Abstract: In this paper, we give some conditions for a weighted composition operator in the space of analytic functions on a region of the plane domain have an eigenvalue.

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1. Introduction

Let $\Omega$ be a domain in the complex plane, then the space $H(\Omega)$ of all complex-valued functions analytic on $\Omega$ can be made into a F-space by a complete metric for which a sequence $\{f_n\}$ in $H(\Omega)$ converges to $f \in H(\Omega)$ if and only if $f_n \rightarrow f$ uniformly on every compact subsets of $\Omega$. Each $\varphi \in H(\Omega)$ and analytic self-map $\psi$ of $\Omega$ induces a linear weighted composition operator $C_{\varphi,\psi} : H(\Omega) \rightarrow H(\Omega)$ by

$$C_{\varphi,\psi}(f)(z) = \varphi(z)f(\psi(z))$$

for every $f \in H(\Omega)$ and $z \in \Omega$. Indeed, $C_{\varphi,\psi} = M_{\varphi}C_{\psi}$ where $M_{\varphi}$ denotes the operator of multiplication by $\varphi$ and $C_{\psi}$ is a composition operator by means
of the definition $C_\psi(f) = f \circ \psi$ for every $f \in H(\Omega)$. For some details on composition operators one can see [1–10].

2. Main Results

By $H^\infty(\Omega)$ we denote the space of bounded analytic functions on $\Omega$. We will denote the open unit disk by $U$ and for a set $A$, we will write $\overline{A}$ to denote the closure of $A$. Also, $\psi_n$ means the $n$th iterate of $\psi$ and $\|f\|_\Omega$ denotes the supremum norm of $f$ on $\Omega$.

**Lemma 2.1.** Let $\varphi$ be a holomorphic self-map of the open unit disc $U$. Then for every pair of points $p, q \in U$ we have $d(\varphi(p), \varphi(q)) \leq d(p, q)$. Moreover, there is equality here for some pair of points if and only if there is equality for all points, and this happens if and only if $\varphi$ is a conformal automorphism of $U$.

**Proof.** See [5, p. 60].

**Theorem 2.2.** Let $\psi$ be an analytic self-map of a bounded simply connected domain $\Omega$ with a fixed point in $\Omega$. Also, let $\varphi \in H^\infty(\Omega)$ be such that $\varphi$ is not zero at the fixed point of $\psi$. If the closure of $\psi(\Omega)$ is contained in $\Omega$, then $\varphi(w)$ is an eigenvalue for $C_{\varphi, \psi}$ acting on $H(\Omega)$.

**Proof.** Let $w$ be the fixed point of $\psi$. By the Riemann Mapping Theorem there exists an $R \in H(U)$ such that $R$ is univalent in $U$ and $R(U) = \Omega$ (see [4, Theorem 14.8, p. 283]). Clearly

$$\{f \circ R : f \in H(\Omega)\} \subseteq H(U).$$

Also for any $f \in H(U)$ clearly $f \circ R^{-1} \in H(\Omega)$, thus indeed

$$H(U) = \{f \circ R : f \in H(\Omega)\}.$$

Put $\Phi = \varphi \circ R$ and $\Psi = R^{-1} \circ \psi \circ R$. Then $\Phi \in H^\infty(U)$ and $\Psi$ is an analytic self-map of the open unit disc $U$. Since $\overline{\psi(\Omega)}$ is compact, hence clearly $R^{-1}(\overline{\psi(\Omega)})$ is closed and so we have

$$\overline{\Psi(U)} = R^{-1} \circ \overline{\psi(\Omega)} \subseteq R^{-1}(\overline{\psi(\Omega)}) \subseteq R^{-1}(\Omega) = U.$$

This implies that there exists $0 < \lambda < 1$ such that $\Psi(U) \subseteq \lambda U$. So by the Schwarz’s Lemma we have $|\Psi(z)| \leq \lambda |z|$ for all $z \in U$. Set

$$h(z) = (\Phi(z) - \Phi(0))/2\|\Phi\|_U.$$
Then \( h \) is a self-map of \( U \) and \( h(0) = 0 \). Thus by the Schwarz’s Lemma ([1, p. 130]), \(|h(z)| \leq |z|\) which implies that

\[
|\Phi(z)| \leq 2||\Phi||_U |z| + |\Phi(0)|
\]

for every \( z \in U \). Now by substituting \( \Phi(z) \) instead of \( z \) in the above inequality we get

\[
|\Phi(\Psi(z))| \leq 2\lambda^n||\Phi||_U |z| + |\Phi(0)|.
\]

