} \underset{1}{\mathbf{T}^{\mathcal{W}_{D^2}}}M \\ &\xrightarrow{\dot{\zeta}^{-}(M)} \\ &\mathbf{T}^{\mathcal{W}_{D}}M \end{aligned}$$

sum up only to vanish, where the numbers $1, 2$ *under* $\mathbf{T}^{\mathcal{W}_{D^2}}M$ *are given simply so as for the reader to easily relate each occurrence of* $\mathbf{T}^{\mathcal{W}_{D^2}}M$ *to another.*

Proposition 7. *The diagram*

$$\begin{array}{ccccc} & \mathbf{T}^{\mathcal{W}_{D^4\{(2,4),(3,4)\}}}M & \xrightarrow{\alpha_{\mathcal{W}_{\varphi_1^3}}(M)} & \underset{2}{\mathbf{T}^{\mathcal{W}_{D^3}}}M & \\ \alpha_{\mathcal{W}_{\psi_1^3}}(M) & \downarrow & & \downarrow & \alpha_{\mathcal{W}_{i_{D^3\{(2,3)\}}^{D^3}}}(M) \\ & \underset{1}{\mathbf{T}^{\mathcal{W}_{D^3}}}M & \xrightarrow[\alpha_{\mathcal{W}_{i_{D^3\{(2,3)\}}^{D^3}}}(M)]{} & \mathbf{T}^{\mathcal{W}_{D^3\{(2,3)\}}}M & \end{array}$$

is a pullback diagram, where the assumptive mapping

$$\varphi_1^3 : D^3 \to D^4\{(2,4),(3,4)\}$$

is

$$(d_1, d_2, d_3) \in D^3 \mapsto (d_1, d_2, d_3, 0) \in D^4\{(2,4),(3,4)\},$$

while the assumptive mapping

$$\psi_1^3 : D^3 \to D^4\{(2,4),(3,4)\}$$

is

$$(d_1, d_2, d_3) \in D^3 \mapsto (d_1, d_2, d_3, d_2 d_3) \in D^4\{(2,4),(3,4)\}$$

Remark 8. The numbers $1, 2$ under $\mathbf{T}^{\mathcal{W}_{D^3}}M$ are given simply so as for the reader to easily relate each occurrence of $\mathbf{T}^{\mathcal{W}_{D^3}}M$ in the above Proposition to its corresponding occurrence in the following Corollary.

Corollary 9. *We have*

$$\underset{1}{\mathbf{T}^{\mathcal{W}_{D^3}}M} \times_{\mathbf{T}^{\mathcal{W}_{D^3\{(2,3)\}}}M} \underset{2}{\mathbf{T}^{\mathcal{W}_{D^3}}M}$$
$$= \mathbf{T}^{\mathcal{W}_{D^4\{(2,4),(3,4)\}}}M$$

Notation 10. We will write

$$\overset{\cdot}{\zeta}^{\overline{}}_{1}(M) : \mathbf{T}^{\mathcal{W}_{D^3}}M \times_{\mathbf{T}^{\mathcal{W}_{D^3\{(2,3)\}}}M} \mathbf{T}^{\mathcal{W}_{D^3}}M \to \mathbf{T}^{\mathcal{W}_{D^2}}M$$

for the morphism

$$\mathbf{T}^{\mathcal{W}_{D^3}}M \times_{\mathbf{T}^{\mathcal{W}_{D^3\{(2,3)\}}}M} \mathbf{T}^{\mathcal{W}_{D^3}}M$$
$$= \mathbf{T}^{\mathcal{W}_{D^4\{(2,4),(3,4)\}}}M$$
$$\xrightarrow{\alpha_{\mathcal{W}_{(d_1,d_2)\in D^2 \mapsto (d_1,0,0,d_2)\in D^4\{(2,4),(3,4)\}}}(M)}$$
$$\mathbf{T}^{\mathcal{W}_{D^2}}M$$

Proposition 11. *The diagram*

$$\begin{array}{ccccc} & & \alpha_{\mathcal{W}_{\varphi_2^3}}(M) & & \\ & \mathbf{T}^{\mathcal{W}_{D^4\{(1,4),(3,4)\}}}M & \to & \underset{2}{\mathbf{T}^{\mathcal{W}_{D^3}}M} & \\ \alpha_{\mathcal{W}_{\psi_2^3}}(M) & \downarrow & & \downarrow & \alpha_{\mathcal{W}_{i_{D^3\{(1,3)\}}^{D^3}}}(M) \\ & \underset{1}{\mathbf{T}^{\mathcal{W}_{D^3}}M} & \to & \mathbf{T}^{\mathcal{W}_{D^3\{(1,3)\}}}M & \\ & & \alpha_{\mathcal{W}_{i_{D^3\{(1,3)\}}^{D^3}}}(M) & & \end{array}$$

is a pullback diagram, where the assumptive mapping

$$\varphi_2^3 : D^3 \to D^4\{(1,4),(3,4)\}$$

is

$$(d_1, d_2, d_3) \in D^3 \mapsto (d_1, d_2, d_3, 0) \in D^4\{(1,4),(3,4)\}$$

while the assumptive mapping

$$\psi_2^3 : D^3 \to D^4\{(1,4),(3,4)\}$$

is

$$(d_1, d_2, d_3) \in D^3 \mapsto (d_1, d_2, d_3, d_1 d_3) \in D^4\{(1,4),(3,4)\}$$

Remark 12. The numbers $1, 2$ under $\mathbf{T}^{\mathcal{W}_{D^3}} M$ are given simply so as for the reader to easily relate each occurrence of $\mathbf{T}^{\mathcal{W}_{D^3}} M$ in the above Proposition to its corresponding occurrence in the following Corollary.

Corollary 13. *We have*

$$\underset{1}{\mathbf{T}^{\mathcal{W}_{D^3}} M} \times_{\mathbf{T}^{\mathcal{W}_{D^3\{(1,3)\}}} M} \underset{2}{\mathbf{T}^{\mathcal{W}_{D^3}} M} \\ = \mathbf{T}^{\mathcal{W}_{D^4\{(1,4),(3,4)\}}} M$$

Notation 14. We will write

$$\dot{\overline{\zeta}}^{2}(M) : \mathbf{T}^{\mathcal{W}_{D^3}} M \times_{\mathbf{T}^{\mathcal{W}_{D^3\{(1,3)\}}} M} \mathbf{T}^{\mathcal{W}_{D^3}} M \to \mathbf{T}^{\mathcal{W}_{D^2}} M$$

for the morphism

$$\mathbf{T}^{\mathcal{W}_{D^3}} M \times_{\mathbf{T}^{\mathcal{W}_{D^3\{(1,3)\}}} M} \mathbf{T}^{\mathcal{W}_{D^3}} M \\ = \mathbf{T}^{\mathcal{W}_{D^4\{(1,4),(3,4)\}}} M \\ \xrightarrow{\alpha_{\mathcal{W}_{(d_1,d_2) \in D^2 \mapsto (0,d_1,0,d_2) \in D^4\{(1,4),(3,4)\}}}(M)} \\ \mathbf{T}^{\mathcal{W}_{D^2}} M$$

Proposition 15. *The diagram*

$$\begin{array}{ccccc} & \mathbf{T}^{\mathcal{W}_{D^4\{(1,4),(2,4)\}}} M & \xrightarrow{\alpha_{\mathcal{W}_{\varphi_3^3}}(M)} & \underset{2}{\mathbf{T}^{\mathcal{W}_{D^3}} M} & \\ \alpha_{\mathcal{W}_{\psi_3^3}}(M) & \downarrow & & \downarrow & \alpha_{\mathcal{W}_{i_{D^3\{(1,2)\}}^{D^3}}}(M) \\ & \underset{1}{\mathbf{T}^{\mathcal{W}_{D^3}} M} & \xrightarrow[\alpha_{\mathcal{W}_{i_{D^3\{(1,2)\}}^{D^3}}}(M)]{} & \mathbf{T}^{\mathcal{W}_{D^3\{(1,2)\}}} M & \end{array}$$

is a pullback diagram, where the assumptive mapping

$$\varphi_3^3 : D^3 \to D^4\{(1,4),(2,4)\}$$

is

$$(d_1, d_2, d_3) \in D^3 \mapsto (d_1, d_2, d_3, 0) \in D^4\{(1,4),(2,4)\}$$

while the assumptive mapping

$$\psi_3^3 : D^3 \to D^4\{(1,4),(2,4)\}$$

is

$$(d_1, d_2, d_3) \in D^3 \mapsto (d_1, d_2, d_3, d_1 d_2) \in D^4\{(1,4),(2,4)\}$$

Remark 16. The numbers $1, 2$ under $\mathbf{T}^{\mathcal{W}_{D^3}} M$ are given simply so as for the reader to easily relate each occurrence of $\mathbf{T}^{\mathcal{W}_{D^3}} M$ in the above Proposition to its corresponding occurrence in the following Corollary.

Corollary 17. *We have*

$$\begin{aligned}
&\underset{1}{\mathbf{T}^{\mathcal{W}_{D^3}} M} \times_{\mathbf{T}^{\mathcal{W}_{D^3\{(1,2)\}}} M} \underset{2}{\mathbf{T}^{\mathcal{W}_{D^3}} M} \\
&= \mathbf{T}^{\mathcal{W}_{D^4\{(1,4),(2,4)\}}} M
\end{aligned}$$

Notation 18. We will write

$$\zeta^{\dot{\overline{\ }}_3} (M) : \mathbf{T}^{\mathcal{W}_{D^3}} M \times_{\mathbf{T}^{\mathcal{W}_{D^3\{(1,2)\}}} M} \mathbf{T}^{\mathcal{W}_{D^3}} M \to \mathbf{T}^{\mathcal{W}_{D^2}} M$$

for the morphism

$$\begin{aligned}
&\mathbf{T}^{\mathcal{W}_{D^3}} M \times_{\mathbf{T}^{\mathcal{W}_{D^3\{(1,2)\}}} M} \mathbf{T}^{\mathcal{W}_{D^3}} M \\
&= \mathbf{T}^{\mathcal{W}_{D^4\{(1,4),(2,4)\}}} M \\
&\alpha_{\mathcal{W}_{(d_1,d_2) \in D^2 \mapsto (0,0,d_1,d_2) \in D^4\{(1,4),(3,4)\}}} (M) \\
&\mathbf{T}^{\mathcal{W}_{D^2}} M
\end{aligned}$$

Notation 19. We will introduce three notations.

1. We will write

$$\zeta^{(*_{123} \underset{1}{\dot{-}} *_{132}) \dot{-} (*_{231} \underset{1}{\dot{-}} *_{321})} (M) : \triangle (M) \to \mathbf{T}^{\mathcal{W}_D} M$$

for the composition of morphisms

$$\pi^{\triangle(M)}_{\left(\left(\mathbf{T}^{\mathcal{W}_{D^3}}_{321}M\times_{\mathbf{T}^{\mathcal{W}_{D^3\{(2,3)\}}}M}\mathbf{T}^{\mathcal{W}_{D^3}}_{231}M\right)\times_{\mathbf{T}^{\mathcal{W}_{D(2)}}M}\left(\mathbf{T}^{\mathcal{W}_{D^3}}_{132}M\times_{\mathbf{T}^{\mathcal{W}_{D^3\{(2,3)\}}}M}\mathbf{T}^{\mathcal{W}_{D^3}}_{123}M\right)\right)}$$

$$:\triangle(M)\rightarrow\begin{pmatrix}\mathbf{T}^{\mathcal{W}_{D^3}}_{321}M\\ \times_{\mathbf{T}^{\mathcal{W}_{D^3\{(2,3)\}}}M}\\ \mathbf{T}^{\mathcal{W}_{D^3}}_{231}M\end{pmatrix}\times_{\mathbf{T}^{\mathcal{W}_{D(2)}}M}\begin{pmatrix}\mathbf{T}^{\mathcal{W}_{D^3}}_{132}M\\ \times_{\mathbf{T}^{\mathcal{W}_{D^3\{(2,3)\}}}M}\\ \mathbf{T}^{\mathcal{W}_{D^3}}_{123}M\end{pmatrix}$$

$$\dot{\overline{\zeta_1}}(M)\times_{\mathbf{T}^{\mathcal{W}_{D(2)}}M}\dot{\overline{\zeta_1}}(M):$$

$$:\begin{pmatrix}\mathbf{T}^{\mathcal{W}_{D^3}}_{321}M\\ \times_{\mathbf{T}^{\mathcal{W}_{D^3\{(2,3)\}}}M}\\ \mathbf{T}^{\mathcal{W}_{D^3}}_{231}M\end{pmatrix}\times_{\mathbf{T}^{\mathcal{W}_{D(2)}}M}\begin{pmatrix}\mathbf{T}^{\mathcal{W}_{D^3}}_{132}M\\ \times_{\mathbf{T}^{\mathcal{W}_{D^3\{(2,3)\}}}M}\\ \mathbf{T}^{\mathcal{W}_{D^3}}_{123}M\end{pmatrix}$$

$$\rightarrow\mathbf{T}^{\mathcal{W}_{D^2}}M\times_{\mathbf{T}^{\mathcal{W}_{D(2)}}M}\mathbf{T}^{\mathcal{W}_{D^2}}M$$

$$\dot{\zeta^{-}}(M):\mathbf{T}^{\mathcal{W}_{D^2}}M\times_{\mathbf{T}^{\mathcal{W}_{D(2)}}M}\mathbf{T}^{\mathcal{W}_{D^2}}M\rightarrow\mathbf{T}^{\mathcal{W}_{D}}M$$

in succession, where $\triangle(M)$ denotes

$$\left[\begin{array}{ccc} & \begin{pmatrix}\mathbf{T}^{\mathcal{W}_{D^3}}_{321}M\\ \times_{\mathbf{T}^{\mathcal{W}_{D^3\{(2,3)\}}}M}\\ \mathbf{T}^{\mathcal{W}_{D^3}}_{231}M\end{pmatrix} & \\ & \times_{\mathbf{T}^{\mathcal{W}_{D(2)}}M} & \\ & \begin{pmatrix}\mathbf{T}^{\mathcal{W}_{D^3}}_{132}M\\ \times_{\mathbf{T}^{\mathcal{W}_{D^3\{(2,3)\}}}M}\\ \mathbf{T}^{\mathcal{W}_{D^3}}_{123}M\end{pmatrix} & \\ \times_{\mathbf{T}^{\mathcal{W}_{D^3\oplus D^3}}M} & & \times_{\mathbf{T}^{\mathcal{W}_{D^3\oplus D^3}}M} \\ \begin{pmatrix}\mathbf{T}^{\mathcal{W}_{D^3}}_{132}M\\ \times_{\mathbf{T}^{\mathcal{W}_{D^3\{(1,3)\}}}M}\\ \mathbf{T}^{\mathcal{W}_{D^3}}_{312}M\end{pmatrix} & & \begin{pmatrix}\mathbf{T}^{\mathcal{W}_{D^3}}_{213}M\\ \times_{\mathbf{T}^{\mathcal{W}_{D^3\{(1,2)\}}}M}\\ \mathbf{T}^{\mathcal{W}_{D^3}}_{123}M\end{pmatrix} \\ \times_{\mathbf{T}^{\mathcal{W}_{D(2)}}M} & \times_{\mathbf{T}^{\mathcal{W}_{D^3\oplus D^3}}M} & \times_{\mathbf{T}^{\mathcal{W}_{D(2)}}M} \\ \begin{pmatrix}\mathbf{T}^{\mathcal{W}_{D^3}}_{213}M\\ \times_{\mathbf{T}^{\mathcal{W}_{D^3\{(1,3)\}}}M}\\ \mathbf{T}^{\mathcal{W}_{D^3}}_{231}M\end{pmatrix} & & \begin{pmatrix}\mathbf{T}^{\mathcal{W}_{D^3}}_{321}M\\ \times_{\mathbf{T}^{\mathcal{W}_{D^3\{(1,2)\}}}M}\\ \mathbf{T}^{\mathcal{W}_{D^3}}_{312}M\end{pmatrix}\end{array}\right]$$

2. We will write the morphism

$$\zeta^{\left(\ast 231\overset{\cdot}{\underset{2}{-}}\ast 213\right)\overset{\cdot}{-}\left(\ast 312\overset{\cdot}{\underset{2}{-}}\ast 132\right)}\left(M\right):\triangle\left(M\right)\rightarrow\mathbf{T}^{\mathcal{W}_{D}}M$$

for the composition of morphisms

$$\pi_{\left(\left(\mathbf{T}_{132}^{\mathcal{W}_{D^{3}}}M\times_{\mathbf{T}^{\mathcal{W}_{D^{3}\{(1,3)\}}}M}\mathbf{T}_{312}^{\mathcal{W}_{D^{3}}}M\right)\times_{\mathbf{T}^{\mathcal{W}_{D(2)}}M}\left(\mathbf{T}_{213}^{\mathcal{W}_{D^{3}}}M\times_{\mathbf{T}^{\mathcal{W}_{D^{3}\{(1,3)\}}}M}\mathbf{T}_{231}^{\mathcal{W}_{D^{3}}}M\right)\right)}^{\triangle(M)}$$
$$:\triangle\left(M\right)\rightarrow\left(\begin{array}{c}\mathbf{T}_{132}^{\mathcal{W}_{D^{3}}}M\\ \times_{\mathbf{T}^{\mathcal{W}_{D^{3}\{(1,3)\}}}M}\\ \mathbf{T}_{312}^{\mathcal{W}_{D^{3}}}M\end{array}\right)\times_{\mathbf{T}^{\mathcal{W}_{D(2)}}M}\left(\begin{array}{c}\mathbf{T}_{213}^{\mathcal{W}_{D^{3}}}M\\ \times_{\mathbf{T}^{\mathcal{W}_{D^{3}\{(1,3)\}}}M}\\ \mathbf{T}_{231}^{\mathcal{W}_{D^{3}}}M\end{array}\right)$$

$$\zeta^{\overset{\cdot}{\underset{2}{-}}}\left(M\right)\times_{\mathbf{T}^{\mathcal{W}_{D(2)}}M}\zeta^{\overset{\cdot}{\underset{2}{-}}}\left(M\right):$$
$$:\left(\begin{array}{c}\mathbf{T}_{132}^{\mathcal{W}_{D^{3}}}M\\ \times_{\mathbf{T}^{\mathcal{W}_{D^{3}\{(1,3)\}}}M}\\ \mathbf{T}_{312}^{\mathcal{W}_{D^{3}}}M\end{array}\right)\times_{\mathbf{T}^{\mathcal{W}_{D(2)}}M}\left(\begin{array}{c}\mathbf{T}_{213}^{\mathcal{W}_{D^{3}}}M\\ \times_{\mathbf{T}^{\mathcal{W}_{D^{3}\{(1,3)\}}}M}\\ \mathbf{T}_{231}^{\mathcal{W}_{D^{3}}}M\end{array}\right)$$
$$\rightarrow\mathbf{T}^{\mathcal{W}_{D^{2}}}M\times_{\mathbf{T}^{\mathcal{W}_{D(2)}}M}\mathbf{T}^{\mathcal{W}_{D^{2}}}M$$

$$\zeta^{\overset{\cdot}{-}}\left(M\right):\mathbf{T}^{\mathcal{W}_{D^{2}}}M\times_{\mathbf{T}^{\mathcal{W}_{D(2)}}M}\mathbf{T}^{\mathcal{W}_{D^{2}}}M\rightarrow\mathbf{T}^{\mathcal{W}_{D}}M$$

in succession.

