TUPLES OF OPERATORS AND SUPERCYCLICITY

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Abstract: In this paper, we give sufficient conditions for a tuple of operators to be supercyclic.

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1. Introduction

By an n -tuple of operators we mean a finite sequence of length n of commuting continuous linear operators on a Banach space X .

Definition 1.1. Let $\mathcal{T} = (T_1, T_2, \dots, T_n)$ be an n -tuple of operators acting on an infinite dimensional Banach space X . We will let

$$\mathcal{F}_{\mathcal{T}} = \{T_1^{k_1} T_2^{k_2}, \dots, T_n^{k_n} : k_i \geq 0, i = 1, \dots, n\}$$

be the semigroup generated by $\mathcal{T}$. For $x \in X$, the orbit of x under the tuple $\mathcal{T}$ is the set $Orb(\mathcal{T}, x) = \{Sx : S \in \mathcal{F}_{\mathcal{T}}\}$. A vector x is called a hypercyclic vector for $\mathcal{T}$ if $Orb(\mathcal{T}, x)$ is dense in X and in this case the tuple $\mathcal{T}$ is called hypercyclic. Also, a vector x is called a supercyclic vector for $\mathcal{T}$ if $\mathbf{C}Orb(\mathcal{T}, x)$

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is dense in X and in this case the tuple $\mathcal{T}$ is called supercyclic. The set of all supercyclic vectors for a tuple $\mathcal{T}$ is denoted by $SC(\mathcal{T})$.

Supercyclic operators were introduced by Hilden and wallen. They proved that all unilateral backward weighted shifts on a Hilbert space have a supercyclic vector. The study of supercyclic operators has experimented a great development during the last years. Salas gave a characterization of supercyclic bilateral backward weighted shifts via the Supercyclicity Criterion. This Criterion is an important tool in much recent works on supercyclic operators. Here, we want to extend a property of hypercyclic operators to a pair of commuting operators, and although the techniques work for any n -tuple of operators but for simplicity we prove our results only for the case $n = 2$. For some topics we refer to [1–18].

2. Main Results

In this section, we prove that if a tuple $\mathcal{T} = (T_1, T_2, \dots, T_n)$ is supercyclic and $\mathcal{F}_{\mathcal{T}}$ is a bounded set, then $T_1^{k(1)}T_2^{k(2)} \dots T_n^{k(n)} \rightarrow 0$ as $k(i) \rightarrow \infty$ for $i = 1, \dots, n$.

Lemma 2.1. *Let $\mathcal{T} = (T_1, T_2, \dots, T_n)$ be a supercyclic tuple of continuous linear operators acting on a separable infinite dimensional Banach space X . If $SC(\mathcal{T}) \neq \emptyset$, then $SC(\mathcal{T})$ is dense in X .*

Proof. Fix an enumeration $\{B_n : n \in \mathbb{N}\}$ of the open balls in X with rational radii, and centers in a countable dense subset of X . By the continuity of the operators $T_1, T_2, \dots, T_n$, the sets

$$G_j = \bigcup \{ \lambda T_1^{-k(1)} T_2^{-k(2)} \dots T_n^{-k(n)} (B_j) : k(i) \geq 0; i = 1, \dots, n; \lambda \in \mathbb{C} \}$$

are open. Since for any two non-void open sets U and V , there exist $k(i) \geq 1$ for $i = 1, \dots, n$ and $\lambda \in \mathbb{C} \setminus \{0\}$ such that $\lambda T_1^{k(1)} T_2^{k(2)} \dots T_n^{k(n)} (U) \cap V \neq \emptyset$, thus one can see that $SC(\mathcal{T})$ is exactly equal to $\bigcap \{G_j : j \in \mathbb{N}\}$ that is dense in X . This completes the proof. $\square$

Theorem 2.2. *Let $\mathcal{T} = (T_1, T_2, \dots, T_n)$ be supercyclic tuple of continuous linear operators acting on a separable infinite dimensional Banach space X . If $\mathcal{F}_{\mathcal{T}}$ is a bounded set, then $T_1^{k(1)}T_2^{k(2)} \dots T_n^{k(n)} \rightarrow 0$ as $k(i) \rightarrow \infty$ for $i = 1, \dots, n$.*