But \( \Phi(0) \neq 0 \), thus

\[
\frac{|\Phi(\Psi(z))|}{|\Phi(0)|} \leq \exp\left(\frac{2||\Phi||_U}{|\Phi(0)|} \lambda^n\right),
\]

since \( 1 + x \leq e^x \) for all \( x \in \mathbb{R} \). Hence

\[
\prod_{n=0}^{\infty} \frac{1}{\Phi(0)} \Phi(z) \leq \exp\left(\sum_{n=0}^{\infty} \frac{2||\Phi||_U}{|\Phi(0)|} \lambda^n\right) = \exp\left(\frac{2||\Phi||_U}{|\Phi(0)|} \frac{1}{1 - \lambda}\right)
\]

for every \( z \in U \). Set

\[
G(z) = \prod_{n=0}^{\infty} \frac{1}{\Phi(0)} \Phi(\Psi(z)),
\]

then \( G \) is nonzero and belongs to \( H^\infty(U) \subseteq H(U) \). Also, note that

\[
G(z) = \frac{1}{\Phi(0)} \Phi(z) \prod_{n=1}^{\infty} \frac{1}{\Phi(0)} \Phi(\Psi(z))
\]

and consequently

\[
\Phi(0)G(z) = \Phi(z) \prod_{n=1}^{\infty} \frac{1}{\Phi(0)} \Phi(\Psi(z)).
\]

But

\[
G \circ \Psi(z) = \prod_{n=0}^{\infty} \frac{1}{\Phi(0)} \Phi(\Psi_{n+1}(z)) = \prod_{n=0}^{\infty} \frac{1}{\Phi(0)} \Phi(\Psi_n(z)),
\]
thus indeed

\[ C_{\Phi, \Psi} G = \Phi(0)G. \]

If \( u \neq 0 \), consider the self maps \( \Psi_1 = \alpha_u \circ \Psi \circ \alpha_u \) and \( \Phi_1 = \Phi \circ \alpha_u \) where \( \alpha_u(z) = \frac{(u - z)}{(1 - \overline{u}z)} \). Clearly we can see that \( \Psi_1(0) = 0 \) and \( \Phi_1(0) \neq 0 \). Also, note that by the Lemma 2.1, \( \alpha_u(\lambda U) \subseteq \lambda U \) which implies that \( \Psi_1(U) \subseteq U \). So the first part of the proof shows that there exists a nonzero bounded analytic function \( G_1 \) on \( U \) such that

\[ C_{\Phi_1, \Psi_1} G_1 = \Phi_1(0)G_1. \]

Therefore

\[
\Phi \circ \alpha_u(z) \cdot G_1 \circ (\alpha_u \circ \Psi \circ \alpha_u)(z) = (\Phi \circ \alpha_u(0))G_1(z) = \Phi(u)G_1(z).
\]

Note that \( \alpha_u \circ \alpha_u(z) = z \). By substituting \( \alpha_u(z) \) instead of \( z \) in the above equality we get \( C_{\Phi, \Psi} G = \Phi(u)G \) where \( G = G_1 \circ \alpha_u \) is a nonzero function in \( H(U) \). Hence we have proved that \( \Phi \cdot G \circ \Psi = \Phi(u) \cdot G \). This implies that

\[
\varphi \circ R(z) \cdot G \circ R^{-1} \circ \psi \circ R(z) = \varphi \circ R(0) \cdot G(z) = \varphi(w)G(z)
\]

and so by substituting \( R^{-1}(z) \) instead of \( z \) in the above relation we get \( \varphi \cdot g \circ \psi = \varphi(w) \cdot g \), where \( g = G \circ R^{-1} \in H(\Omega) \). Thus \( \varphi(w) \) is an eigenvalue for \( C_{\varphi, \psi} \) acting on \( H(\Omega) \) and so the proof is complete.

\[ \square \]

References