3. We will write the morphism

$$\zeta^{\left(\ast 312\overset{\cdot}{\underset{3}{-}}\ast 321\right)\overset{\cdot}{-}\left(\ast 123\overset{\cdot}{\underset{3}{-}}\ast 213\right)}\left(M\right):\triangle\left(M\right)\rightarrow\mathbf{T}^{\mathcal{W}_{D}}M$$

for the composition of morphisms

$$\pi_{\left(\left(\mathbf{T}_{213}^{\mathcal{W}_{D^{3}}}M\times_{\mathbf{T}^{\mathcal{W}_{D^{3}\{(1,2)\}}}M}\mathbf{T}_{123}^{\mathcal{W}_{D^{3}}}M\right)\times_{\mathbf{T}^{\mathcal{W}_{D(2)}}M}\left(\mathbf{T}_{321}^{\mathcal{W}_{D^{3}}}M\times_{\mathbf{T}^{\mathcal{W}_{D^{3}\{(1,2)\}}}M}\mathbf{T}_{312}^{\mathcal{W}_{D^{3}}}M\right)\right)}^{\triangle(M)}$$
$$:\triangle\left(M\right)\rightarrow\left(\begin{array}{c}\mathbf{T}_{213}^{\mathcal{W}_{D^{3}}}M\\ \times_{\mathbf{T}^{\mathcal{W}_{D^{3}\{(1,2)\}}}M}\\ \mathbf{T}_{123}^{\mathcal{W}_{D^{3}}}M\end{array}\right)\times_{\mathbf{T}^{\mathcal{W}_{D(2)}}M}\left(\begin{array}{c}\mathbf{T}_{321}^{\mathcal{W}_{D^{3}}}M\\ \times_{\mathbf{T}^{\mathcal{W}_{D^{3}\{(1,2)\}}}M}\\ \mathbf{T}_{312}^{\mathcal{W}_{D^{3}}}M\end{array}\right)$$

$$\zeta^{\overset{\cdot}{\underset{3}{-}}}\left(M\right)\times_{\mathbf{T}^{\mathcal{W}_{D(2)}}M}\zeta^{\overset{\cdot}{\underset{3}{-}}}\left(M\right)$$

$$: \left(\begin{array}{c} \mathbf{T}^{\mathcal{W}_{D^3}} M \\ 213 \\ \times_{\mathbf{T}^{\mathcal{W}_{D^3\{(1,2)\}}} M} \\ \mathbf{T}^{\mathcal{W}_{D^3}} M \\ 123 \end{array} \right) \times_{\mathbf{T}^{\mathcal{W}_{D(2)}} M} \left(\begin{array}{c} \mathbf{T}^{\mathcal{W}_{D^3}} M \\ 321 \\ \times_{\mathbf{T}^{\mathcal{W}_{D^3\{(1,2)\}}} M} \\ \mathbf{T}^{\mathcal{W}_{D^3}} M \\ 312 \end{array} \right)$$

$$\rightarrow \mathbf{T}^{\mathcal{W}_{D^2}} M \times_{\mathbf{T}^{\mathcal{W}_{D(2)}} M} \mathbf{T}^{\mathcal{W}_{D^2}} M$$

$$\zeta^{\dot{-}} (M) : \mathbf{T}^{\mathcal{W}_{D^2}} M \times_{\mathbf{T}^{\mathcal{W}_{D(2)}} M} \mathbf{T}^{\mathcal{W}_{D^2}} M \rightarrow \mathbf{T}^{\mathcal{W}_D} M$$

in succession.

Theorem 20. *(The general Jacobi Identity) The three morphisms*

$$\zeta^{(*123 \underset{1}{\dot{-}} *132) \dot{-} (*231 \underset{1}{\dot{-}} *321)} (M) : \triangle (M) \rightarrow \mathbf{T}^{\mathcal{W}_D} M$$

$$\zeta^{(*231 \underset{2}{\dot{-}} *213) \dot{-} (*312 \underset{2}{\dot{-}} *132)} (M) : \triangle (M) \rightarrow \mathbf{T}^{\mathcal{W}_D} M$$

$$\zeta^{(*312 \underset{3}{\dot{-}} *321) \dot{-} (*123 \underset{3}{\dot{-}} *213)} (M) : \triangle (M) \rightarrow \mathbf{T}^{\mathcal{W}_D} M$$

sum up only to vanish.

4. Tangent-Vector-Valued Forms

Notation 21. We write

$$\underline{\Omega}^{(p,1)}_{(0)} (M)$$

for

$$\left[\mathbf{T}^{\mathcal{W}_{D^p}} M \rightarrow \mathbf{T}^{\mathcal{W}_D} M \right],$$

while we write

$$\overline{\Omega}^{(p,1)}_{(0)} (M)$$

for

$$\mathbf{T}^{\mathcal{W}_D} \left[\mathbf{T}^{\mathcal{W}_{D^p}} M \rightarrow M \right]$$

It is easy to see that

Proposition 22. *We have*

$$\underline{\Omega}^{(p,1)}_{(0)} (M) = \overline{\Omega}^{(p,1)}_{(0)} (M)$$

Notation 23. We write

$$\underline{\Omega}_{(1)}^{(p,1)}\left(M\right)$$

for the equalizer of the exponential transpose of

$$\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\rightarrow\mathbf{T}^{\mathcal{W}_D}M\right]\times\mathbf{T}^{\mathcal{W}_{D^p}}M\\ \underrightarrow{\mathrm{ev}_{\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\rightarrow\mathbf{T}^{\mathcal{W}_D}M\right]\times\mathbf{T}^{\mathcal{W}_{D^p}}M}}\\ \mathbf{T}^{\mathcal{W}_D}M\\ \underrightarrow{\alpha_{\mathcal{W}_{1\rightarrow D}}}\\ M$$

and that of

$$\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\rightarrow\mathbf{T}^{\mathcal{W}_D}M\right]\times\mathbf{T}^{\mathcal{W}_{D^p}}M\\ \underrightarrow{\pi_{\mathbf{T}^{\mathcal{W}_{D^p}}M}^{\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\rightarrow\mathbf{T}^{\mathcal{W}_D}M\right]\times\mathbf{T}^{\mathcal{W}_{D^p}}M}}\\ \mathbf{T}^{\mathcal{W}_{D^p}}M\\ \underrightarrow{\alpha_{\mathcal{W}_{1\rightarrow D}}}\\ M$$

while we write

$$\overline{\Omega}_{(1)}^{(p,1)}\left(M\right)$$

for the equalizer of the exponential transpose of

$$\mathbf{T}^{\mathcal{W}_D}\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\rightarrow M\right]\times\mathbf{T}^{\mathcal{W}_{D^p}}M\\ \underrightarrow{\alpha_{\mathcal{W}_{1\rightarrow D}}\left(\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\rightarrow M\right]\right)}\\ \left[\mathbf{T}^{\mathcal{W}_{D^p}}M\rightarrow M\right]\times\mathbf{T}^{\mathcal{W}_{D^p}}M\\ \underrightarrow{\mathrm{ev}_{\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\rightarrow M\right]\times\mathbf{T}^{\mathcal{W}_{D^p}}M}}\\ M$$

and that of

$$\mathbf{T}^{\mathcal{W}_D}\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\rightarrow M\right]\times\mathbf{T}^{\mathcal{W}_{D^p}}M\\ \underrightarrow{\pi_{\mathbf{T}^{\mathcal{W}_{D^p}}M}^{\mathbf{T}^{\mathcal{W}_D}\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\rightarrow M\right]\times\mathbf{T}^{\mathcal{W}_{D^p}}M}}$$

$$\mathbf{T}^{\mathcal{W}_{D^p}}M \xrightarrow{\alpha_{\mathcal{W}_{1\rightarrow D^p}}(M)} M$$

It is easy to see that

Proposition 24. *We have*

$$\underline{\Omega}_{(1)}^{(p,1)}(M) = \overline{\Omega}_{(1)}^{(p,1)}(M)$$

Notation 25. We write

$$\underline{\Omega}_{(12)}^{(p,1)}(M)$$

for the intersection of $\underline{\Omega}_{(1)}^{(p,1)}(M)$ and the equalizer of the exponential transpose of

$$\left[\mathbf{T}^{\mathcal{W}_{D^p}}M \rightarrow \mathbf{T}^{\mathcal{W}_{D}}M\right] \times \mathbf{T}^{\mathcal{W}_{D^p}}M \times \mathbb{R} \xrightarrow{\mathrm{id}_{\left[\mathbf{T}^{\mathcal{W}_{D^p}}M \rightarrow \mathbf{T}^{\mathcal{W}_{D}}M\right]} \times \left(\begin{array}{c}\cdot \\ i\end{array}\right)_{\mathbf{T}^{\mathcal{W}_{D^p}}M\times\mathbb{R}}} \left[\mathbf{T}^{\mathcal{W}_{D^p}}M \rightarrow \mathbf{T}^{\mathcal{W}_{D}}M\right] \times \mathbf{T}^{\mathcal{W}_{D^p}}M \xrightarrow{\mathrm{ev}_{\left[\mathbf{T}^{\mathcal{W}_{D^p}}M \rightarrow \mathbf{T}^{\mathcal{W}_{D}}M\right]\times\mathbf{T}^{\mathcal{W}_{D^p}}M}} \mathbf{T}^{\mathcal{W}_{D}}M$$

and that of

$$\left[\mathbf{T}^{\mathcal{W}_{D^p}}M \rightarrow \mathbf{T}^{\mathcal{W}_{D}}M\right] \times \mathbf{T}^{\mathcal{W}_{D^p}}M \times \mathbb{R} \xrightarrow{\mathrm{ev}_{\left[\mathbf{T}^{\mathcal{W}_{D^p}}M \rightarrow \mathbf{T}^{\mathcal{W}_{D}}M\right]\times\mathbf{T}^{\mathcal{W}_{D^p}}M} \times \mathrm{id}_{\mathbb{R}}} \mathbf{T}^{\mathcal{W}_{D}}M \times \mathbb{R} \xrightarrow{(\cdot)_{\mathbf{T}^{\mathcal{W}_{D}}M\times\mathbb{R}}} \mathbf{T}^{\mathcal{W}_{D}}M$$

where

1. The morphism

$$\left(\begin{array}{c}\cdot \\ i\end{array}\right)_{\mathbf{T}^{\mathcal{W}_{D^p}}M\times\mathbb{R}} : \mathbf{T}^{\mathcal{W}_{D^p}}M \times \mathbb{R} \rightarrow \mathbf{T}^{\mathcal{W}_{D^p}}M$$

stands for the morphism whose exponential transpose is

$$\begin{aligned} &\mathbf{T}^{\mathcal{W}_{D^p}}M \\ &\xrightarrow{\alpha_{\mathcal{W}_{(d_1,\ldots,d_p,a)\in D^p\times\mathsf{R}\to(d_1,\ldots,ad_i,\ldots,d_p)\in D^p}}(M)} \\ &\mathbf{T}^{\mathcal{W}_{D^p\times\mathsf{R}}}M \\ &=\left[\mathbb{R}\to\mathbf{T}^{\mathcal{W}_{D^p}}M\right] \end{aligned}$$

2. The morphism

$$(\cdot)_{\mathbf{T}^{\mathcal{W}_D}M\times\mathbb{R}}:\mathbf{T}^{\mathcal{W}_D}M\times\mathbb{R}\to\mathbf{T}^{\mathcal{W}_D}M$$

stands for the morphism whose exponential transpose is

$$\begin{aligned} &\mathbf{T}^{\mathcal{W}_D}M \\ &\xrightarrow{\alpha_{\mathcal{W}_{(d,a)\in D\times\mathsf{R}\to ad\in D}}(M)} \\ &\mathbf{T}^{\mathcal{W}_{D\times\mathsf{R}}}M \\ &=\left[\mathbb{R}\to\mathbf{T}^{\mathcal{W}_D}M\right] \end{aligned}$$

Notation 26. We write

$$\overline{\Omega}_{(12)}^{(p,1)}(M)$$

for the intersection of $\overline{\Omega}_{(1)}^{(p,1)}(M)$ and the equalizer of the exponential transpose of

$$\begin{aligned} &\mathbf{T}^{\mathcal{W}_D}\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\to M\right]\times\mathbf{T}^{\mathcal{W}_{D^p}}M\times\mathbb{R} \\ &\xrightarrow{\mathrm{id}_{\mathbf{T}^{\mathcal{W}_D}\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\to M\right]}\times\left(\begin{smallmatrix}\cdot\\ i\end{smallmatrix}\right)_{\mathbf{T}^{\mathcal{W}_{D^p}}M\times\mathbb{R}}} \\ &\mathbf{T}^{\mathcal{W}_D}\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\to M\right]\times\mathbf{T}^{\mathcal{W}_{D^p}}M \\ &\xrightarrow{\mathrm{id}_{\mathbf{T}^{\mathcal{W}_D}\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\to M\right]}\times\alpha_{\mathcal{W}_{1\to D}}\left(\mathbf{T}^{\mathcal{W}_{D^p}}M\right)} \\ &\mathbf{T}^{\mathcal{W}_D}\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\to M\right]\times\mathbf{T}^{\mathcal{W}_D}\left(\mathbf{T}^{\mathcal{W}_{D^p}}M\right) \\ &=\mathbf{T}^{\mathcal{W}_D}\left(\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\to M\right]\times\mathbf{T}^{\mathcal{W}_{D^p}}M\right) \\ &\xrightarrow{\mathbf{T}^{\mathcal{W}_D}\mathrm{ev}_{\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\to M\right]\times\mathbf{T}^{\mathcal{W}_{D^p}}M}} \\ &\mathbf{T}^{\mathcal{W}_D}M \end{aligned}$$

and that of

$$\mathbf{T}^{\mathcal{W}_D}\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\rightarrow M\right]\times\mathbf{T}^{\mathcal{W}_{D^p}}M\times\mathbb{R}$$
$$\underrightarrow{\mathrm{id}_{\mathbf{T}^{\mathcal{W}_D}\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\rightarrow M\right]}\times\alpha_{\mathcal{W}_{1\rightarrow D}}\left(\mathbf{T}^{\mathcal{W}_{D^p}}M\right)\times\mathrm{id}_{\mathbb{R}}}$$
$$\mathbf{T}^{\mathcal{W}_D}\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\rightarrow M\right]\times\mathbf{T}^{\mathcal{W}_D}\left(\mathbf{T}^{\mathcal{W}_{D^p}}M\right)\times\mathbb{R}$$
$$\underrightarrow{\mathbf{T}^{\mathcal{W}_D}\mathrm{ev}_M^{\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\rightarrow M\right]\times\mathbf{T}^{\mathcal{W}_{D^p}}M}\times\mathrm{id}_{\mathbb{R}}}$$
$$\mathbf{T}^{\mathcal{W}_D}M\times\mathrm{id}_{\mathbb{R}}$$
$$\underrightarrow{(\cdot)_{\mathbf{T}^{\mathcal{W}_D}M\times\mathbb{R}}}$$
$$\mathbf{T}^{\mathcal{W}_D}M$$

It is easy to see that

Proposition 27. *We have*

$$\underline{\Omega}_{(12)}^{(p,1)}(M)=\overline{\Omega}_{(12)}^{(p,1)}(M)$$

Notation 28. We write

$$\underline{\Omega}_{(13)}^{(p,1)}(M)$$

for the intersection of $\underline{\Omega}_{(1)}^{(p,1)}(M)$ and the equalizers, with all $\sigma\in\mathbb{S}_p$, of the exponential transpose of

$$\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\rightarrow\mathbf{T}^{\mathcal{W}_D}M\right]\times\mathbf{T}^{\mathcal{W}_{D^p}}M$$
$$\underrightarrow{\mathrm{id}_{\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\rightarrow\mathbf{T}^{\mathcal{W}_D}M\right]}\times(\cdot^{\sigma})_{\mathbf{T}^{\mathcal{W}_{D^p}}M}}$$
$$\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\rightarrow\mathbf{T}^{\mathcal{W}_D}M\right]\times\mathbf{T}^{\mathcal{W}_{D^p}}M$$
$$\underrightarrow{\mathrm{ev}_{\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\rightarrow\mathbf{T}^{\mathcal{W}_D}M\right]\times\mathbf{T}^{\mathcal{W}_{D^p}}M}}$$
$$\mathbf{T}^{\mathcal{W}_D}M$$

and that of

$$\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\rightarrow\mathbf{T}^{\mathcal{W}_D}M\right]\times\mathbf{T}^{\mathcal{W}_{D^p}}M$$
$$\underrightarrow{\mathrm{ev}_{\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\rightarrow\mathbf{T}^{\mathcal{W}_D}M\right]\times\mathbf{T}^{\mathcal{W}_{D^p}}M}}$$
$$\mathbf{T}^{\mathcal{W}_D}M$$
$$\underrightarrow{\varepsilon_\sigma}$$

$$\mathbf{T}^{\mathcal{W}_D}M$$

where the morphism

$$(\cdot^{\sigma})_{\mathbf{T}^{\mathcal{W}_{D^p}}M}:\mathbf{T}^{\mathcal{W}_{D^p}}M\rightarrow\mathbf{T}^{\mathcal{W}_{D^p}}M$$

stands for the morphism

$$\mathbf{T}^{\mathcal{W}_{D^p}}M \xrightarrow{\alpha_{\mathcal{W}_{(d_1,...,d_p)\in D^p\mapsto\left(d_{\sigma(1)},...,d_{\sigma(p)}\right)\in D^p}}(M)} \mathbf{T}^{\mathcal{W}_{D^p}}M$$

We write

$$\overline{\Omega}_{(13)}^{(p,1)}(M)$$

for the intersection of $\underline{\Omega}_{(1)}^{(p,1)}(M)$ and the equalizers, with all $\sigma\in\mathbb{S}_p$, of the exponential transpose of

$$\mathbf{T}^{\mathcal{W}_D}\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\rightarrow M\right]\times\mathbf{T}^{\mathcal{W}_{D^p}}M$$
$$\xrightarrow{\mathrm{id}_{\mathbf{T}^{\mathcal{W}_D}\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\rightarrow M\right]}\times(\cdot^{\sigma})_{\mathbf{T}^{\mathcal{W}_{D^p}}M}}$$
$$\mathbf{T}^{\mathcal{W}_D}\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\rightarrow M\right]\times\mathbf{T}^{\mathcal{W}_{D^p}}M$$
$$\xrightarrow{\mathrm{id}_{\mathbf{T}^{\mathcal{W}_D}\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\rightarrow M\right]}\times\alpha_{\mathcal{W}_{1\rightarrow D}}\left(\mathbf{T}^{\mathcal{W}_{D^p}}M\right)}$$
$$\mathbf{T}^{\mathcal{W}_D}\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\rightarrow M\right]\times\mathbf{T}^{\mathcal{W}_D}\left(\mathbf{T}^{\mathcal{W}_{D^p}}M\right)$$
$$=\mathbf{T}^{\mathcal{W}_D}\left(\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\rightarrow M\right]\times\mathbf{T}^{\mathcal{W}_{D^p}}M\right)$$
$$\xrightarrow{\mathbf{T}^{\mathcal{W}_D}\mathrm{ev}_{\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\rightarrow M\right]\times\mathbf{T}^{\mathcal{W}_{D^p}}M}}$$
$$\mathbf{T}^{\mathcal{W}_D}M$$

and that of

$$\mathbf{T}^{\mathcal{W}_D}\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\rightarrow M\right]\times\mathbf{T}^{\mathcal{W}_{D^p}}M$$
$$\xrightarrow{\mathrm{id}_{\mathbf{T}^{\mathcal{W}_D}\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\rightarrow M\right]}\times\alpha_{\mathcal{W}_{1\rightarrow D}}\left(\mathbf{T}^{\mathcal{W}_{D^p}}M\right)}$$
$$\mathbf{T}^{\mathcal{W}_D}\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\rightarrow M\right]\times\mathbf{T}^{\mathcal{W}_D}\left(\mathbf{T}^{\mathcal{W}_{D^p}}M\right)$$
$$\xrightarrow{\mathbf{T}^{\mathcal{W}_D}\mathrm{ev}_{\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\rightarrow M\right]\times\mathbf{T}^{\mathcal{W}_{D^p}}M}}$$
$$\mathbf{T}^{\mathcal{W}_D}M$$

$$\overset{\varepsilon_{q}}{\rightarrow}$$
$$\mathbf{T}^{\mathcal{W}_D} M$$

It is easy to see that

Proposition 29. *We have*

$$\underline{\Omega}_{(13)}^{(p,1)}(M) = \overline{\Omega}_{(13)}^{(p,1)}(M)$$

Notation 30. We write

$$\underline{\Omega}_{(123)}^{(p,1)}(M)$$

for the intersection of $\underline{\Omega}_{(12)}^{(p,1)}(M)$ and $\underline{\Omega}_{(13)}^{(p,1)}(M)$, while we write

$$\overline{\Omega}_{(123)}^{(p,1)}(M)$$

for the intersection of $\overline{\Omega}_{(12)}^{(p,1)}(M)$ and $\overline{\Omega}_{(13)}^{(p,1)}(M)$.