Proof. Since $\mathcal{F}_{\mathcal{T}}$ is a bounded set, there exists $M > 0$, such that

$$\|T_1^{k(1)}T_2^{k(2)} \dots T_n^{k(n)}\| \leq M$$

for all $k(i) \geq 0$ for $i = 1, \dots, n$. Let x be a supercyclic vector for $\mathcal{T}$ such that

$$\lim_{k(1), \dots, k(n)} T_1^{k(1)} T_2^{k(2)} \dots T_n^{k(n)} x \neq 0.$$

So there exists $r > 0$ such that $\|T_1^{k(1)} T_2^{k(2)} \dots T_n^{k(n)} x\| > r > 0$ for every integers $k(i) \geq 0, i = 1, \dots, n$. Now, since $SC(\mathcal{T}) \neq \emptyset$, thus there exists a nonzero $y \in X$ with $\|y\| = 1$ such that

$$\lim_{k(1), \dots, k(n)} T_1^{k(1)} T_2^{k(2)} \dots T_n^{k(n)} y \neq 0.$$

Since $Orb(\mathcal{T}, x)$ is dense in X , there exist sequences of scalars $\{c_j\} \subset \mathbf{C}$, $\{k_j(i)\}_j \subset \mathbf{N}; i = 1, \dots, n$ such that $c_j T_1^{k_j(1)} T_2^{k_j(2)} \dots T_n^{k_j(n)} x \rightarrow y$ as $j \rightarrow \infty$. Thus, there exists a natural number k_0 such that for all $j \geq k_0$ we have

$$\|c_j T_1^{k_j(1)} T_2^{k_j(2)} \dots T_n^{k_j(n)} x - y\| < 1/2.$$

Thus $(2M\|x\|)^{-1} \leq 2^{-1} \|T_1^{k_j(1)} T_2^{k_j(2)} \dots T_n^{k_j(n)} x\|^{-1} < |c_j|$ for all $j \geq k_0$. Define $\alpha = r(3M^2\|x\|)^{-1}$. Since y is a supercyclic vector for $\mathcal{T}$, there exists $j_0 > k_0$, $c_{j_0} \in \mathbf{C}$, and $k_{j_0}(i) \in \mathbf{N}; i = 1, \dots, n$ such that

$$\|c_{j_0} T_1^{k_{j_0}(1)} T_2^{k_{j_0}(2)} \dots T_n^{k_{j_0}(n)} x - y\| < \alpha/2.$$

Since

$$\lim_{k(1), \dots, k(n)} T_1^{k(1)} T_2^{k(2)} \dots T_n^{k(n)} y \neq 0,$$

thus for some $k(1), k(2), \dots, k(n) \in \mathbf{N}$, we have $\|T_1^{k(1)} T_2^{k(2)} \dots T_n^{k(n)} y\| \leq M\alpha/2$. Put

$$L = c_{j_0} T_1^{k_{j_0}(1)+k(1)} T_2^{k_{j_0}(2)+k(2)} \dots T_n^{k_{j_0}(n)+k(n)} x - T_1^{k(1)} T_2^{k(2)} \dots T_n^{k(n)} y.$$

So we get

$$\begin{aligned} \|L\| &= \|T_1^{k(1)} T_2^{k(2)} \dots T_n^{k(n)} (c_{j_0} T_1^{k_{j_0}(1)} T_2^{k_{j_0}(2)} \dots T_n^{k_{j_0}(n)} x - y)\| \\ &\leq \|T_1^{k(1)} T_2^{k(2)} \dots T_n^{k(n)}\| \|c_{j_0} T_1^{k_{j_0}(1)} T_2^{k_{j_0}(2)} \dots T_n^{k_{j_0}(n)} x - y\| \\ &< M\alpha/2. \end{aligned}$$

Note that

$$\begin{aligned} r(2M\|x\|)^{-1} &\leq \|c_{j_0} T_1^{k_{j_0}(1)} T_2^{k_{j_0}(2)} \dots T_n^{k_{j_0}(n)} x\| \\ &\leq \|c_{j_0} T_1^{k_{j_0}(1)} T_2^{k_{j_0}(2)} \dots T_n^{k_{j_0}(n)} x - T_1^{k(1)} T_2^{k(2)} \dots T_n^{k(n)} y\| \end{aligned}$$

$$\begin{aligned}
& + \|T_1^{k(1)}T_2^{k(2)}\dots T_n^{k(n)}y\| \\
& \leq r(3M\|x\|)^{-1},
\end{aligned}$$

that is a contradiction. Thus, indeed $T_1^{k(1)}T_2^{k(2)}\dots T_n^{k(n)} \rightarrow 0$ as $k(i) \rightarrow \infty$ for $i = 1, \dots, n$ whenever $x \in SC(\mathcal{T})$. Now, since $SC(\mathcal{T})$ is dense in X , thus it should be $T_1^{k(1)}T_2^{k(2)}\dots T_n^{k(n)} \rightarrow 0$ as $k(i) \rightarrow \infty$ for $i = 1, \dots, n$ for all $x \in X$. This completes the proof. $\square$

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