It is easy to see that

Proposition 31. *We have*

$$\underline{\Omega}_{(123)}^{(p,1)}(M) = \overline{\Omega}_{(123)}^{(p,1)}(M)$$

5. The Jacobi Identity in the Frölicher-Nijenhuis Algebra

5.1. Preparatory Considerations

Let us begin this subsection by adding the following definition to our lexicon.

Definition 32. Given an object M in the category $\mathcal{K}$ and natural numbers p, q, we define a morphism

$$\underline{\mathrm{Conv}}_{p,q}^{M} : \left[\mathbf{T}^{\mathcal{W}_{D^p}} M \to M\right] \times \left[\mathbf{T}^{\mathcal{W}_{D^q}} M \to M\right] \to \left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}} M \to M\right]$$

in the category $\mathcal{K}$ to be the exponential transpose of

$$\begin{aligned}
&\left[\mathbf{T}^{\mathcal{W}_{D^p}} M \to M\right] \times \left[\mathbf{T}^{\mathcal{W}_{D^q}} M \to M\right] \times \mathbf{T}^{\mathcal{W}_{D^{p+q}}} M \\
&= \left[\mathbf{T}^{\mathcal{W}_{D^p}} M \to M\right] \times \left[\mathbf{T}^{\mathcal{W}_{D^q}} M \to M\right] \times \mathbf{T}^{\mathcal{W}_{D^p} \otimes_k \mathcal{W}_{D^q}} M \\
&= \left[\mathbf{T}^{\mathcal{W}_{D^p}} M \to M\right] \times \left[\mathbf{T}^{\mathcal{W}_{D^q}} M \to M\right] \times \mathbf{T}^{\mathcal{W}_{D^q} \otimes_k \mathcal{W}_{D^p}} M
\end{aligned}$$

$$= \left[\mathbf{T}^{\mathcal{W}_{D^p}} M \to M\right] \times \left[\mathbf{T}^{\mathcal{W}_{D^q}} M \to M\right] \times \mathbf{T}^{\mathcal{W}_{D^p}} \left(\mathbf{T}^{\mathcal{W}_{D^q}} M\right)$$

$$\xrightarrow{\mathrm{id}_{\left[\mathbf{T}^{\mathcal{W}_{D^p}} M \to M\right]} \times \alpha_{\mathcal{W}_{D^p \to 1}} \left(\left[\mathbf{T}^{\mathcal{W}_{D^q}} M \to M\right]\right) \times \mathrm{id}_{\mathbf{T}^{\mathcal{W}_{D^p}}\left(\mathbf{T}^{\mathcal{W}_{D^q}} M\right)}}$$

$$\left[\mathbf{T}^{\mathcal{W}_{D^p}} M \to M\right] \times \mathbf{T}^{\mathcal{W}_{D^p}} \left[\mathbf{T}^{\mathcal{W}_{D^q}} M \to M\right] \times \mathbf{T}^{\mathcal{W}_{D^p}} \left(\mathbf{T}^{\mathcal{W}_{D^q}} M\right)$$

$$= \left[\mathbf{T}^{\mathcal{W}_{D^p}} M \to M\right] \times \mathbf{T}^{\mathcal{W}_{D^p}} \left(\left[\mathbf{T}^{\mathcal{W}_{D^q}} M \to M\right] \times \mathbf{T}^{\mathcal{W}_{D^q}} M\right)$$

$$\xrightarrow{\mathrm{id}_{\left[\mathbf{T}^{\mathcal{W}_{D^p}} M \to M\right]} \times \mathbf{T}^{\mathcal{W}_{D^p}} \mathrm{ev}_{\left[\mathbf{T}^{\mathcal{W}_{D^q}} M \to M\right] \times \mathbf{T}^{\mathcal{W}_{D^q}} M}}$$

$$\left[\mathbf{T}^{\mathcal{W}_{D^p}} M \to M\right] \times \mathbf{T}^{\mathcal{W}_{D^p}} M$$

$$\xrightarrow{\mathrm{ev}_{\left[\mathbf{T}^{\mathcal{W}_{D^p}} M \to M\right] \times \mathbf{T}^{\mathcal{W}_{D^p}} M}}$$

$$M,$$

while we define another morphism

$$\overline{\mathrm{Conv}}_{p,q}^{M} : \left[\mathbf{T}^{\mathcal{W}_{D^p}} M \to M\right] \times \left[\mathbf{T}^{\mathcal{W}_{D^q}} M \to M\right] \to \left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}} M \to M\right]$$

in the category $\mathcal{K}$ to be the exponential transpose of

$$\left[\mathbf{T}^{\mathcal{W}_{D^p}} M \to M\right] \times \left[\mathbf{T}^{\mathcal{W}_{D^q}} M \to M\right] \times \mathbf{T}^{\mathcal{W}_{D^{p+q}}} M$$

$$= \left[\mathbf{T}^{\mathcal{W}_{D^p}} M \to M\right] \times \left[\mathbf{T}^{\mathcal{W}_{D^q}} M \to M\right] \times \mathbf{T}^{\mathcal{W}_{D^p} \otimes_k \mathcal{W}_{D^q}} M$$

$$= \left[\mathbf{T}^{\mathcal{W}_{D^q}} M \to M\right] \times \left[\mathbf{T}^{\mathcal{W}_{D^p}} M \to M\right] \times \mathbf{T}^{\mathcal{W}_{D^q}} \left(\mathbf{T}^{\mathcal{W}_{D^p}} M\right)$$

$$\xrightarrow{\mathrm{id}_{\left[\mathbf{T}^{\mathcal{W}_{D^q}} M \to M\right]} \times \alpha_{\mathcal{W}_{D^q \to 1}} \left(\left[\mathbf{T}^{\mathcal{W}_{D^p}} M \to M\right]\right) \times \mathrm{id}_{\mathbf{T}^{\mathcal{W}_{D^q}}\left(\mathbf{T}^{\mathcal{W}_{D^p}} M\right)}}$$

$$\left[\mathbf{T}^{\mathcal{W}_{D^q}} M \to M\right] \times \mathbf{T}^{\mathcal{W}_{D^q}} \left[\mathbf{T}^{\mathcal{W}_{D^p}} M \to M\right] \times \mathbf{T}^{\mathcal{W}_{D^q}} \left(\mathbf{T}^{\mathcal{W}_{D^p}} M\right)$$

$$= \left[\mathbf{T}^{\mathcal{W}_{D^q}} M \to M\right] \times \mathbf{T}^{\mathcal{W}_{D^q}} \left(\left[\mathbf{T}^{\mathcal{W}_{D^p}} M \to M\right] \times \mathbf{T}^{\mathcal{W}_{D^p}} M\right)$$

$$\xrightarrow{\mathrm{id}_{\left[\mathbf{T}^{\mathcal{W}_{D^q}} M \to M\right]} \times \mathbf{T}^{\mathcal{W}_{D^q}} \mathrm{ev}_{\left[\mathbf{T}^{\mathcal{W}_{D^p}} M \to M\right] \times \mathbf{T}^{\mathcal{W}_{D^p}} M}}$$

$$\left[\mathbf{T}^{\mathcal{W}_{D^q}} M \to M\right] \times \mathbf{T}^{\mathcal{W}_{D^q}} M$$

$$\xrightarrow{\mathrm{ev}_{\left[\mathbf{T}^{\mathcal{W}_{D^q}} M \to M\right] \times \mathbf{T}^{\mathcal{W}_{D^q}} M}}$$

$$M$$

Remark 33. 1. Our two convolutions $\underline{\mathrm{Conv}}_{p,q}^{M}$ and $\overline{\mathrm{Conv}}_{p,q}^{M}$ are reminiscent of the familiar ones in abstract harmonic analysis and the theory of Schwartz distributions.

2. In case that $p = q = 0$, it obtains that

$$M \otimes \mathcal{W}_{D^p} = M \otimes \mathcal{W}_{D^q} = M \otimes \mathcal{W}_{D^{p+q}} = M$$

so that

$$[M \otimes \mathcal{W}_{D^p} \to M] = [M \otimes \mathcal{W}_{D^q} \to M] = [M \otimes \mathcal{W}_{D^{p+q}} \to M] = [M \to M],$$

in which we have

$$\underline{\mathrm{Conv}}_{p,q}^{M} = \mathrm{ass}_M$$

and the morphism $\overline{\mathrm{Conv}}_{p,q}^{M}$ is identical to the morphism

$$\begin{aligned}
&[M \underset{1}{\to} M] \times [M \underset{2}{\to} M] \\
&= [M \underset{2}{\to} M] \times [M \underset{1}{\to} M] \\
&\xrightarrow{\mathrm{ass}_M} \\
&[M \to M]
\end{aligned}$$

where ass_M stands for composition.

It should be obvious that

Proposition 34. *Given natural numbers p, q, the morphism*

$$\begin{aligned}
&\left[\mathbf{T}^{\mathcal{W}_{D^p}} M \to M\right] \times \left[\mathbf{T}^{\mathcal{W}_{D^q}} M \to M\right] \\
&\xrightarrow{\underline{\mathrm{Conv}}_{p,q}^{M}} \\
&\left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}} M \to M\right] \\
&\xrightarrow{(\cdot^{\sigma_{p,q}})_{\left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}} M \to M\right]}} \\
&\left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}} M \to M\right]
\end{aligned}$$

is identical to the morphism

$$\begin{aligned}
&\left[\mathbf{T}^{\mathcal{W}_{D^p}} M \to M\right] \times \left[\mathbf{T}^{\mathcal{W}_{D^q}} M \to M\right] \\
&= \left[\mathbf{T}^{\mathcal{W}_{D^q}} M \to M\right] \times \left[\mathbf{T}^{\mathcal{W}_{D^p}} M \to M\right] \\
&\xrightarrow{\overline{\mathrm{Conv}}_{q,p}^{M}} \\
&\left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}} M \to M\right],
\end{aligned}$$

while the morphism

$$\left[\mathbf{T}^{\mathcal{W}_{D^p}} M \to M\right] \times \left[\mathbf{T}^{\mathcal{W}_{D^q}} M \to M\right]$$

$$\begin{array}{l} \overset{\overline{\mathrm{Conv}}_{p,q}^{M}}{\longrightarrow} \\ \left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}}M\rightarrow M\right] \\ \left(\cdot^{\sigma_{p,q}}\right)_{\mathbf{T}^{\mathcal{W}_{D^{p+q}}}M}\rightarrow \\ \left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}}M\rightarrow M\right] \end{array}$$

is identical to the morphism

$$\begin{array}{l} \left[\mathbf{T}^{\mathcal{W}_{D^{p}}}M\rightarrow M\right]\times\left[\mathbf{T}^{\mathcal{W}_{D^{q}}}M\rightarrow M\right] \\ =\left[\mathbf{T}^{\mathcal{W}_{D^{q}}}M\rightarrow M\right]\times\left[\mathbf{T}^{\mathcal{W}_{D^{p}}}M\rightarrow M\right] \\ \overset{\underline{\mathrm{Conv}}_{q,p}^{M}}{\longrightarrow} \\ \left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}}M\rightarrow M\right] \end{array}$$

where $\sigma_{p,q}$ *is the permutation mapping the sequence* $1,...,q,q+1,...,p+q$ *to the sequence* $q+1,...,q+p,1,...,q$*, namely,*

$$\sigma_{p,q}=\left(\begin{array}{cccccc} 1 & ... & p & p+1 & ... & p+q \\ q+1 & ... & q+p & 1 & ... & q \end{array}\right)$$

It should be obvious that

Proposition 35. *Both* $\underline{\mathrm{Conv}}^{M}$ *and* $\overline{\mathrm{Conv}}^{M}$ *are associative in the sense that, given an object* M *in the category* $\mathcal{K}$ *and natural numbers* p,q,r*, the morphism*

$$\begin{array}{l} \left[\mathbf{T}^{\mathcal{W}_{D^{p}}}M\rightarrow M\right]\times\left[\mathbf{T}^{\mathcal{W}_{D^{q}}}M\rightarrow M\right]\times\left[\mathbf{T}^{\mathcal{W}_{D^{r}}}M\rightarrow M\right] \\ \xrightarrow{\underline{\mathrm{Conv}}_{p,q}^{M}\times\mathrm{id}_{\left[\mathbf{T}^{\mathcal{W}_{D^{r}}}M\rightarrow M\right]}} \\ \left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}}M\rightarrow M\right]\times\left[\mathbf{T}^{\mathcal{W}_{D^{r}}}M\rightarrow M\right] \\ \underline{\mathrm{Conv}}_{p+q,r}^{M} \\ \left[\mathbf{T}^{\mathcal{W}_{D^{p+q+r}}}M\rightarrow M\right] \end{array}$$

is identical to the morphism

$$\begin{array}{l} \left[\mathbf{T}^{\mathcal{W}_{D^{p}}}M\rightarrow M\right]\times\left[\mathbf{T}^{\mathcal{W}_{D^{q}}}M\rightarrow M\right]\times\left[\mathbf{T}^{\mathcal{W}_{D^{r}}}M\rightarrow M\right] \\ \xrightarrow{\mathrm{id}_{\left[\mathbf{T}^{\mathcal{W}_{D^{p}}}M\rightarrow M\right]}\times\underline{\mathrm{Conv}}_{q,r}^{M}} \\ \left[\mathbf{T}^{\mathcal{W}_{D^{p}}}M\rightarrow M\right]\times\left[\mathbf{T}^{\mathcal{W}_{D^{q+r}}}M\rightarrow M\right] \\ \overset{\underline{\mathrm{Conv}}_{p,q+r}^{M}}{\longrightarrow} \end{array}$$

$$\left[\mathbf{T}^{\mathcal{W}_{D^{p+q+r}}}M \to M\right],$$

and we have a similar identification for $\overline{\mathrm{Conv}}^M$.

Remark 36. This proposition enables us to write

$$\begin{aligned}\underline{\mathrm{Conv}}^M_{p.q.r} : \left[\mathbf{T}^{\mathcal{W}_{D^p}}M \to M\right] \times \left[\mathbf{T}^{\mathcal{W}_{D^q}}M \to M\right] \times \left[\mathbf{T}^{\mathcal{W}_{D^r}}M \to M\right] \\ \to \left[\mathbf{T}^{\mathcal{W}_{D^{p+q+r}}}M \to M\right]\end{aligned}$$

to denote one of the above two identical morphisms without any ambiguity, and similarly for

$$\begin{aligned}\overline{\mathrm{Conv}}^M_{p.q.r} : \left[\mathbf{T}^{\mathcal{W}_{D^p}}M \to M\right] \times \left[\mathbf{T}^{\mathcal{W}_{D^q}}M \to M\right] \times \left[\mathbf{T}^{\mathcal{W}_{D^r}}M \to M\right] \\ \to \left[\mathbf{T}^{\mathcal{W}_{D^{p+q+r}}}M \to M\right]\end{aligned}$$

Notation 37. Given a natural number p, we write

$$\left[\mathbf{T}^{\mathcal{W}_{D^p}}M \to M\right]_{\mathrm{id}_M}$$

for the pullback of

$$\begin{array}{ccc}\left[\mathbf{T}^{\mathcal{W}_{D^p}}M \to M\right]_{\mathrm{id}_M} & \to & \left[\mathbf{T}^{\mathcal{W}_{D^p}}M \to M\right] \\ \downarrow & & \downarrow \\ 1 & \to & [M \to M]\end{array},$$

where the right vertical arrow is

$$[\alpha_{\mathcal{W}_{D^p \to 1}} \to \mathrm{id}_M] : \left[\mathbf{T}^{\mathcal{W}_{D^p}}M \to M\right] \to [M \to M]$$

is the canonical projection, while the bottom horizontal arrow is the exponential transpose of

$$\mathrm{id}_M : 1 \times M = M \to M.$$

The following proposition should be evident.

Proposition 38. *The morphism*

$$\begin{aligned}&\left[\mathbf{T}^{\mathcal{W}_{D^p}}M \to M\right]_{\mathrm{id}_M} \times \left[\mathbf{T}^{\mathcal{W}_{D^q}}M \to M\right] \\ &\xrightarrow{i^{\left[\mathbf{T}^{\mathcal{W}_{D^p}}M \to M\right]}_{\left[\mathbf{T}^{\mathcal{W}_{D^p}}M \to M\right]_{\mathrm{id}_M}} \times \mathrm{id}_{\left[\mathbf{T}^{\mathcal{W}_{D^q}}M \to M\right]}} \\ &\left[\mathbf{T}^{\mathcal{W}_{D^p}}M \to M\right] \times \left[\mathbf{T}^{\mathcal{W}_{D^q}}M \to M\right]\end{aligned}$$

$$\underrightarrow{\mathrm{Conv}_{p,q}^{M}}$$
$$\left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}}M \rightarrow M\right]$$

is identical to the morphism

$$\left[\mathbf{T}^{\mathcal{W}_{D^{p}}}M \rightarrow M\right]_{\mathrm{id}_M} \times \left[\mathbf{T}^{\mathcal{W}_{D^{q}}}M \rightarrow M\right]$$
$$\xrightarrow{i_{\left[\mathbf{T}^{\mathcal{W}_{D^{p}}}M \rightarrow M\right]_{\mathrm{id}_M}}^{\left[\mathbf{T}^{\mathcal{W}_{D^{p}}}M \rightarrow M\right]} \times \mathrm{id}_{\left[\mathbf{T}^{\mathcal{W}_{D^{q}}}M \rightarrow M\right]}}$$
$$\left[\mathbf{T}^{\mathcal{W}_{D^{p}}}M \rightarrow M\right] \times \left[\mathbf{T}^{\mathcal{W}_{D^{q}}}M \rightarrow M\right]$$
$$\xrightarrow{\overline{\mathrm{Conv}}_{p,q}^{M}}$$
$$\left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}}M \rightarrow M\right]$$

Similarly, both of the morphisms

$$\left[\mathbf{T}^{\mathcal{W}_{D^{p}}}M \rightarrow M\right] \times \left[\mathbf{T}^{\mathcal{W}_{D^{q}}}M \rightarrow M\right]_{\mathrm{id}_M}$$
$$\xrightarrow{\mathrm{id}_{\left[\mathbf{T}^{\mathcal{W}_{D^{p}}}M \rightarrow M\right]} \times i_{\left[\mathbf{T}^{\mathcal{W}_{D^{q}}}M \rightarrow M\right]_{\mathrm{id}_M}}^{\left[\mathbf{T}^{\mathcal{W}_{D^{q}}}M \rightarrow M\right]}}$$
$$\left[\mathbf{T}^{\mathcal{W}_{D^{p}}}M \rightarrow M\right] \times \left[\mathbf{T}^{\mathcal{W}_{D^{q}}}M \rightarrow M\right]$$
$$\underrightarrow{\mathrm{Conv}_{p,q}^{M}}$$
$$\left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}}M \rightarrow M\right]$$

and

$$\left[\mathbf{T}^{\mathcal{W}_{D^{p}}}M \rightarrow M\right] \times \left[\mathbf{T}^{\mathcal{W}_{D^{q}}}M \rightarrow M\right]_{\mathrm{id}_M}$$
$$\xrightarrow{\mathrm{id}_{\left[\mathbf{T}^{\mathcal{W}_{D^{p}}}M \rightarrow M\right]} \times i_{\left[\mathbf{T}^{\mathcal{W}_{D^{q}}}M \rightarrow M\right]_{\mathrm{id}_M}}^{\left[\mathbf{T}^{\mathcal{W}_{D^{q}}}M \rightarrow M\right]}}$$
$$\left[\mathbf{T}^{\mathcal{W}_{D^{p}}}M \rightarrow M\right] \times \left[\mathbf{T}^{\mathcal{W}_{D^{q}}}M \rightarrow M\right]$$
$$\xrightarrow{\overline{\mathrm{Conv}}_{p,q}^{M}}$$
$$\left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}}M \rightarrow M\right]$$

are identical. Besides, the morphism

$$\left[\mathbf{T}^{\mathcal{W}_{D^{p}}}M \rightarrow M\right]_{\mathrm{id}_M} \times \left[\mathbf{T}^{\mathcal{W}_{D^{q}}}M \rightarrow M\right]_{\mathrm{id}_M}$$

$$\underrightarrow{i_{\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\to M\right]_{\mathrm{id}_M}}^{\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\to M\right]}\times i_{\left[\mathbf{T}^{\mathcal{W}_{D^q}}M\to M\right]_{\mathrm{id}_M}}^{\left[\mathbf{T}^{\mathcal{W}_{D^q}}M\to M\right]}}$$
$$\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\to M\right]\times\left[\mathbf{T}^{\mathcal{W}_{D^q}}M\to M\right]$$
$$\underrightarrow{\underline{\mathrm{Conv}}_{p,q}^{M}}$$
$$\left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}}M\to M\right],$$

which is identical to the morphism

$$\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\to M\right]_{\mathrm{id}_M}\times\left[\mathbf{T}^{\mathcal{W}_{D^q}}M\to M\right]_{\mathrm{id}_M}$$
$$\underrightarrow{i_{\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\to M\right]_{\mathrm{id}_M}}^{\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\to M\right]}\times i_{\left[\mathbf{T}^{\mathcal{W}_{D^q}}M\to M\right]_{\mathrm{id}_M}}^{\left[\mathbf{T}^{\mathcal{W}_{D^q}}M\to M\right]}}$$
$$\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\to M\right]\times\left[\mathbf{T}^{\mathcal{W}_{D^q}}M\to M\right]$$
$$\underrightarrow{\overline{\mathrm{Conv}}_{p,q}^{M}}$$
$$\left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}}M\to M\right],$$

is to be factored through the canonical injection

$$i_{\left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}}M\to M\right]_{\mathrm{id}_M}}^{\left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}}M\to M\right]}:\left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}}M\to M\right]_{\mathrm{id}_M}\to\left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}}M\to M\right]$$

Definition 39. We define a morphism

$$\underline{\mathrm{Pr\,od}}_{(p,m),(q,n)}^{M}:\mathbf{T}^{\mathcal{W}_{D^m}}\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\to M\right]\times\mathbf{T}^{\mathcal{W}_{D^n}}\left[\mathbf{T}^{\mathcal{W}_{D^q}}M\to M\right]$$
$$\to\mathbf{T}^{\mathcal{W}_{D^{m+n}}}\left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}}M\to M\right]$$

to be

$$\mathbf{T}^{\mathcal{W}_{D^m}}\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\to M\right]\times\mathbf{T}^{\mathcal{W}_{D^n}}\left[\mathbf{T}^{\mathcal{W}_{D^q}}M\to M\right]$$
$$\underrightarrow{\frac{\alpha_{\mathcal{W}_{(d_1,\ldots,d_m,d_{m+1},\ldots,d_{m+n})\in D^{m+n}\mapsto(d_1,\ldots,d_m)\in D^m}}\left(\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\to M\right]\right)\times}{\alpha_{\mathcal{W}_{(d_1,\ldots,d_m,d_{m+1},\ldots,d_{m+n})\in D^{m+n}\mapsto(d_{m+1},\ldots,d_{m+n})\in D^n}}\left(\left[\mathbf{T}^{\mathcal{W}_{D^q}}M\to M\right]\right)}}$$
$$\mathbf{T}^{\mathcal{W}_{D^{m+n}}}\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\to M\right]\times\mathbf{T}^{\mathcal{W}_{D^{m+n}}}\left[\mathbf{T}^{\mathcal{W}_{D^q}}M\to M\right]$$
$$=\mathbf{T}^{\mathcal{W}_{D^{m+n}}}\left(\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\to M\right]\times\left[\mathbf{T}^{\mathcal{W}_{D^q}}M\to M\right]\right)$$
$$\underrightarrow{\mathbf{T}^{\mathcal{W}_{D^{m+n}}}\underline{\mathrm{Conv}}_{p,q}^{M}}$$
$$\mathbf{T}^{\mathcal{W}_{D^{m+n}}}\left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}}M\to M\right],$$

while we define a morphism

$$\overline{\mathrm{Prod}}^{M}_{(p,m),(q,n)}:\mathbf{T}^{\mathcal{W}_{D^m}}\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\to M\right]\times\mathbf{T}^{\mathcal{W}_{D^n}}\left[\mathbf{T}^{\mathcal{W}_{D^q}}M\to M\right]$$
$$\to\mathbf{T}^{\mathcal{W}_{D^{m+n}}}\left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}}M\to M\right]$$

to be

$$\mathbf{T}^{\mathcal{W}_{D^m}}\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\to M\right]\times\mathbf{T}^{\mathcal{W}_{D^n}}\left[\mathbf{T}^{\mathcal{W}_{D^q}}M\to M\right]$$
$$\xrightarrow{\frac{\alpha_{\mathcal{W}_{(d_1,\ldots,d_m,d_{m+1},\ldots,d_{m+n})\in D^{m+n}\mapsto(d_1,\ldots,d_m)\in D^m}}\left(\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\to M\right]\right)\times}{\alpha_{\mathcal{W}_{(d_1,\ldots,d_m,d_{m+1},\ldots,d_{m+n})\in D^{m+n}\mapsto(d_{m+1},\ldots,d_{m+n})\in D^n}}\left(\left[\mathbf{T}^{\mathcal{W}_{D^q}}M\to M\right]\right)}}$$
$$\mathbf{T}^{\mathcal{W}_{D^{m+n}}}\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\to M\right]\times\mathbf{T}^{\mathcal{W}_{D^{m+n}}}\left[\mathbf{T}^{\mathcal{W}_{D^q}}M\to M\right]$$
$$=\mathbf{T}^{\mathcal{W}_{D^{m+n}}}\left(\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\to M\right]\times\left[\mathbf{T}^{\mathcal{W}_{D^q}}M\to M\right]\right)$$
$$\xrightarrow{\mathbf{T}^{\mathcal{W}_{D^{m+n}}}\overline{\mathrm{Conv}}^{M}_{p,q}}$$
$$\mathbf{T}^{\mathcal{W}_{D^{m+n}}}\left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}}M\to M\right]$$

It should be obvious that

Proposition 40. *Both* $\underline{\mathrm{Prod}}^M$ *and* $\overline{\mathrm{Prod}}^M$ *are associative in the sense that, given an object* M *in the category* $\mathcal{K}$ *and natural numbers* l,m,n,p,q,r, *the morphism*

$$\mathbf{T}^{\mathcal{W}_{D^l}}\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\to M\right]\times\mathbf{T}^{\mathcal{W}_{D^m}}\left[\mathbf{T}^{\mathcal{W}_{D^q}}M\to M\right]\times\mathbf{T}^{\mathcal{W}_{D^n}}\left[\mathbf{T}^{\mathcal{W}_{D^r}}M\to M\right]$$
$$\xrightarrow{\underline{\mathrm{Prod}}^{M}_{(p,l),(q,m)}\times\mathrm{id}_{\mathbf{T}^{\mathcal{W}_{D^n}}\left[\mathbf{T}^{\mathcal{W}_{D^r}}M\to M\right]}}$$
$$\mathbf{T}^{\mathcal{W}_{D^{m+n}}}\left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}}M\to M\right]\times\mathbf{T}^{\mathcal{W}_{D^n}}\left[\mathbf{T}^{\mathcal{W}_{D^r}}M\to M\right]$$
$$\underline{\mathrm{Prod}}^{M}_{(p+q,l+m),(r,n)}$$
$$\mathbf{T}^{\mathcal{W}_{D^{l+m+n}}}\left[\mathbf{T}^{\mathcal{W}_{D^{p+q+r}}}M\to M\right]\tag{1}$$

is identical to the morphism

$$\mathbf{T}^{\mathcal{W}_{D^l}}\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\to M\right]\times\mathbf{T}^{\mathcal{W}_{D^m}}\left[\mathbf{T}^{\mathcal{W}_{D^q}}M\to M\right]\times\mathbf{T}^{\mathcal{W}_{D^n}}\left[\mathbf{T}^{\mathcal{W}_{D^r}}M\to M\right]$$
$$\xrightarrow{\mathrm{id}_{\mathbf{T}^{\mathcal{W}_{D^l}}\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\to M\right]}\times\underline{\mathrm{Prod}}^{M}_{(q,m),(r,n)}}$$
$$\mathbf{T}^{\mathcal{W}_{D^l}}\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\to M\right]\times\mathbf{T}^{\mathcal{W}_{D^{m+n}}}\left[\mathbf{T}^{\mathcal{W}_{D^{q+r}}}M\to M\right]$$
$$\xrightarrow{\underline{\mathrm{Prod}}^{M}_{(p,l),(q+r,m+n)}}$$

$$\mathbf{T}^{\mathcal{W}_{D^{l+m+n}}}\left[\mathbf{T}^{\mathcal{W}_{D^{p+q+r}}}M\rightarrow M\right], \tag{2}$$

and similarly for $\overline{\Pr\text{od}}^{M}$.

Proof. To prove the first statement, we note that the morphism (1) is identical to the morphism

$$\begin{aligned}
&\mathbf{T}^{\mathcal{W}_{D^l}}\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\rightarrow M\right]\times\mathbf{T}^{\mathcal{W}_{D^m}}\left[\mathbf{T}^{\mathcal{W}_{D^q}}M\rightarrow M\right]\times\mathbf{T}^{\mathcal{W}_{D^n}}\left[\mathbf{T}^{\mathcal{W}_{D^r}}M\rightarrow M\right]\\
&\underline{\alpha_{\mathcal{W}_{(d_1,\ldots,d_l,d_{l+1},\ldots,d_{l+m},d_{l+m+1},\ldots,d_{l+m+n})\in D^{l+m+n}\mapsto(d_1,\ldots,d_l)\in D^l}}\left(\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\rightarrow M\right]\right)\times}\\
&\underline{\alpha_{\mathcal{W}_{(d_1,\ldots,d_l,d_{l+1},\ldots,d_{l+m},d_{l+m+1},\ldots,d_{l+m+n})\in D^{l+m+n}\mapsto(d_{l+1},\ldots,d_{l+m})\in D^m}}\left(\left[\mathbf{T}^{\mathcal{W}_{D^q}}M\rightarrow M\right]\right)\times}\\
&\xrightarrow{\alpha_{\mathcal{W}_{(d_1,\ldots,d_l,d_{l+1},\ldots,d_{l+m},d_{l+m+1},\ldots,d_{l+m+n})\in D^{l+m+n}\mapsto(d_{l+m+1},\ldots,d_{l+m+n})\in D^n}}\left(\left[\mathbf{T}^{\mathcal{W}_{D^r}}M\rightarrow M\right]\right)}\\
&\mathbf{T}^{\mathcal{W}_{D^{l+m+n}}}\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\rightarrow M\right]\times\mathbf{T}^{\mathcal{W}_{D^{l+m+n}}}\left[\mathbf{T}^{\mathcal{W}_{D^q}}M\rightarrow M\right]\times\mathbf{T}^{\mathcal{W}_{D^{l+m+n}}}\left[\mathbf{T}^{\mathcal{W}_{D^r}}M\rightarrow M\right]\\
&=\mathbf{T}^{\mathcal{W}_{D^{l+m+n}}}\left(\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\rightarrow M\right]\times\left[\mathbf{T}^{\mathcal{W}_{D^q}}M\rightarrow M\right]\times\left[\mathbf{T}^{\mathcal{W}_{D^r}}M\rightarrow M\right]\right)\\
&\xrightarrow{\mathbf{T}^{\mathcal{W}_{D^{l+m+n}}}\left(\underline{\mathrm{Conv}}_{p,q}^{M}\times\mathrm{id}_{\left[\mathbf{T}^{\mathcal{W}_{D^r}}M\rightarrow M\right]}\right)}\\
&\mathbf{T}^{\mathcal{W}_{D^{l+m+n}}}\left(\left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}}M\rightarrow M\right]\times\left[\mathbf{T}^{\mathcal{W}_{D^r}}M\rightarrow M\right]\right)\\
&\xrightarrow{\mathbf{T}^{\mathcal{W}_{D^{l+m+n}}}\left(\underline{\mathrm{Conv}}_{p+q,r}^{M}\right)}\\
&\mathbf{T}^{\mathcal{W}_{D^{l+m+n}}}\left[\mathbf{T}^{\mathcal{W}_{D^{p+q+r}}}M\rightarrow M\right]
\end{aligned}$$

while the morphism (2) is identical to the morphism

$$\begin{aligned}
&\mathbf{T}^{\mathcal{W}_{D^l}}\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\rightarrow M\right]\times\mathbf{T}^{\mathcal{W}_{D^m}}\left[\mathbf{T}^{\mathcal{W}_{D^q}}M\rightarrow M\right]\times\mathbf{T}^{\mathcal{W}_{D^n}}\left[\mathbf{T}^{\mathcal{W}_{D^r}}M\rightarrow M\right]\\
&\underline{\alpha_{\mathcal{W}_{(d_1,\ldots,d_l,d_{l+1},\ldots,d_{l+m},d_{l+m+1},\ldots,d_{l+m+n})\in D^{l+m+n}\mapsto(d_1,\ldots,d_l)\in D^l}}\left(\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\rightarrow M\right]\right)\times}\\
&\underline{\alpha_{\mathcal{W}_{(d_1,\ldots,d_l,d_{l+1},\ldots,d_{l+m},d_{l+m+1},\ldots,d_{l+m+n})\in D^{l+m+n}\mapsto(d_{l+1},\ldots,d_{l+m})\in D^m}}\left(\left[\mathbf{T}^{\mathcal{W}_{D^q}}M\rightarrow M\right]\right)\times}\\
&\xrightarrow{\alpha_{\mathcal{W}_{(d_1,\ldots,d_l,d_{l+1},\ldots,d_{l+m},d_{l+m+1},\ldots,d_{l+m+n})\in D^{l+m+n}\mapsto(d_{l+m+1},\ldots,d_{l+m+n})\in D^n}}\left(\left[\mathbf{T}^{\mathcal{W}_{D^r}}M\rightarrow M\right]\right)}\\
&\mathbf{T}^{\mathcal{W}_{D^{l+m+n}}}\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\rightarrow M\right]\times\mathbf{T}^{\mathcal{W}_{D^{l+m+n}}}\left[\mathbf{T}^{\mathcal{W}_{D^q}}M\rightarrow M\right]\times\mathbf{T}^{\mathcal{W}_{D^{l+m+n}}}\left[\mathbf{T}^{\mathcal{W}_{D^r}}M\rightarrow M\right]\\
&=\mathbf{T}^{\mathcal{W}_{D^{l+m+n}}}\left(\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\rightarrow M\right]\times\left[\mathbf{T}^{\mathcal{W}_{D^q}}M\rightarrow M\right]\times\left[\mathbf{T}^{\mathcal{W}_{D^r}}M\rightarrow M\right]\right)\\
&\xrightarrow{\mathbf{T}^{\mathcal{W}_{D^{l+m+n}}}\left(\mathrm{id}_{\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\rightarrow M\right]}\times\underline{\mathrm{Conv}}_{q,r}^{M}\right)}\\
&\mathbf{T}^{\mathcal{W}_{D^{l+m+n}}}\left(\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\rightarrow M\right]\times\left[\mathbf{T}^{\mathcal{W}_{D^{q+r}}}M\rightarrow M\right]\right)\\
&\xrightarrow{\mathbf{T}^{\mathcal{W}_{D^{l+m+n}}}\left(\underline{\mathrm{Conv}}_{p,q+r}^{M}\right)}\\
&\mathbf{T}^{\mathcal{W}_{D^{l+m+n}}}\left[\mathbf{T}^{\mathcal{W}_{D^{p+q+r}}}M\rightarrow M\right]
\end{aligned}$$

Therefore the desired result follows directly from Proposition 35. Similarly for the second statement. □

Remark 41. This proposition enables us to write

$$\begin{array}{l}\underline{\Pr\mathrm{od}}_{(p,l),(q,m),(r,n)}^{M}\\ :\mathbf{T}^{\mathcal{W}_{D^{l}}}\left[\mathbf{T}^{\mathcal{W}_{D^{p}}}M\rightarrow M\right]\times\mathbf{T}^{\mathcal{W}_{D^{m}}}\left[\mathbf{T}^{\mathcal{W}_{D^{q}}}M\rightarrow M\right]\times\mathbf{T}^{\mathcal{W}_{D^{n}}}\left[\mathbf{T}^{\mathcal{W}_{D^{r}}}M\rightarrow M\right]\\ \rightarrow\mathbf{T}^{\mathcal{W}_{D^{l+m+n}}}\left[\mathbf{T}^{\mathcal{W}_{D^{p+q+r}}}M\rightarrow M\right]\end{array}$$

to denote one of the above two identical morphisms without any ambiguity, and similarly for

$$\begin{array}{l}\overline{\Pr\mathrm{od}}_{(p,l),(q,m),(r,n)}^{M}\\ :\mathbf{T}^{\mathcal{W}_{D^{l}}}\left[\mathbf{T}^{\mathcal{W}_{D^{p}}}M\rightarrow M\right]\times\mathbf{T}^{\mathcal{W}_{D^{m}}}\left[\mathbf{T}^{\mathcal{W}_{D^{q}}}M\rightarrow M\right]\times\mathbf{T}^{\mathcal{W}_{D^{n}}}\left[\mathbf{T}^{\mathcal{W}_{D^{r}}}M\rightarrow M\right]\\ \rightarrow\mathbf{T}^{\mathcal{W}_{D^{l+m+n}}}\left[\mathbf{T}^{\mathcal{W}_{D^{p+q+r}}}M\rightarrow M\right]\end{array}$$

5.2. The First Consideration

In this subsection we are concerned with the Lie algebra structure of $\overline{\Omega}_{(1)}^{(p,1)}(M)$'s. Let us begin with

Lemma 42. *The morphism*

$$\begin{array}{l}\overline{\Omega}_{(1)}^{(p,1)}(M)\times\overline{\Omega}_{(1)}^{(q,1)}(M)\\ \xrightarrow{i_{\overline{\Omega}_{(1)}^{(p,1)}(M)}^{\overline{\Omega}_{(0)}^{(p,1)}(M)}\times i_{\overline{\Omega}_{(1)}^{(q,1)}(M)}^{\overline{\Omega}_{(0)}^{(q,1)}(M)}}\\ \overline{\Omega}_{(0)}^{(p,1)}(M)\times\overline{\Omega}_{(0)}^{(q,1)}(M)\\ \xrightarrow{\underline{\Pr\mathrm{od}}_{(p,1),(q,1)}^{M}}\\ \mathbf{T}^{\mathcal{W}_{D^{2}}}\left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}}M\rightarrow M\right]\\ \xrightarrow{\alpha_{\mathcal{W}_{(d_{1},d_{2})\in D(2)\mapsto(d_{1},d_{2})\in D^{2}}}\left(\left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}}M\rightarrow M\right]\right)}\\ \mathbf{T}^{\mathcal{W}_{D(2)}}\left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}}M\rightarrow M\right]\end{array}$$

is identical to the morphism

$$\begin{array}{l}\overline{\Omega}_{(1)}^{(p,1)}(M)\times\overline{\Omega}_{(1)}^{(q,1)}(M)\\ \xrightarrow{i_{\overline{\Omega}_{(1)}^{(p,1)}(M)}^{\overline{\Omega}_{(0)}^{(p,1)}(M)}\times i_{\overline{\Omega}_{(1)}^{(q,1)}(M)}^{\overline{\Omega}_{(0)}^{(q,1)}(M)}}\\ \overline{\Omega}_{(0)}^{(p,1)}(M)\times\overline{\Omega}_{(0)}^{(q,1)}(M)\end{array}$$

$$\xrightarrow{\overline{\mathrm{Prod}}^M_{(p,1),(q,1)}} \mathbf{T}^{\mathcal{W}_{D^2}}\left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}}M \to M\right] \xrightarrow{\alpha_{\mathcal{W}_{(d_1,d_2)\in D(2)\mapsto(d_1,d_2)\in D^2}}\left(\left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}}M \to M\right]\right)} \mathbf{T}^{\mathcal{W}_{D(2)}}\left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}}M \to M\right]$$

Proof. By Proposition 38, the morphism

$$\overline{\Omega}^{(p,1)}_{(1)}(M) \times \overline{\Omega}^{(q,1)}_{(1)}(M) \xrightarrow{i^{\overline{\Omega}^{(p,1)}_{(0)}(M)}_{\overline{\Omega}^{(p,1)}_{(1)}(M)} \times i^{\overline{\Omega}^{(q,1)}_{(0)}(M)}_{\overline{\Omega}^{(q,1)}_{(1)}(M)}} \overline{\Omega}^{(p,1)}_{(0)}(M) \times \overline{\Omega}^{(q,1)}_{(0)}(M) \xrightarrow{\underline{\mathrm{Prod}}^M_{(p,1),(q,1)}} \mathbf{T}^{\mathcal{W}_{D^2}}\left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}}M \to M\right]$$

is identical to the morphism

$$\overline{\Omega}^{(p,1)}_{(1)}(M) \times \overline{\Omega}^{(q,1)}_{(1)}(M) \xrightarrow{i^{\overline{\Omega}^{(p,1)}_{(0)}(M)}_{\overline{\Omega}^{(p,1)}_{(1)}(M)} \times i^{\overline{\Omega}^{(q,1)}_{(0)}(M)}_{\overline{\Omega}^{(q,1)}_{(1)}(M)}} \overline{\Omega}^{(p,1)}_{(0)}(M) \times \overline{\Omega}^{(q,1)}_{(0)}(M) \xrightarrow{\overline{\mathrm{Prod}}^M_{(p,1),(q,1)}} \mathbf{T}^{\mathcal{W}_{D^2}}\left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}}M \to M\right]$$

so that their compositions with the morphism

$$\mathbf{T}^{\mathcal{W}_{D^2}}\left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}}M \to M\right] \xrightarrow{\alpha_{\mathcal{W}_{(d_1,d_2)\in D(2)\mapsto(d_1,d_2)\in D^2}}\left(\left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}}M \to M\right]\right)} \mathbf{T}^{\mathcal{W}_{D(2)}}\left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}}M \to M\right]$$

should evidently be identical. □

Corollary 43. *The morphism*

$$\overline{\Omega}^{(p,1)}_{(1)}(M) \times \overline{\Omega}^{(q,1)}_{(1)}(M)$$

$$\xrightarrow{i^{\overline{\Omega}_{(0)}^{(p,1)}(M)}_{\overline{\Omega}_{(1)}^{(p,1)}(M)} \times i^{\overline{\Omega}_{(0)}^{(q,1)}(M)}_{\overline{\Omega}_{(1)}^{(q,1)}(M)}}$$

$$\overline{\Omega}_{(0)}^{(p,1)}(M) \times \overline{\Omega}_{(0)}^{(q,1)}(M)$$

$$\xrightarrow{\left(\underline{\Pr\text{od}}_{(p,1),(q,1)}^{M}, \overline{\Pr\text{od}}_{(p,1),(q,1)}^{M}\right)}$$

$$\mathbf{T}^{\mathcal{W}_{D^2}}\left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}}M \to M\right] \times \mathbf{T}^{\mathcal{W}_{D^2}}\left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}}M \to M\right]$$

is to be factored through the canonical injection

$$\mathbf{T}^{\mathcal{W}_{D^2}}\left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}}M \to M\right] \times_{\mathbf{T}^{\mathcal{W}_{D(2)}}\left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}}M \to M\right]} \mathbf{T}^{\mathcal{W}_{D^2}}\left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}}M \to M\right]$$

$$\to \mathbf{T}^{\mathcal{W}_{D^2}}\left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}}M \to M\right] \times \mathbf{T}^{\mathcal{W}_{D^2}}\left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}}M \to M\right]$$

It is easy to see that

Proposition 44. *The factored morphism*

$$\overline{\Omega}_{(1)}^{(p,1)}(M) \times \overline{\Omega}_{(1)}^{(q,1)}(M)$$

$$\to \mathbf{T}^{\mathcal{W}_{D^2}}\left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}}M \to M\right] \times_{\mathbf{T}^{\mathcal{W}_{D(2)}}\left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}}M \to M\right]} \mathbf{T}^{\mathcal{W}_{D^2}}\left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}}M \to M\right]$$

in Corollary 43 followed by the morphism

$$\mathbf{T}^{\mathcal{W}_{D^2}}\left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}}M \to M\right] \times_{\mathbf{T}^{\mathcal{W}_{D(2)}}\left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}}M \to M\right]} \mathbf{T}^{\mathcal{W}_{D^2}}\left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}}M \to M\right]$$

$$\xrightarrow{\dot{\zeta}^{-}\left(\left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}}M \to M\right]\right)}$$

$$\mathbf{T}^{\mathcal{W}_{D}}\left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}}M \to M\right]$$

is to be factored uniquely into a morphism

$$\zeta_{p,q}^{L_1}(M) : \overline{\Omega}_{(1)}^{(p,1)}(M) \times \overline{\Omega}_{(1)}^{(q,1)}(M) \to \overline{\Omega}_{(1)}^{(p+q,1)}(M)$$

and the canonical injection

$$i^{\mathbf{T}^{\mathcal{W}_{D}}\left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}}M \to M\right]}_{\overline{\Omega}_{(1)}^{(p+q,1)}(M)} : \overline{\Omega}_{(1)}^{(p+q,1)}(M) \to \mathbf{T}^{\mathcal{W}_{D}}\left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}}M \to M\right]$$

Notation 45. Given $\xi \in [M \otimes \mathcal{W}_{D^p} \to M] \otimes \mathcal{W}_{D^n}$ and $\sigma \in \mathbb{S}_p$, ξ^σ denotes

$$\left(()^{\sigma}_{[M \otimes \mathcal{W}_{D^p} \to M]} \otimes \mathrm{id}_{\mathcal{W}_{D^n}}\right)(\xi)$$

where $(\,)^{\sigma}_{[M\otimes\mathcal{W}_{D^p}\to M]} : [M\otimes\mathcal{W}_{D^p}\to M]\to[M\otimes\mathcal{W}_{D^p}\to M]$ denotes the operation

$$\eta\in[M\otimes\mathcal{W}_{D^p}\to M]\mapsto\eta\circ\left(\mathrm{id}_M\otimes\mathcal{W}_{(d_1,\ldots,d_p)\in D^p\mapsto\left(d_{\sigma(1)},\ldots,d_{\sigma(p)}\right)\in D^p}\right)$$

We will show that the morphism

$$\zeta^{L_1}_{p,q}(M):\overline{\Omega}^{(p,1)}_{(1)}(M)\times\overline{\Omega}^{(q,1)}_{(1)}(M)\to\overline{\Omega}^{(p+q,1)}_{(1)}(M)$$

is antisymmetric in the following sense.

Proposition 46. *The morphism*

$$\begin{aligned}&\overline{\Omega}^{(p,1)}_{(1)}(M)\times\overline{\Omega}^{(q,1)}_{(1)}(M)\\&\xrightarrow{\zeta^{L_1}_{p,q}(M)}\\&\overline{\Omega}^{(p+q,1)}_{(1)}(M)\end{aligned}\tag{3}$$

and the morphism

$$\begin{aligned}&\overline{\Omega}^{(p,1)}_{(1)}(M)\times\overline{\Omega}^{(q,1)}_{(1)}(M)\\&=\overline{\Omega}^{(q,1)}_{(1)}(M)\times\overline{\Omega}^{(p,1)}_{(1)}(M)\\&\xrightarrow{\zeta^{L_1}_{q,p}(M)}\\&\overline{\Omega}^{(p+q,1)}_{(1)}(M)\\&\xrightarrow{(\cdot^{\sigma_{p,q}})_{\overline{\Omega}^{(p+q,1)}_{(1)}(M)}}\\&\overline{\Omega}^{(p+q,1)}_{(1)}(M)\end{aligned}\tag{4}$$

sum up only to vanish.

Proof. This follows easily from Propositions 6 and 34.It is easy to see that the morphism

$$\begin{aligned}&\overline{\Omega}^{(p,1)}_{(1)}(M)\times\overline{\Omega}^{(q,1)}_{(1)}(M)\\&\xrightarrow{i^{\overline{\Omega}^{(p,1)}_{(0)}(M)}_{\overline{\Omega}^{(p,1)}_{(1)}(M)}\times i^{\overline{\Omega}^{(q,1)}_{(0)}(M)}_{\overline{\Omega}^{(q,1)}_{(1)}(M)}}\\&\overline{\Omega}^{(q,1)}_{(0)}(M)\times\overline{\Omega}^{(p,1)}_{(0)}(M)\end{aligned}$$

$$\xrightarrow{\left(\underline{\Pr\mathrm{od}}^M_{(q,1),(p,1)},\overline{\Pr\mathrm{od}}^M_{(q,1),(p,1)}\right)}$$

$$\mathbf{T}^{\mathcal{W}_{D^2}}\left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}}M\rightarrow M\right]\times_{\mathbf{T}^{\mathcal{W}_{D(2)}}\left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}}M\rightarrow M\right]}\mathbf{T}^{\mathcal{W}_{D^2}}\left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}}M\rightarrow M\right]$$

$$\xrightarrow{\zeta^{\dot{-}}\left(\left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}}M\rightarrow M\right]\right)}$$

$$\overline{\Omega}^{(p+q,1)}_{(0)}(M)$$

$$\xrightarrow{\left(\cdot^{\sigma_{p,q}}\right)_{\overline{\Omega}^{(p+q,1)}_{(0)}(M)}}$$

$$\overline{\Omega}^{(p+q,1)}_{(0)}(M)$$

is identical to the morphism

$$\overline{\Omega}^{(p,1)}_{(1)}(M)\times\overline{\Omega}^{(q,1)}_{(1)}(M)$$

$$\xrightarrow{i^{\overline{\Omega}^{(p,1)}_{(0)}(M)}_{\overline{\Omega}^{(p,1)}_{(1)}(M)}\times i^{\overline{\Omega}^{(q,1)}_{(0)}(M)}_{\overline{\Omega}^{(q,1)}_{(1)}(M)}}$$

$$\overline{\Omega}^{(q,1)}_{(0)}(M)\times\overline{\Omega}^{(p,1)}_{(0)}(M)$$

$$\xrightarrow{\left(\left(\cdot^{\sigma_{p,q}}\right)_{\overline{\Omega}^{(p+q,1)}_{(0)}(M)}\circ\underline{\Pr\mathrm{od}}^M_{(q,1),(p,1)},\left(\cdot^{\sigma_{p,q}}\right)_{\overline{\Omega}^{(p+q,1)}_{(0)}(M)}\circ\overline{\Pr\mathrm{od}}^M_{(q,1),(p,1)}\right)}$$

$$\mathbf{T}^{\mathcal{W}_{D^2}}\left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}}M\rightarrow M\right]\times_{\mathbf{T}^{\mathcal{W}_{D(2)}}\left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}}M\rightarrow M\right]}\mathbf{T}^{\mathcal{W}_{D^2}}\left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}}M\rightarrow M\right]$$

$$\xrightarrow{\zeta^{\dot{-}}\left(\left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}}M\rightarrow M\right]\right)}$$

$$\overline{\Omega}^{(p+q,1)}_{(0)}(M)$$

However we know well by Proposition 34 that

This follows from Propositions 4 and 6 in §3.4 of Lavendhomme [5]. More specifically we have

$$\begin{aligned}
&[\xi_1,\xi_2]_L+\left([\xi_2,\xi_1]_L\right)^{\sigma_{p,q}}\\
&=\left(\xi_1\widetilde{\circledast}\xi_2\dot{-}\xi_1\circledast\xi_2\right)+\left(\left(\xi_2\widetilde{\circledast}\xi_1\right)^{\sigma_{p,q}}\dot{-}\left(\xi_2\circledast\xi_1\right)^{\sigma_{p,q}}\right)\\
&=\left(\xi_1\widetilde{\circledast}\xi_2\dot{-}\xi_1\circledast\xi_2\right)+\left(\xi_1\circledast\xi_2\dot{-}\xi_1\widetilde{\circledast}\xi_2\right)\\
&\text{[By Proposition ??]}\\
&=0.
\end{aligned}$$

□

We have the following Jacobi identity.

Theorem 47. *The three morphisms*

$$\begin{aligned}
&\overline{\Omega}_{(1)}^{(p,1)}(M)\times\overline{\Omega}_{(1)}^{(q,1)}(M)\times\overline{\Omega}_{(1)}^{(r,1)}(M)\\
&\underrightarrow{\mathrm{id}_{\overline{\Omega}_{(1)}^{(p,1)}(M)}\times\zeta_{q,r}^{L_1}(M)}\\
&\overline{\Omega}_{(1)}^{(p,1)}(M)\times\overline{\Omega}_{(1)}^{(q+r,1)}(M)\\
&\underrightarrow{\zeta_{p,q+r}^{L_1}(M)}\\
&\overline{\Omega}_{(1)}^{(p+q+r,1)}(M)\,,
\end{aligned}\tag{5}$$

$$\begin{aligned}
&\overline{\Omega}_{(1)}^{(p,1)}(M)\times\overline{\Omega}_{(1)}^{(q,1)}(M)\times\overline{\Omega}_{(1)}^{(r,1)}(M)\\
&=\overline{\Omega}_{(1)}^{(q,1)}(M)\times\overline{\Omega}_{(1)}^{(r,1)}(M)\times\overline{\Omega}_{(1)}^{(p,1)}(M)\\
&\underrightarrow{\mathrm{id}_{\overline{\Omega}_{(1)}^{(q,1)}(M)}\times\zeta_{r,p}^{L_1}(M)}\\
&\overline{\Omega}_{(1)}^{(q,1)}(M)\times\overline{\Omega}_{(1)}^{(r+p,1)}(M)\\
&\underrightarrow{\zeta_{q,r+p}^{L_1}(M)}\\
&\overline{\Omega}_{(1)}^{(p+q+r,1)}(M)\\
&\underrightarrow{(\cdot^{\sigma_{p,q+r}})_{\overline{\Omega}_{(1)}^{(p+q+r,1)}(M)}}\\
&\overline{\Omega}_{(1)}^{(p+q+r,1)}(M)
\end{aligned}\tag{6}$$

and

$$\begin{aligned}
&\overline{\Omega}_{(1)}^{(p,1)}(M)\times\overline{\Omega}_{(1)}^{(q,1)}(M)\times\overline{\Omega}_{(1)}^{(r,1)}(M)\\
&=\overline{\Omega}_{(1)}^{(r,1)}(M)\times\overline{\Omega}_{(1)}^{(p,1)}(M)\times\overline{\Omega}_{(1)}^{(q,1)}(M)\\
&\underrightarrow{\mathrm{id}_{\overline{\Omega}_{(1)}^{(r,1)}(M)}\times\zeta_{p,q}^{L_1}(M)}\\
&\overline{\Omega}_{(1)}^{(r,1)}(M)\times\overline{\Omega}_{(1)}^{(p+q,1)}(M)\\
&\underrightarrow{\zeta_{r,p+q}^{L_1}(M)}\\
&\overline{\Omega}_{(1)}^{(p+q+r,1)}(M)
\end{aligned}$$

$$\xrightarrow{\left(\cdot^{\sigma_{r,p+q}}\right)_{\overline{\Omega}_{(1)}^{(p+q+r,1)}(M)}} \overline{\Omega}_{(1)}^{(p+q+r,1)}(M) \tag{7}$$

sum up only to vanish.

In order to establish the above theorem, we need the following simple lemma, which is a tiny generalization of Proposition 2.6 of [9].

Lemma 48. *The morphism*

$$\overline{\Omega}_{(1)}^{(p,1)}(M) \times \left(\overline{\Omega}_{(1)}^{(q,2)}(M) \times_{\mathbf{T}^{\mathcal{W}_{D(2)}}\left[\mathbf{T}^{\mathcal{W}_{D^q}}M \to M\right]} \overline{\Omega}_{(1)}^{(q,2)}(M)\right)$$
$$\xrightarrow{\mathrm{id}_{\overline{\Omega}_{(1)}^{(p,1)}(M)} \times \dot{\zeta}^{-}\left(\left[\mathbf{T}^{\mathcal{W}_{D^q}}M \to M\right]\right)}$$
$$\overline{\Omega}_{(1)}^{(p,1)}(M) \times \overline{\Omega}_{(1)}^{(q,1)}(M)$$
$$\xrightarrow{\mathrm{Pr\,od}_{(p,1),(q,1)}^{M}}$$
$$\overline{\Omega}_{(1)}^{(p+q,2)}(M)$$

is identical to

$$\overline{\Omega}_{(1)}^{(p,1)}(M) \times \left(\overline{\Omega}_{(1)}^{(q,2)}(M) \times_{\mathbf{T}^{\mathcal{W}_{D(2)}}\left[\mathbf{T}^{\mathcal{W}_{D^q}}M \to M\right]} \overline{\Omega}_{(1)}^{(q,2)}(M)\right)$$
$$= \left(\overline{\Omega}_{(1)}^{(p,1)}(M) \times \overline{\Omega}_{(1)}^{(q,2)}(M)\right) \times_{\mathbf{T}^{\mathcal{W}_{D(2)}}\left[\mathbf{T}^{\mathcal{W}_{D^q}}M \to M\right]} \left(\overline{\Omega}_{(1)}^{(p,1)}(M) \times \overline{\Omega}_{(1)}^{(q,2)}(M)\right)$$
$$\xrightarrow{\mathrm{Pr\,od}_{(p,1),(q,2)}^{M} \times_{\mathbf{T}^{\mathcal{W}_{D(2)}}\left[\mathbf{T}^{\mathcal{W}_{D^q}}M \to M\right]} \mathrm{Pr\,od}_{(p,1),(q,2)}^{M}}$$
$$\overline{\Omega}_{(1)}^{(p+q,3)}(M) \times_{\mathbf{T}^{\mathcal{W}_{D^3\{(2,3)\}}}\left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}}M \to M\right]} \overline{\Omega}_{(1)}^{(p+q,3)}(M)$$
$$\xrightarrow{\dot{\zeta}^{\overline{1}}\left(\left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}}M \to M\right]\right)}$$
$$\overline{\Omega}_{(1)}^{(p+q,2)}(M),$$

while the morphism

$$\left(\overline{\Omega}_{(1)}^{(p,2)}(M) \times_{\mathbf{T}^{\mathcal{W}_{D(2)}}\left[\mathbf{T}^{\mathcal{W}_{D^p}}M \to M\right]} \overline{\Omega}_{(1)}^{(p,2)}(M)\right) \times \overline{\Omega}_{(1)}^{(q,1)}(M)$$
$$\xrightarrow{\dot{\zeta}^{-}\left(\left[\mathbf{T}^{\mathcal{W}_{D^p}}M \to M\right]\right) \times \mathrm{id}_{\overline{\Omega}_{(1)}^{(q,1)}(M)}}$$

$$\overline{\Omega}_{(1)}^{(p,1)}(M)\times\overline{\Omega}_{(1)}^{(q,1)}(M)\xrightarrow{\mathrm{Prod}_{(p,1),(q,1)}^{M}}\overline{\Omega}_{(1)}^{(p+q,2)}(M)$$

is identical to

$$\left(\overline{\Omega}_{(1)}^{(p,2)}(M)\times_{\mathbf{T}^{\mathcal{W}_{D(2)}}\left[\mathbf{T}^{\mathcal{W}_{D^p}}M\to M\right]}\overline{\Omega}_{(1)}^{(p,2)}(M)\right)\times\overline{\Omega}_{(1)}^{(q,1)}(M)$$
$$=\left(\overline{\Omega}_{(1)}^{(p,2)}(M)\times\overline{\Omega}_{(1)}^{(q,1)}(M)\right)\times_{\mathbf{T}^{\mathcal{W}_{D(2)}}\left[\mathbf{T}^{\mathcal{W}_{D^q}}M\to M\right]}\left(\overline{\Omega}_{(1)}^{(p,2)}(M)\times\overline{\Omega}_{(1)}^{(q,1)}(M)\right)$$
$$\xrightarrow{\mathrm{Prod}_{(p,2),(q,1)}^{M}\times_{\mathbf{T}^{\mathcal{W}_{D(2)}}\left[\mathbf{T}^{\mathcal{W}_{D^q}}M\to M\right]}\mathrm{Prod}_{(p,2),(q,1)}^{M}}$$
$$\overline{\Omega}_{(1)}^{(p+q,3)}(M)\times_{\mathbf{T}^{\mathcal{W}_{D^3\{1,(2)\}}}\left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}}M\to M\right]}\overline{\Omega}_{(1)}^{(p+q,3)}(M)$$
$$\xrightarrow{\zeta_{3}^{\overset{\cdot}{-}}\left(\left[\mathbf{T}^{\mathcal{W}_{D^{p+q}}}M\to M\right]\right)}$$
$$\overline{\Omega}_{(1)}^{(p+q,2)}(M)$$

Notation 49. For the sake of the proof of Theorem 47, we introduce the following six notations:

1. We write ξ_{123} for the morphism

$$\overline{\Omega}_{(1)}^{(p,1)}(M)\times\overline{\Omega}_{(1)}^{(q,1)}(M)\times\overline{\Omega}_{(1)}^{(r,1)}(M)\xrightarrow{\underline{\mathrm{Prod}}_{(p,1),(q,1),(r,1)}^{M}}\overline{\Omega}_{(1)}^{(p+q+r,3)}(M)$$

2. We write ξ_{132} for the morphism

$$\overline{\Omega}_{(1)}^{(p,1)}(M)\times\overline{\Omega}_{(1)}^{(q,1)}(M)\times\overline{\Omega}_{(1)}^{(r,1)}(M)\xrightarrow{\mathrm{id}_{\overline{\Omega}_{(1)}^{(p,1)}(M)}\times\overline{\mathrm{Prod}}_{(q,1),(r,1)}^{M}}\overline{\Omega}_{(1)}^{(p,1)}(M)\times\overline{\Omega}_{(1)}^{(q+r,1)}(M)\xrightarrow{\underline{\mathrm{Prod}}_{(p,1),(q+r,1)}^{M}}\overline{\Omega}_{(1)}^{(p+q+r,3)}(M)$$

3. We write ξ_{213} for the morphism

$$\overline{\Omega}_{(1)}^{(p,1)}(M)\times\overline{\Omega}_{(1)}^{(q,1)}(M)\times\overline{\Omega}_{(1)}^{(r,1)}(M)$$
$$\xrightarrow{\overline{\mathrm{Pr\,od}}_{(p,1),(q,1)}^{M}\times\mathrm{id}_{\overline{\Omega}_{(1)}^{(r,1)}(M)}}$$
$$\overline{\Omega}_{(1)}^{(p+q,1)}(M)\times\overline{\Omega}_{(1)}^{(r,1)}(M)$$
$$\xrightarrow{\underline{\mathrm{Pr\,od}}_{(p+q,1),(r,1)}^{M}}$$
$$\overline{\Omega}_{(1)}^{(p+q+r,3)}(M)$$

4. We write ξ_{231} for the morphism

$$\overline{\Omega}_{(1)}^{(p,1)}(M)\times\overline{\Omega}_{(1)}^{(q,1)}(M)\times\overline{\Omega}_{(1)}^{(r,1)}(M)$$
$$\xrightarrow{\mathrm{id}_{\overline{\Omega}_{(1)}^{(p,1)}(M)}\times\underline{\mathrm{Pr\,od}}_{(q,1),(r,1)}^{M}}$$
$$\overline{\Omega}_{(1)}^{(p,1)}(M)\times\overline{\Omega}_{(1)}^{(q+r,1)}(M)$$
$$\xrightarrow{\overline{\mathrm{Pr\,od}}_{(p,1),(q+r,1)}^{M}}$$
$$\overline{\Omega}_{(1)}^{(p+q+r,3)}(M)$$

5. We write ξ_{312} for the morphism

$$\overline{\Omega}_{(1)}^{(p,1)}(M)\times\overline{\Omega}_{(1)}^{(q,1)}(M)\times\overline{\Omega}_{(1)}^{(r,1)}(M)$$
$$\xrightarrow{\underline{\mathrm{Pr\,od}}_{(p,1),(q,1)}^{M}\times\mathrm{id}_{\overline{\Omega}_{(1)}^{(r,1)}(M)}}$$
$$\overline{\Omega}_{(1)}^{(p+q,1)}(M)\times\overline{\Omega}_{(1)}^{(r,1)}(M)$$
$$\xrightarrow{\overline{\mathrm{Pr\,od}}_{(p+q,1),(r,1)}^{M}}$$
$$\overline{\Omega}_{(1)}^{(p+q+r,3)}(M)$$

6. We write ξ_{321} for the morphism

$$\overline{\Omega}_{(1)}^{(p,1)}(M)\times\overline{\Omega}_{(1)}^{(q,1)}(M)\times\overline{\Omega}_{(1)}^{(r,1)}(M)$$
$$\xrightarrow{\overline{\mathrm{Pr\,od}}_{(p,1),(q,1),(r,1)}^{M}}$$
$$\overline{\Omega}_{(1)}^{(p+q+r,3)}(M)$$

Lemma 50. *We have the following statements:*

1. *The morphism*

$$\overline{\Omega}_{(1)}^{(p,1)}(M)\times\overline{\Omega}_{(1)}^{(q,1)}(M)\times\overline{\Omega}_{(1)}^{(r,1)}(M) \xrightarrow{(\xi_{123},\xi_{132})} \overline{\Omega}_{(1)}^{(p+q+r,3)}(M)\times\overline{\Omega}_{(1)}^{(p+q+r,3)}(M)$$

is to be factored uniquely through the canonical injection

$$\overline{\Omega}_{(1)}^{(p+q+r,3)}(M)\times_{\mathbf{T}^{\mathcal{W}_{D^3\{(2,3)\}}}\left[\mathbf{T}^{\mathcal{W}_{D^{p+q+r}}}M\to M\right]}\overline{\Omega}_{(1)}^{(p+q+r,3)}(M) \to \overline{\Omega}_{(1)}^{(p+q+r,3)}(M)\times\overline{\Omega}_{(1)}^{(p+q+r,3)}(M)$$

into

$$\overline{\Omega}_{(1)}^{(p,1)}(M)\times\overline{\Omega}_{(1)}^{(q,1)}(M)\times\overline{\Omega}_{(1)}^{(r,1)}(M) \to \overline{\Omega}_{(1)}^{(p+q+r,3)}(M)\times_{\mathbf{T}^{\mathcal{W}_{D^3\{(2,3)\}}}\left[\mathbf{T}^{\mathcal{W}_{D^{p+q+r}}}M\to M\right]}\overline{\Omega}_{(1)}^{(p+q+r,3)}(M) \qquad (8)$$

2. *The morphism*

$$\overline{\Omega}_{(1)}^{(p,1)}(M)\times\overline{\Omega}_{(1)}^{(q,1)}(M)\times\overline{\Omega}_{(1)}^{(r,1)}(M) \xrightarrow{(\xi_{231},\xi_{321})} \overline{\Omega}_{(1)}^{(p+q+r,3)}(M)\times\overline{\Omega}_{(1)}^{(p+q+r,3)}(M)$$

is to be factored uniquely through the canonical injection

$$\overline{\Omega}_{(1)}^{(p+q+r,3)}(M)\times_{\mathbf{T}^{\mathcal{W}_{D^3\{(2,3)\}}}\left[\mathbf{T}^{\mathcal{W}_{D^{p+q+r}}}M\to M\right]}\overline{\Omega}_{(1)}^{(p+q+r,3)}(M) \to \overline{\Omega}_{(1)}^{(p+q+r,3)}(M)\times\overline{\Omega}_{(1)}^{(p+q+r,3)}(M)$$

into

$$\overline{\Omega}_{(1)}^{(p,1)}(M)\times\overline{\Omega}_{(1)}^{(q,1)}(M)\times\overline{\Omega}_{(1)}^{(r,1)}(M) \to \overline{\Omega}_{(1)}^{(p+q+r,3)}(M)\times_{\mathbf{T}^{\mathcal{W}_{D^3\{(2,3)\}}}\left[\mathbf{T}^{\mathcal{W}_{D_{p+q+r}}}M\to M\right]}\overline{\Omega}_{(1)}^{(p+q+r,3)}(M) \qquad (9)$$

3. *The morphism*

$$\overline{\Omega}_{(1)}^{(p,1)}(M) \times \overline{\Omega}_{(1)}^{(q,1)}(M) \times \overline{\Omega}_{(1)}^{(r,1)}(M) \xrightarrow{(\xi_{231},\xi_{213})} \overline{\Omega}_{(1)}^{(p+q+r,3)}(M) \times \overline{\Omega}_{(1)}^{(p+q+r,3)}(M)$$

is to be factored uniquely through the canonical injection

$$\overline{\Omega}_{(1)}^{(p+q+r,3)}(M) \times_{\mathbf{T}^{\mathcal{W}_{D^3\{(1,3)\}}}\left[\mathbf{T}^{\mathcal{W}_{D^{p+q+r}}}M \to M\right]} \overline{\Omega}_{(1)}^{(p+q+r,3)}(M) \to \overline{\Omega}_{(1)}^{(p+q+r,3)}(M) \times \overline{\Omega}_{(1)}^{(p+q+r,3)}(M)$$

into

$$\overline{\Omega}_{(1)}^{(p,1)}(M) \times \overline{\Omega}_{(1)}^{(q,1)}(M) \times \overline{\Omega}_{(1)}^{(r,1)}(M) \to \overline{\Omega}_{(1)}^{(p+q+r,3)}(M) \times_{\mathbf{T}^{\mathcal{W}_{D^3\{(1,3)\}}}\left[\mathbf{T}^{\mathcal{W}_{D^{p+q+r}}}M \to M\right]} \overline{\Omega}_{(1)}^{(p+q+r,3)}(M) \quad (10)$$

4. *The morphism*

$$\overline{\Omega}_{(1)}^{(p,1)}(M) \times \overline{\Omega}_{(1)}^{(q,1)}(M) \times \overline{\Omega}_{(1)}^{(r,1)}(M) \xrightarrow{(\xi_{312},\xi_{132})} \overline{\Omega}_{(1)}^{(p+q+r,3)}(M) \times \overline{\Omega}_{(1)}^{(p+q+r,3)}(M)$$

is to be factored uniquely through the canonical injection

$$\overline{\Omega}_{(1)}^{(p+q+r,3)}(M) \times_{\mathbf{T}^{\mathcal{W}_{D^3\{(1,3)\}}}\left[\mathbf{T}^{\mathcal{W}_{D^{p+q+r}}}M \to M\right]} \overline{\Omega}_{(1)}^{(p+q+r,3)}(M) \to \overline{\Omega}_{(1)}^{(p+q+r,3)}(M) \times \overline{\Omega}_{(1)}^{(p+q+r,3)}(M)$$

into

$$\overline{\Omega}_{(1)}^{(p,1)}(M) \times \overline{\Omega}_{(1)}^{(q,1)}(M) \times \overline{\Omega}_{(1)}^{(r,1)}(M) \to \overline{\Omega}_{(1)}^{(p+q+r,3)}(M) \times_{\mathbf{T}^{\mathcal{W}_{D^3\{(1,3)\}}}\left[\mathbf{T}^{\mathcal{W}_{D^{p+q+r}}}M \to M\right]} \overline{\Omega}_{(1)}^{(p+q+r,3)}(M) \quad (11)$$

5. *The morphism*

$$\overline{\Omega}_{(1)}^{(p,1)}(M) \times \overline{\Omega}_{(1)}^{(q,1)}(M) \times \overline{\Omega}_{(1)}^{(r,1)}(M)$$

$$\xrightarrow{(\xi_{312},\xi_{132})} \overline{\Omega}_{(1)}^{(p+q+r,3)}(M)\times\overline{\Omega}_{(1)}^{(p+q+r,3)}(M)$$

is to be factored uniquely through the canonical injection

$$\overline{\Omega}_{(1)}^{(p+q+r,3)}(M)\times_{\mathbf{T}^{\mathcal{W}_{D^3\{(1,2)\}}}\left[\mathbf{T}^{\mathcal{W}_{D^{p+q+r}}}M\rightarrow M\right]}\overline{\Omega}_{(1)}^{(p+q+r,3)}(M)$$
$$\rightarrow\overline{\Omega}_{(1)}^{(p+q+r,3)}(M)\times\overline{\Omega}_{(1)}^{(p+q+r,3)}(M)$$

into

$$\overline{\Omega}_{(1)}^{(p,1)}(M)\times\overline{\Omega}_{(1)}^{(q,1)}(M)\times\overline{\Omega}_{(1)}^{(r,1)}(M)$$
$$\rightarrow\overline{\Omega}_{(1)}^{(p+q+r,3)}(M)\times_{\mathbf{T}^{\mathcal{W}_{D^3\{(1,2)\}}}\left[\mathbf{T}^{\mathcal{W}_{D^{p+q+r}}}M\rightarrow M\right]}\overline{\Omega}_{(1)}^{(p+q+r,3)}(M) \qquad (12)$$

6. *The morphism*

$$\overline{\Omega}_{(1)}^{(p,1)}(M)\times\overline{\Omega}_{(1)}^{(q,1)}(M)\times\overline{\Omega}_{(1)}^{(r,1)}(M)$$
$$\xrightarrow{(\xi_{312},\xi_{132})}$$
$$\overline{\Omega}_{(1)}^{(p+q+r,3)}(M)\times\overline{\Omega}_{(1)}^{(p+q+r,3)}(M)$$

is to be factored uniquely through the canonical injection

$$\overline{\Omega}_{(1)}^{(p+q+r,3)}(M)\times_{\mathbf{T}^{\mathcal{W}_{D^3\{(1,2)\}}}\left[\mathbf{T}^{\mathcal{W}_{D^{p+q+r}}}M\rightarrow M\right]}\overline{\Omega}_{(1)}^{(p+q+r,3)}(M)$$
$$\rightarrow\overline{\Omega}_{(1)}^{(p+q+r,3)}(M)\times\overline{\Omega}_{(1)}^{(p+q+r,3)}(M)$$

into

$$\overline{\Omega}_{(1)}^{(p,1)}(M)\times\overline{\Omega}_{(1)}^{(q,1)}(M)\times\overline{\Omega}_{(1)}^{(r,1)}(M)$$
$$\rightarrow\overline{\Omega}_{(1)}^{(p+q+r,3)}(M)\times_{\mathbf{T}^{\mathcal{W}_{D^3\{(1,2)\}}}\left[\mathbf{T}^{\mathcal{W}_{D^{p+q+r}}}M\rightarrow M\right]}\overline{\Omega}_{(1)}^{(p+q+r,3)}(M) \qquad (13)$$

Notation 51. We introduce the following six notations:

1. The composition of (8) with

$$\overline{\Omega}_{(1)}^{(p+q+r,3)}(M)\times_{\mathbf{T}^{\mathcal{W}_{D^3\{(2,3)\}}}\left[\mathbf{T}^{\mathcal{W}_{D^{p+q+r}}}M\rightarrow M\right]}\overline{\Omega}_{(1)}^{(p+q+r,3)}(M)$$

$$\xrightarrow{\zeta^{\underset{1}{\overset{\cdot}{-}}}\left(\left[\mathbf{T}^{\mathcal{W}_{D_{p+q+r}}}M\to M\right]\right)}$$
$$\overline{\Omega}_{(1)}^{(p+q+r,2)}(M)$$

is denoted

$$\zeta^{*123\underset{1}{\overset{\cdot}{-}}*132}$$

2. The composition of (9) with

$$\overline{\Omega}_{(1)}^{(p+q+r,3)}(M)\times_{\mathbf{T}^{\mathcal{W}_{D^{3}\{(2,3)\}}}\left[\mathbf{T}^{\mathcal{W}_{D_{p+q+r}}}M\to M\right]}\overline{\Omega}_{(1)}^{(p+q+r,3)}(M)$$
$$\xrightarrow{\zeta^{\underset{1}{\overset{\cdot}{-}}}\left(\left[\mathbf{T}^{\mathcal{W}_{D_{p+q+r}}}M\to M\right]\right)}$$
$$\overline{\Omega}_{(1)}^{(p+q+r,2)}(M)$$

is denoted

$$\zeta^{*231\underset{1}{\overset{\cdot}{-}}*321}$$

3. The composition of (10) with

$$\overline{\Omega}_{(1)}^{(p+q+r,3)}(M)\times_{\mathbf{T}^{\mathcal{W}_{D^{3}\{(1,3)\}}}\left[\mathbf{T}^{\mathcal{W}_{D_{p+q+r}}}M\to M\right]}\overline{\Omega}_{(1)}^{(p+q+r,3)}(M)$$
$$\xrightarrow{\zeta^{\underset{2}{\overset{\cdot}{-}}}\left(\left[\mathbf{T}^{\mathcal{W}_{D_{p+q+r}}}M\to M\right]\right)}$$
$$\overline{\Omega}_{(1)}^{(p+q+r,2)}(M)$$

is denoted

$$\zeta^{*231\underset{2}{\overset{\cdot}{-}}*213}$$

4. The composition of (11) with

$$\overline{\Omega}_{(1)}^{(p+q+r,3)}(M)\times_{\mathbf{T}^{\mathcal{W}_{D^{3}\{(1,3)\}}}\left[\mathbf{T}^{\mathcal{W}_{D_{p+q+r}}}M\to M\right]}\overline{\Omega}_{(1)}^{(p+q+r,3)}(M)$$
$$\xrightarrow{\zeta^{\underset{2}{\overset{\cdot}{-}}}\left(\left[\mathbf{T}^{\mathcal{W}_{D_{p+q+r}}}M\to M\right]\right)}$$
$$\overline{\Omega}_{(1)}^{(p+q+r,2)}(M)$$

is denoted

$$\zeta^{*312\underset{2}{\overset{\cdot}{-}}*132}$$

5. The composition of (12) with

$$\overline{\Omega}_{(1)}^{(p+q+r,3)}(M)\times_{\mathbf{T}^{\mathcal{W}_{D^3\{(1,3)\}}}\left[\mathbf{T}^{\mathcal{W}_{D^{p+q+r}}}M\to M\right]}\overline{\Omega}_{(1)}^{(p+q+r,3)}(M)$$
$$\underrightarrow{\zeta^{\overset{\cdot}{-}_3}\left(\left[\mathbf{T}^{\mathcal{W}_{D^{p+q+r}}}M\to M\right]\right)}$$
$$\overline{\Omega}_{(1)}^{(p+q+r,2)}(M)$$

is denoted

$$\zeta^{*312\overset{\cdot}{\underset{3}{-}}*321}$$

6. The composition of (13) with

$$\overline{\Omega}_{(1)}^{(p+q+r,3)}(M)\times_{\mathbf{T}^{\mathcal{W}_{D^3\{(1,3)\}}}\left[\mathbf{T}^{\mathcal{W}_{D^{p+q+r}}}M\to M\right]}\overline{\Omega}_{(1)}^{(p+q+r,3)}(M)$$
$$\underrightarrow{\zeta^{\overset{\cdot}{-}_3}\left(\left[\mathbf{T}^{\mathcal{W}_{D^{p+q+r}}}M\to M\right]\right)}$$
$$\overline{\Omega}_{(1)}^{(p+q+r,2)}(M)$$

is denoted

$$\zeta^{*123\overset{\cdot}{\underset{3}{-}}*213}$$

Lemma 52. *We have the following three statements:*

1. *The morphism*

$$\overline{\Omega}_{(1)}^{(p,1)}(M)\times\overline{\Omega}_{(1)}^{(q,1)}(M)\times\overline{\Omega}_{(1)}^{(r,1)}(M)$$
$$\underrightarrow{\left(\zeta^{*123\overset{\cdot}{\underset{1}{-}}*132},\zeta^{*231\overset{\cdot}{\underset{1}{-}}*321}\right)}$$
$$\overline{\Omega}_{(1)}^{(p+q+r,2)}(M)\times\overline{\Omega}_{(1)}^{(p+q+r,2)}(M)$$

is to be factored through the canonical injection

$$\overline{\Omega}_{(1)}^{(p+q+r,2)}(M)\times_{\mathbf{T}^{\mathcal{W}_{D(2)}}\left[\mathbf{T}^{\mathcal{W}_{D^{p+q+r}}}M\to M\right]}\overline{\Omega}_{(1)}^{(p+q+r,2)}(M)$$
$$\to\overline{\Omega}_{(1)}^{(p+q+r,2)}(M)\times\overline{\Omega}_{(1)}^{(p+q+r,2)}(M)$$

into

$$\overline{\Omega}_{(1)}^{(p,1)}(M)\times\overline{\Omega}_{(1)}^{(q,1)}(M)\times\overline{\Omega}_{(1)}^{(r,1)}(M)$$
$$\rightarrow\overline{\Omega}_{(1)}^{(p+q+r,2)}(M)\times_{\mathbf{T}^{\mathcal{W}_{D(2)}}\left[\mathbf{T}^{\mathcal{W}_{D^{p+q+r}}}M\rightarrow M\right]}\overline{\Omega}_{(1)}^{(p+q+r,2)}(M) \tag{14}$$

2. *The morphism*

$$\overline{\Omega}_{(1)}^{(p,1)}(M)\times\overline{\Omega}_{(1)}^{(q,1)}(M)\times\overline{\Omega}_{(1)}^{(r,1)}(M)\xrightarrow{\left(\zeta^{*231\underset{2}{\overset{\cdot}{-}}*213},\zeta^{*312\underset{2}{\overset{\cdot}{-}}*132}\right)}\overline{\Omega}_{(1)}^{(p+q+r,2)}(M)\times\overline{\Omega}_{(1)}^{(p+q+r,2)}(M)$$

is to be factored through the canonical injection

$$\overline{\Omega}_{(1)}^{(p+q+r,2)}(M)\times_{\mathbf{T}^{\mathcal{W}_{D(2)}}\left[\mathbf{T}^{\mathcal{W}_{D^{p+q+r}}}M\rightarrow M\right]}\overline{\Omega}_{(1)}^{(p+q+r,2)}(M)$$
$$\rightarrow\overline{\Omega}_{(1)}^{(p+q+r,2)}(M)\times\overline{\Omega}_{(1)}^{(p+q+r,2)}(M)$$

into

$$\overline{\Omega}_{(1)}^{(p,1)}(M)\times\overline{\Omega}_{(1)}^{(q,1)}(M)\times\overline{\Omega}_{(1)}^{(r,1)}(M)$$
$$\rightarrow\overline{\Omega}_{(1)}^{(p+q+r,2)}(M)\times_{\mathbf{T}^{\mathcal{W}_{D(2)}}\left[\mathbf{T}^{\mathcal{W}_{D^{p+q+r}}}M\rightarrow M\right]}\overline{\Omega}_{(1)}^{(p+q+r,2)}(M) \tag{15}$$

3. *The morphism*

$$\overline{\Omega}_{(1)}^{(p,1)}(M)\times\overline{\Omega}_{(1)}^{(q,1)}(M)\times\overline{\Omega}_{(1)}^{(r,1)}(M)\xrightarrow{\left(\zeta^{*312\underset{3}{\overset{\cdot}{-}}*321},\zeta^{*123\underset{3}{\overset{\cdot}{-}}*213}\right)}\overline{\Omega}_{(1)}^{(p+q+r,2)}(M)\times\overline{\Omega}_{(1)}^{(p+q+r,2)}(M)$$

is to be factored through the canonical injection

$$\overline{\Omega}_{(1)}^{(p+q+r,2)}(M)\times_{\mathbf{T}^{\mathcal{W}_{D(2)}}\left[\mathbf{T}^{\mathcal{W}_{D^{p+q+r}}}M\rightarrow M\right]}\overline{\Omega}_{(1)}^{(p+q+r,2)}(M)$$
$$\rightarrow\overline{\Omega}_{(1)}^{(p+q+r,2)}(M)\times\overline{\Omega}_{(1)}^{(p+q+r,2)}(M)$$

into

$$
\begin{aligned}
&\overline{\Omega}^{(p,1)}_{(1)}(M) \times \overline{\Omega}^{(q,1)}_{(1)}(M) \times \overline{\Omega}^{(r,1)}_{(1)}(M) \\
&\to \overline{\Omega}^{(p+q+r,2)}_{(1)}(M) \times_{\mathbf{T}^{\mathcal{W}_{D(2)}}\left[\mathbf{T}^{\mathcal{W}_{Dp+q+r}} M \to M\right]} \overline{\Omega}^{(p+q+r,2)}_{(1)}(M)
\end{aligned}
\tag{16}
$$

Notation 53. We introduce the following three notations:

1. The composition of (14) with

$$
\begin{aligned}
&\overline{\Omega}^{(p+q+r,2)}_{(1)}(M) \times_{\mathbf{T}^{\mathcal{W}_{D(2)}}\left[\mathbf{T}^{\mathcal{W}_{Dp+q+r}} M \to M\right]} \overline{\Omega}^{(p+q+r,2)}_{(1)}(M) \\
&\xrightarrow{\zeta^{\overset{\cdot}{-}}\left(\left[\mathbf{T}^{\mathcal{W}_{Dp+q+r}} M \to M\right]\right)} \\
&\overline{\Omega}^{(p+q+r,1)}_{(1)}(M)
\end{aligned}
$$

is denoted

$$
\zeta^{(*123 \underset{1}{\overset{\cdot}{-}} *132) \overset{\cdot}{-} (*231 \underset{1}{\overset{\cdot}{-}} *321)}
$$

2. The composition of (15) with

$$
\begin{aligned}
&\overline{\Omega}^{(p+q+r,2)}_{(1)}(M) \times_{\mathbf{T}^{\mathcal{W}_{D(2)}}\left[\mathbf{T}^{\mathcal{W}_{Dp+q+r}} M \to M\right]} \overline{\Omega}^{(p+q+r,2)}_{(1)}(M) \\
&\xrightarrow{\zeta^{\overset{\cdot}{-}}\left(\left[\mathbf{T}^{\mathcal{W}_{Dp+q+r}} M \to M\right]\right)} \\
&\overline{\Omega}^{(p+q+r,1)}_{(1)}(M)
\end{aligned}
$$

is denoted

$$
\zeta^{(*231 \underset{2}{\overset{\cdot}{-}} *213) \overset{\cdot}{-} (*312 \underset{2}{\overset{\cdot}{-}} *132)}
$$

3. The composition of (16) with

$$
\begin{aligned}
&\overline{\Omega}^{(p+q+r,2)}_{(1)}(M) \times_{\mathbf{T}^{\mathcal{W}_{D(2)}}\left[\mathbf{T}^{\mathcal{W}_{Dp+q+r}} M \to M\right]} \overline{\Omega}^{(p+q+r,2)}_{(1)}(M) \\
&\xrightarrow{\zeta^{\overset{\cdot}{-}}\left(\left[\mathbf{T}^{\mathcal{W}_{Dp+q+r}} M \to M\right]\right)} \\
&\overline{\Omega}^{(p+q+r,1)}_{(1)}(M)
\end{aligned}
$$

is denoted

$$
\zeta^{(*312 \underset{3}{\overset{\cdot}{-}} *321) \overset{\cdot}{-} (*123 \underset{3}{\overset{\cdot}{-}} *213)}
$$

Now we are ready to present a proof of Theorem 47.

Proof. (of Theorem 47). By the morphisms (5)-(7) are identical to the morphisms

$$\zeta^{(*123 \underset{1}{\dot{-}} *132) \dot{-} (*231 \underset{1}{\dot{-}} *321)}$$

$$\zeta^{(*231 \underset{2}{\dot{-}} *213) \dot{-} (*312 \underset{2}{\dot{-}} *132)}$$

$$\zeta^{(*312 \underset{3}{\dot{-}} *321) \dot{-} (*123 \underset{3}{\dot{-}} *213)}$$

respectively. Therefore Theorem 47 follows from Theorem 20. □

5.3. The Second Consideration

In this subsection we are concerned with the Lie algebra structure of $\overline{\Omega}_{(12)}^{(p,1)}(M)$'s, where p ranges over natural numbers. It is easy to see that

Lemma 54. *The morphism*

$$\overline{\Omega}_{(12)}^{(p,1)}(M) \times \overline{\Omega}_{(12)}^{(q,1)}(M)$$

$$\xrightarrow{i_{\overline{\Omega}_{(12)}^{(p,1)}(M)}^{\overline{\Omega}_{(1)}^{(p,1)}(M)} \times i_{\overline{\Omega}_{(12)}^{(q,1)}(M)}^{\overline{\Omega}_{(1)}^{(q,1)}(M)}}$$

$$\overline{\Omega}_{(1)}^{(p,1)}(M) \times \overline{\Omega}_{(1)}^{(q,1)}(M)$$

$$\xrightarrow{\zeta_{p,q}^{L_1}(M)}$$

$$\overline{\Omega}_{(1)}^{(p+q,1)}(M)$$

is to be factored uniquely through the canonical injection

$$i_{\overline{\Omega}_{(12)}^{(p+q,1)}(M)}^{\overline{\Omega}_{(1)}^{(p+q,1)}(M)} : \overline{\Omega}_{(12)}^{(p+q,1)}(M) \to \overline{\Omega}_{(1)}^{(p+q,1)}(M)$$

into a morphism

$$\overline{\Omega}_{(12)}^{(p,1)}(M) \times \overline{\Omega}_{(12)}^{(q,1)}(M) \to \overline{\Omega}_{(12)}^{(p+q,1)}(M)$$

Notation 55. The morphism in (18) is denoted

$$\zeta_{p,q}^{L_{12}}(M) : \overline{\Omega}_{(12)}^{(p,1)}(M) \times \overline{\Omega}_{(12)}^{(q,1)}(M) \to \overline{\Omega}_{(12)}^{(p+q,1)}(M)$$

Proposition 56. *The morphism*

$$\overline{\Omega}_{(12)}^{(p,1)}(M) \times \overline{\Omega}_{(12)}^{(q,1)}(M) \xrightarrow{\zeta_{p,q}^{L_{12}}(M)} \overline{\Omega}_{(12)}^{(p+q,1)}(M)$$

and the morphism

$$\overline{\Omega}_{(12)}^{(p,1)}(M) \times \overline{\Omega}_{(12)}^{(q,1)}(M) = \overline{\Omega}_{(12)}^{(q,1)}(M) \times \overline{\Omega}_{(12)}^{(p,1)}(M) \xrightarrow{\zeta_{q,p}^{L_{12}}(M)} \overline{\Omega}_{(12)}^{(p+q,1)}(M) \xrightarrow{(\cdot^{\sigma_{p,q}})_{\overline{\Omega}_{(12)}^{(p+q,1)}(M)}} \overline{\Omega}_{(12)}^{(p+q,1)}(M)$$

sum up only to vanish.

Proof. This follows directly from Proposition 46. □

Theorem 57. *The three morphisms*

$$\overline{\Omega}_{(12)}^{(p,1)}(M) \times \overline{\Omega}_{(12)}^{(q,1)}(M) \times \overline{\Omega}_{(12)}^{(r,1)}(M) \xrightarrow{\mathrm{id}_{\overline{\Omega}_{(12)}^{(p,1)}(M)} \times \zeta_{q,r}^{L_1}(M)} \overline{\Omega}_{(12)}^{(p,1)}(M) \times \overline{\Omega}_{(12)}^{(q+r,1)}(M) \xrightarrow{\zeta_{p,q+r}^{L_{12}}(M)} \overline{\Omega}_{(12)}^{(p+q+r,1)}(M),$$

$$\overline{\Omega}_{(12)}^{(p,1)}(M) \times \overline{\Omega}_{(12)}^{(q,1)}(M) \times \overline{\Omega}_{(12)}^{(r,1)}(M) = \overline{\Omega}_{(12)}^{(q,1)}(M) \times \overline{\Omega}_{(12)}^{(r,1)}(M) \times \overline{\Omega}_{(12)}^{(p,1)}(M) \xrightarrow{\mathrm{id}_{\overline{\Omega}_{(12)}^{(q,1)}(M)} \times \zeta_{r,p}^{L_1}(M)}$$

$$\begin{array}{l} \overline{\Omega}_{(12)}^{(q,1)}(M)\times\overline{\Omega}_{(12)}^{(r+p,1)}(M) \\ \underrightarrow{\zeta_{q,r+p}^{L_{12}}(M)} \\ \overline{\Omega}_{(12)}^{(p+q+r,1)}(M) \\ \underrightarrow{(\cdot^{\sigma_{p,q+r}})_{\overline{\Omega}_{(12)}^{(p+q+r,1)}(M)}} \\ \overline{\Omega}_{(12)}^{(p+q+r,1)}(M) \end{array}$$

and

$$\begin{array}{l} \overline{\Omega}_{(12)}^{(p,1)}(M)\times\overline{\Omega}_{(12)}^{(q,1)}(M)\times\overline{\Omega}_{(12)}^{(r,1)}(M) \\ \quad=\overline{\Omega}_{(12)}^{(r,1)}(M)\times\overline{\Omega}_{(12)}^{(p,1)}(M)\times\overline{\Omega}_{(12)}^{(q,1)}(M) \\ \underrightarrow{\mathrm{id}_{\overline{\Omega}_{(12)}^{(r,1)}(M)}\times\zeta_{p,q}^{L_1}(M)} \\ \overline{\Omega}_{(12)}^{(r,1)}(M)\times\overline{\Omega}_{(12)}^{(p+q,1)}(M) \\ \underrightarrow{\zeta_{r,p+q}^{L_{12}}(M)} \\ \overline{\Omega}_{(12)}^{(p+q+r,1)}(M) \\ \underrightarrow{(\cdot^{\sigma_{r,p+q}})_{\overline{\Omega}_{(12)}^{(p+q+r,1)}(M)}} \\ \overline{\Omega}_{(12)}^{(p+q+r,1)}(M) \end{array}$$

sum up only to vanish.

Proof. This follows directly from Theorem 47. □

5.4. The Third Consideration

In this subsection we are concerned with the Lie algebra structure of $\overline{\Omega}_{(13)}^{(p,1)}(M)$'s, where p ranges over natural numbers.

Notation 58. We introduce the following notations:

1. We denote by
$$\mathcal{A}^p:\overline{\Omega}_{(1)}^{(p,1)}(M)\rightarrow\overline{\Omega}_{(1)}^{(p,1)}(M)$$
the morphism
$$\overline{\Omega}_{(1)}^{(p,1)}(M)$$

$$\xrightarrow{\sum_{\sigma\in\mathbb{S}_p}\varepsilon_\sigma\left(\cdot^{\sigma}\right)_{\overline{\Omega}_{(1)}^{(p,1)}(M)}}$$
$$\overline{\Omega}_{(1)}^{(p,1)}(M)$$

where $\mathbb{S}_p$ is the group of permutations of the set $\{1,2,...,n\}$.

2. We denote by
$$\mathcal{A}_{p,q}^{p+q}:\overline{\Omega}_{(1)}^{(p+q,1)}(M)\to\overline{\Omega}_{(1)}^{(p+q,1)}(M)$$
the morphism
$$\overline{\Omega}_{(1)}^{(p+q,1)}(M)$$
$$\xrightarrow{(1/p!q!)\,\mathcal{A}^{p+q}}$$
$$\overline{\Omega}_{(1)}^{(p+q,1)}(M)$$

3. We denote by
$$\mathcal{A}_{p,q,r}^{p+q+r}:\overline{\Omega}_{(1)}^{(p+q+r,1)}(M)\to\overline{\Omega}_{(1)}^{(p+q+r,1)}(M)$$
the morphism
$$\overline{\Omega}_{(1)}^{(p+q+r,1)}(M)$$
$$\xrightarrow{(1/p!q!r!)\,\mathcal{A}^{p+q+r}}$$
$$\overline{\Omega}_{(1)}^{(p+q+r,1)}(M)$$

It is easy to see that

Lemma 59. *The morphism*

$$\overline{\Omega}_{(13)}^{(p,1)}(M)\times\overline{\Omega}_{(13)}^{(q,1)}(M)$$
$$\xrightarrow{i_{\overline{\Omega}_{(13)}^{(p,1)}(M)}^{\overline{\Omega}_{(1)}^{(p,1)}(M)}\times i_{\overline{\Omega}_{(13)}^{(q,1)}(M)}^{\overline{\Omega}_{(1)}^{(q,1)}(M)}}$$
$$\overline{\Omega}_{(1)}^{(p,1)}(M)\times\overline{\Omega}_{(1)}^{(q,1)}(M)$$
$$\xrightarrow{\zeta_{p,q}^{L_1}(M)}$$
$$\overline{\Omega}_{(1)}^{(p+q,1)}(M)$$

$$\underrightarrow{\mathcal{A}_{p,q}^{p+q}}$$

$$\overline{\Omega}_{(1)}^{(p+q,1)}(M)$$

is to be factored uniquely through the canonical injection

$$i_{\overline{\Omega}_{(13)}^{(p+q,1)}(M)}^{\overline{\Omega}_{(1)}^{(p+q,1)}(M)}:\overline{\Omega}_{(13)}^{(p+q,1)}(M)\rightarrow\overline{\Omega}_{(1)}^{(p+q,1)}(M) \quad (17)$$

into a morphism

$$\overline{\Omega}_{(13)}^{(p,1)}(M)\times\overline{\Omega}_{(13)}^{(q,1)}(M)\rightarrow\overline{\Omega}_{(13)}^{(p+q,1)}(M) \quad (18)$$

Notation 60. The morphism in (18) is denoted

$$\zeta_{p,q}^{FN_{13}}(M):\overline{\Omega}_{(13)}^{(p,1)}(M)\times\overline{\Omega}_{(13)}^{(q,1)}(M)\rightarrow\overline{\Omega}_{(13)}^{(p+q,1)}(M)$$

where FN stands for Frölicher and Nijenhuis.

Lemma 61. *Given $\sigma\in\mathbb{S}_p$, the morphism*

$$\overline{\Omega}_{(1)}^{(p,1)}(M)$$

$$\underrightarrow{\mathcal{A}^{p}}$$

$$\overline{\Omega}_{(1)}^{(p,1)}(M)$$

$$\underrightarrow{(\cdot^{\sigma})_{\overline{\Omega}_{(1)}^{(p,1)}(M)}}$$

$$\overline{\Omega}_{(1)}^{(p,1)}(M)$$

is identical to the morphism

$$\overline{\Omega}_{(1)}^{(p,1)}(M)$$

$$\underrightarrow{(\cdot^{\sigma})_{\overline{\Omega}_{(1)}^{(p,1)}(M)}}$$

$$\overline{\Omega}_{(1)}^{(p,1)}(M)$$

$$\underrightarrow{\mathcal{A}^{p}}$$

$$\overline{\Omega}_{(1)}^{(p,1)}(M)$$

Both of them are identical to the morphism

$$\overline{\Omega}_{(1)}^{(p,1)}(M)$$

$$\underrightarrow{\mathcal{A}^p}$$
$$\overline{\Omega}_{(1)}^{(p,1)}(M)$$
$$\underrightarrow{\varepsilon_{\sigma}}$$
$$\overline{\Omega}_{(1)}^{(p,1)}(M)$$

Proof. By mimicking the familiar token in establishing the antisymmetry of wedge products of differential forms in orthodox differential geometry. □

Proposition 62. *The morphisms*

$$\overline{\Omega}_{(13)}^{(p,1)}(M) \times \overline{\Omega}_{(13)}^{(q,1)}(M)$$
$$\underrightarrow{\zeta_{p,q}^{FN_{13}}(M)}$$
$$\overline{\Omega}_{(13)}^{(p+q,1)}(M) \tag{19}$$

and

$$\overline{\Omega}_{(13)}^{(p,1)}(M) \times \overline{\Omega}_{(13)}^{(q,1)}(M)$$
$$= \overline{\Omega}_{(13)}^{(q,1)}(M) \times \overline{\Omega}_{(13)}^{(p,1)}(M)$$
$$\underrightarrow{\zeta_{q,p}^{FN_{13}}(M)}$$
$$\overline{\Omega}_{(13)}^{(p+q,1)}(M)$$
$$\underrightarrow{(-1)^{pq}}$$
$$\overline{\Omega}_{(13)}^{(p+q,1)}(M) \tag{20}$$

sum up only to vanish.

Proof. The morphism (19) followed by the canonical injection (17) is identical to the morphism

$$\overline{\Omega}_{(13)}^{(p,1)}(M) \times \overline{\Omega}_{(13)}^{(q,1)}(M)$$
$$\underrightarrow{i_{\overline{\Omega}_{(13)}^{(p,1)}(M)}^{\overline{\Omega}_{(1)}^{(p,1)}(M)} \times i_{\overline{\Omega}_{(13)}^{(q,1)}(M)}^{\overline{\Omega}_{(1)}^{(q,1)}(M)}}$$
$$\overline{\Omega}_{(1)}^{(p,1)}(M) \times \overline{\Omega}_{(1)}^{(q,1)}(M)$$
$$\underrightarrow{\zeta_{p,q}^{L_1}(M)}$$

$$\overline{\Omega}_{(1)}^{(p+q,1)}(M) \underrightarrow{\mathcal{A}_{p,q}^{p+q}} \overline{\Omega}_{(1)}^{(p+q,1)}(M) \tag{21}$$

by definition, while the morphism (20) followed by the canonical injection (17) is identical to the morphism

$$\overline{\Omega}_{(13)}^{(p,1)}(M)\times\overline{\Omega}_{(13)}^{(q,1)}(M) \underrightarrow{i_{\overline{\Omega}_{(13)}^{(p,1)}(M)}^{\overline{\Omega}_{(1)}^{(p,1)}(M)}\times i_{\overline{\Omega}_{(13)}^{(q,1)}(M)}^{\overline{\Omega}_{(1)}^{(q,1)}(M)}} \overline{\Omega}_{(1)}^{(p,1)}(M)\times\overline{\Omega}_{(1)}^{(q,1)}(M) \underrightarrow{\zeta_{p,q}^{L_1}(M)} \overline{\Omega}_{(1)}^{(p+q,1)}(M) \underrightarrow{(\cdot^{\sigma_{p,q}})_{\overline{\Omega}_{(1)}^{(p+q,1)}(M)}} \overline{\Omega}_{(1)}^{(p+q,1)}(M) \underrightarrow{\mathcal{A}_{p,q}^{p+q}} \overline{\Omega}_{(1)}^{(p+q,1)}(M) \tag{22}$$

by Lemma 61. The sum of (21) and (22) vanishes by dint of Proposition 46, so that the sum of (19) and (20) also vanishes. □

Lemma 63. *The morphism*

$$\overline{\Omega}_{(13)}^{(p,1)}(M)\times\overline{\Omega}_{(13)}^{(q,1)}(M)\times\overline{\Omega}_{(13)}^{(r,1)}(M) \underrightarrow{i_{\overline{\Omega}_{(13)}^{(p,1)}(M)}^{\overline{\Omega}_{(1)}^{(p,1)}(M)}\times i_{\overline{\Omega}_{(13)}^{(q,1)}(M)}^{\overline{\Omega}_{(1)}^{(q,1)}(M)}\times i_{\overline{\Omega}_{(13)}^{(r,1)}(M)}^{\overline{\Omega}_{(1)}^{(r,1)}(M)}} \overline{\Omega}_{(1)}^{(p,1)}(M)\times\overline{\Omega}_{(1)}^{(q,1)}(M)\times\overline{\Omega}_{(1)}^{(r,1)}(M) \underrightarrow{\mathrm{id}_{\overline{\Omega}_{(1)}^{(p,1)}(M)}\times\zeta_{q.r}^{L_1}(M)} \overline{\Omega}_{(1)}^{(p,1)}(M)\times\overline{\Omega}_{(1)}^{(q+r,1)}(M) \underrightarrow{\mathrm{id}_{\overline{\Omega}_{(1)}^{(p,1)}(M)}\times\mathcal{A}_{q,r}^{q+r}}$$

$$\begin{array}{l}\overline{\Omega}_{(1)}^{(p,1)}(M)\times\overline{\Omega}_{(1)}^{(q+r,1)}(M)\\ \xrightarrow{\zeta_{p,q+r}^{L_1}(M)}\\ \overline{\Omega}_{(1)}^{(p+q+r,1)}(M)\\ \xrightarrow{\mathcal{A}_{p,q+r}^{p+q+r}}\\ \overline{\Omega}_{(1)}^{(p+q+r,1)}(M)\end{array}$$

is identical to the morphism

$$\begin{array}{l}\overline{\Omega}_{(13)}^{(p,1)}(M)\times\overline{\Omega}_{(13)}^{(q,1)}(M)\times\overline{\Omega}_{(13)}^{(r,1)}(M)\\ \xrightarrow{i_{\overline{\Omega}_{(13)}^{(p,1)}(M)}^{\overline{\Omega}_{(1)}^{(p,1)}(M)}\times i_{\overline{\Omega}_{(13)}^{(q,1)}(M)}^{\overline{\Omega}_{(1)}^{(q,1)}(M)}\times i_{\overline{\Omega}_{(13)}^{(r,1)}(M)}^{\overline{\Omega}_{(1)}^{(r,1)}(M)}}\\ \overline{\Omega}_{(1)}^{(p,1)}(M)\times\overline{\Omega}_{(1)}^{(q,1)}(M)\times\overline{\Omega}_{(1)}^{(r,1)}(M)\\ \xrightarrow{\mathrm{id}_{\overline{\Omega}_{(1)}^{(p,1)}(M)}\times\zeta_{q.r}^{L_1}(M)}\\ \overline{\Omega}_{(1)}^{(p,1)}(M)\times\overline{\Omega}_{(1)}^{(q+r,1)}(M)\\ \xrightarrow{\zeta_{p,q+r}^{L_1}(M)}\\ \overline{\Omega}_{(1)}^{(p+q+r,1)}(M)\\ \xrightarrow{\mathcal{A}_{p,q,r}^{p+q+r}}\\ \overline{\Omega}_{(1)}^{(p+q+r,1)}(M)\end{array}$$

Proof. By mimicking the familiar token in establishing the associativity of wedge products of differential forms in orthodox differential geometry. □

Theorem 64. *The three morphisms*

$$\begin{array}{l}\overline{\Omega}_{(13)}^{(p,1)}(M)\times\overline{\Omega}_{(13)}^{(q,1)}(M)\times\overline{\Omega}_{(13)}^{(r,1)}(M)\\ \xrightarrow{\mathrm{id}_{\overline{\Omega}_{(13)}^{(p,1)}(M)}\times\zeta_{q,r}^{FN_{13}}(M)}\\ \overline{\Omega}_{(13)}^{(p,1)}(M)\times\overline{\Omega}_{(13)}^{(q+r,1)}(M)\\ \xrightarrow{\zeta_{p,q+r}^{FN_{13}}(M)}\\ \overline{\Omega}_{(13)}^{(p+q+r,1)}(M)\end{array}\tag{23}$$

$$
\begin{array}{l}
\overline{\Omega}_{(13)}^{(p,1)}(M)\times\overline{\Omega}_{(13)}^{(q,1)}(M)\times\overline{\Omega}_{(13)}^{(r,1)}(M)\\
=\overline{\Omega}_{(13)}^{(q,1)}(M)\times\overline{\Omega}_{(13)}^{(r,1)}(M)\times\overline{\Omega}_{(13)}^{(p,1)}(M)\\
\underrightarrow{\mathrm{id}_{\overline{\Omega}_{(13)}^{(q,1)}(M)}\times\zeta_{r,p}^{FN_{13}}(M)}\\
\overline{\Omega}_{(13)}^{(q,1)}(M)\times\overline{\Omega}_{(13)}^{(p+r,1)}(M)\\
\underrightarrow{\zeta_{q,p+r}^{FN_{13}}(M)}\\
\overline{\Omega}_{(13)}^{(p+q+r,1)}(M)\\
\underrightarrow{(-1)^{p(q+r)}}\\
\overline{\Omega}_{(13)}^{(p+q+r,1)}(M)
\end{array}
\tag{24}
$$

and

$$
\begin{array}{l}
\overline{\Omega}_{(13)}^{(p,1)}(M)\times\overline{\Omega}_{(13)}^{(q,1)}(M)\times\overline{\Omega}_{(13)}^{(r,1)}(M)\\
=\overline{\Omega}_{(13)}^{(r,1)}(M)\times\overline{\Omega}_{(13)}^{(p,1)}(M)\times\overline{\Omega}_{(13)}^{(q,1)}(M)\\
\underrightarrow{\mathrm{id}_{\overline{\Omega}_{(13)}^{(r,1)}(M)}\times\zeta_{p,q}^{FN_{13}}(M)}\\
\overline{\Omega}_{(13)}^{(r,1)}(M)\times\overline{\Omega}_{(13)}^{(p+q,1)}(M)\\
\underrightarrow{\zeta_{r,p+q}^{FN_{13}}(M)}\\
\overline{\Omega}_{(13)}^{(p+q+r,1)}(M)\\
\underrightarrow{(-1)^{r(p+q)}}\\
\overline{\Omega}_{(13)}^{(p+q+r,1)}(M)
\end{array}
\tag{25}
$$

sum up only to vanish.

Proof. The morphism (23) followed by the canonical injection (17) is identical to the morphism

$$
\begin{array}{l}
\overline{\Omega}_{(13)}^{(p,1)}(M)\times\overline{\Omega}_{(13)}^{(q,1)}(M)\times\overline{\Omega}_{(13)}^{(r,1)}(M)\\
\underrightarrow{i_{\overline{\Omega}_{(13)}^{(p,1)}(M)}^{\overline{\Omega}_{(1)}^{(p,1)}(M)}\times i_{\overline{\Omega}_{(13)}^{(q,1)}(M)}^{\overline{\Omega}_{(1)}^{(q,1)}(M)}\times i_{\overline{\Omega}_{(13)}^{(r,1)}(M)}^{\overline{\Omega}_{(1)}^{(r,1)}(M)}}\\
\overline{\Omega}_{(1)}^{(p,1)}(M)\times\overline{\Omega}_{(1)}^{(q,1)}(M)\times\overline{\Omega}_{(1)}^{(r,1)}(M)
\end{array}
$$

$$
\begin{aligned}
&\xrightarrow{\mathrm{id}_{\overline{\Omega}_{(1)}^{(p,1)}(M)}\times\zeta_{q.r}^{L_1}(M)} \\
&\overline{\Omega}_{(1)}^{(p,1)}(M)\times\overline{\Omega}_{(1)}^{(q+r,1)}(M) \\
&\xrightarrow{\zeta_{p,q+r}^{L_1}(M)} \\
&\overline{\Omega}_{(1)}^{(p+q+r,1)}(M) \\
&\xrightarrow{\mathcal{A}_{p,q,r}^{p+q+r}} \\
&\overline{\Omega}_{(1)}^{(p+q+r,1)}(M)
\end{aligned}
\tag{26}
$$

by dint of Lemma 63. The morphism (24) followed by the canonical injection (17) is identical to the morphism

$$
\begin{aligned}
&\overline{\Omega}_{(13)}^{(p,1)}(M)\times\overline{\Omega}_{(13)}^{(q,1)}(M)\times\overline{\Omega}_{(13)}^{(r,1)}(M) \\
&\xrightarrow{i_{\overline{\Omega}_{(13)}^{(p,1)}(M)}^{\overline{\Omega}_{(1)}^{(p,1)}(M)}\times i_{\overline{\Omega}_{(13)}^{(q,1)}(M)}^{\overline{\Omega}_{(1)}^{(q,1)}(M)}\times i_{\overline{\Omega}_{(13)}^{(r,1)}(M)}^{\overline{\Omega}_{(1)}^{(r,1)}(M)}} \\
&\overline{\Omega}_{(1)}^{(p,1)}(M)\times\overline{\Omega}_{(1)}^{(q,1)}(M)\times\overline{\Omega}_{(1)}^{(r,1)}(M) \\
&=\overline{\Omega}_{(1)}^{(q,1)}(M)\times\overline{\Omega}_{(1)}^{(r,1)}(M)\times\overline{\Omega}_{(1)}^{(p,1)}(M) \\
&\xrightarrow{\mathrm{id}_{\overline{\Omega}_{(1)}^{(q,1)}(M)}\times\zeta_{r.p}^{L_1}(M)} \\
&\overline{\Omega}_{(1)}^{(q,1)}(M)\times\overline{\Omega}_{(1)}^{(p+r,1)}(M) \\
&\xrightarrow{\zeta_{q,p+r}^{L_1}(M)} \\
&\overline{\Omega}_{(1)}^{(p+q+r,1)}(M) \\
&\xrightarrow{(\cdot^{\sigma_{p,q+r}})_{\overline{\Omega}_{(1)}^{(p+q+r,1)}(M)}} \\
&\overline{\Omega}_{(1)}^{(p+q+r,1)}(M) \\
&\xrightarrow{\mathcal{A}_{p,q,r}^{p+q+r}} \\
&\overline{\Omega}_{(1)}^{(p+q+r,1)}(M)
\end{aligned}
\tag{27}
$$

by dint of Lemmas 61 and 63 with $\varepsilon_{\sigma_{p,q+r}}=(-1)^{p(q+r)}$. The morphism (25) followed by the canonical injection (17) is identical to the morphism

$$
\overline{\Omega}_{(13)}^{(p,1)}(M)\times\overline{\Omega}_{(13)}^{(q,1)}(M)\times\overline{\Omega}_{(13)}^{(r,1)}(M)
$$

$$\underrightarrow{i_{\overline{\Omega}_{(13)}^{(p,1)}(M)}^{\overline{\Omega}_{(1)}^{(p,1)}(M)} \times i_{\overline{\Omega}_{(13)}^{(q,1)}(M)}^{\overline{\Omega}_{(1)}^{(q,1)}(M)} \times i_{\overline{\Omega}_{(13)}^{(r,1)}(M)}^{\overline{\Omega}_{(1)}^{(r,1)}(M)}}$$

$$\overline{\Omega}_{(1)}^{(p,1)}(M) \times \overline{\Omega}_{(1)}^{(q,1)}(M) \times \overline{\Omega}_{(1)}^{(r,1)}(M)$$

$$= \overline{\Omega}_{(1)}^{(r,1)}(M) \times \overline{\Omega}_{(1)}^{(p,1)}(M) \times \overline{\Omega}_{(1)}^{(q,1)}(M)$$

$$\underrightarrow{\mathrm{id}_{\overline{\Omega}_{(1)}^{(r,1)}(M)} \times \zeta_{p,q}^{L_1}(M)}$$

$$\overline{\Omega}_{(1)}^{(r,1)}(M) \times \overline{\Omega}_{(1)}^{(p+q,1)}(M)$$

$$\underrightarrow{\zeta_{q,p+r}^{L_1}(M)}$$

$$\overline{\Omega}_{(1)}^{(p+q+r,1)}(M)$$

$$\underrightarrow{\left(\cdot^{\sigma_{r,p+q}}\right)_{\overline{\Omega}_{(1)}^{(p+q+r,1)}(M)}}$$

$$\overline{\Omega}_{(1)}^{(p+q+r,1)}(M)$$

$$\underrightarrow{\mathcal{A}_{p,q,r}^{p+q+r}}$$

$$\overline{\Omega}_{(1)}^{(p+q+r,1)}(M) \tag{28}$$

by dint of Lemmas 61 and 63 with $\varepsilon_{\sigma_{r,p+q}} = (-1)^{r(p+q)}$. The sum of the three morphisms (26)-(28) vanishes thanks to Theorem 47, so that the sum of the three morphisms (23)-(25) also vanishes. □

5.5. The Fourth Consideration

In this subsection we are concerned with the Lie algebra structure of $\overline{\Omega}_{(123)}^{(p,1)}(M)$'s, where p ranges over natural numbers. It is easy to see that

Lemma 65. *The morphism*

$$\overline{\Omega}_{(123)}^{(p,1)}(M) \times \overline{\Omega}_{(123)}^{(q,1)}(M)$$

$$\underrightarrow{i_{\overline{\Omega}_{(123)}^{(p,1)}(M)}^{\overline{\Omega}_{(13)}^{(p,1)}(M)} \times i_{\overline{\Omega}_{(123)}^{(q,1)}(M)}^{\overline{\Omega}_{(13)}^{(q,1)}(M)}}$$

$$\overline{\Omega}_{(13)}^{(p,1)}(M) \times \overline{\Omega}_{(13)}^{(q,1)}(M)$$

$$\underrightarrow{\zeta_{p,q}^{FN_{13}}(M)}$$

$$\overline{\Omega}_{(13)}^{(p+q,1)}(M)$$

is to be factored uniquely through the canonical injection

$$i_{\overline{\Omega}_{(123)}^{(p+q,1)}(M)}^{\overline{\Omega}_{(13)}^{(p+q,1)}(M)} : \overline{\Omega}_{(123)}^{(p+q,1)}(M) \to \overline{\Omega}_{(13)}^{(p+q,1)}(M)$$

into a morphism

$$\overline{\Omega}_{(123)}^{(p,1)}(M) \times \overline{\Omega}_{(123)}^{(q,1)}(M) \to \overline{\Omega}_{(123)}^{(p+q,1)}(M) \quad (29)$$

Notation 66. The morphism (29) is denoted

$$\zeta_{p,q}^{FN_{123}}(M) : \overline{\Omega}_{(123)}^{(p,1)}(M) \times \overline{\Omega}_{(123)}^{(q,1)}(M) \to \overline{\Omega}_{(123)}^{(p+q,1)}(M)$$

Proposition 67. *The morphisms*

$$\begin{array}{l} \overline{\Omega}_{(123)}^{(p,1)}(M) \times \overline{\Omega}_{(123)}^{(q,1)}(M) \\ \underrightarrow{\zeta_{p,q}^{FN_{123}}(M)} \\ \overline{\Omega}_{(123)}^{(p+q,1)}(M) \end{array}$$

and

$$\begin{array}{l} \overline{\Omega}_{(123)}^{(p,1)}(M) \times \overline{\Omega}_{(123)}^{(q,1)}(M) \\ \ = \overline{\Omega}_{(123)}^{(q,1)}(M) \times \overline{\Omega}_{(123)}^{(p,1)}(M) \\ \underrightarrow{\zeta_{q,p}^{FN_{123}}(M)} \\ \overline{\Omega}_{(123)}^{(p+q,1)}(M) \\ \underrightarrow{(-1)^{pq}} \\ \overline{\Omega}_{(123)}^{(p+q,1)}(M) \end{array}$$

sum up only to vanish.

Proof. This follows directly from Proposition 62. □

Theorem 68. *The three morphisms*

$$\begin{array}{l} \overline{\Omega}_{(123)}^{(p,1)}(M) \times \overline{\Omega}_{(123)}^{(q,1)}(M) \times \overline{\Omega}_{(123)}^{(r,1)}(M) \\ \underrightarrow{\mathrm{id}_{\overline{\Omega}_{(123)}^{(p,1)}(M)} \times \zeta_{q,r}^{FN_{123}}(M)} \end{array}$$

$$\begin{aligned}&\overline{\Omega}_{(123)}^{(p,1)}(M)\times\overline{\Omega}_{(123)}^{(q+r,1)}(M)\\&\underrightarrow{\zeta_{p,q+r}^{FN_{123}}(M)}\\&\overline{\Omega}_{(123)}^{(p+q+r,1)}(M)\end{aligned}$$

$$\begin{aligned}&\overline{\Omega}_{(123)}^{(p,1)}(M)\times\overline{\Omega}_{(123)}^{(q,1)}(M)\times\overline{\Omega}_{(123)}^{(r,1)}(M)\\&=\overline{\Omega}_{(123)}^{(q,1)}(M)\times\overline{\Omega}_{(123)}^{(r,1)}(M)\times\overline{\Omega}_{(123)}^{(p,1)}(M)\\&\underrightarrow{\mathrm{id}_{\overline{\Omega}_{(123)}^{(q,1)}(M)}\times\zeta_{r,p}^{FN_{123}}(M)}\\&\overline{\Omega}_{(123)}^{(q,1)}(M)\times\overline{\Omega}_{(123)}^{(p+r,1)}(M)\\&\underrightarrow{\zeta_{q,p+r}^{FN_{123}}(M)}\\&\overline{\Omega}_{(123)}^{(p+q+r,1)}(M)\\&\underrightarrow{(-1)^{p(q+r)}}\\&\overline{\Omega}_{(123)}^{(p+q+r,1)}(M)\end{aligned}$$

and

$$\begin{aligned}&\overline{\Omega}_{(123)}^{(p,1)}(M)\times\overline{\Omega}_{(123)}^{(q,1)}(M)\times\overline{\Omega}_{(123)}^{(r,1)}(M)\\&=\overline{\Omega}_{(123)}^{(r,1)}(M)\times\overline{\Omega}_{(123)}^{(p,1)}(M)\times\overline{\Omega}_{(123)}^{(q,1)}(M)\\&\underrightarrow{\mathrm{id}_{\overline{\Omega}_{(123)}^{(r,1)}(M)}\times\zeta_{p,q}^{FN_{123}}(M)}\\&\overline{\Omega}_{(123)}^{(r,1)}(M)\times\overline{\Omega}_{(123)}^{(p+q,1)}(M)\\&\underrightarrow{\zeta_{r,p+q}^{FN_{123}}(M)}\\&\overline{\Omega}_{(123)}^{(p+q+r,1)}(M)\\&\underrightarrow{(-1)^{r(p+q)}}\\&\overline{\Omega}_{(123)}^{(p+q+r,1)}(M)\end{aligned}$$

sum up only to vanish.

Proof. This follows directly from Theorem 64. □

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