A SUBCLASS OF MULTIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS DEFINED BY GENERALIZED FRACTIONAL CALCULUS

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Abstract: In this paper we defined two new subclasses of multivalent functions associated with generalized fractional calculus operator. We obtained coefficients estimates, distortion bounds, extreme points, neighborhood property, inclusion results, modified Hadamard product. Further, we discussed radii of starlikeness and convexity, results on quasi Hadamard product and class preserving integral operator for functions in the newly defined subclass.

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1. Introduction and Definitions

Let $A_p(n)$ denote the class of functions of the form

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$$f(z) = z^p + \sum_{k=n+p} a_k z^k \quad (p, n \in \mathbb{N} = \{1, 2, 3, \dots\}), \quad (1.1)$$

which are analytic and p -valent in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$.

Let $T_p(n)$ be the subclass of $A_p(n)$, consisting of functions of the form

$$f(z) = z^p - \sum_{k=p+n} a_k z^k \quad (p, n \in \mathbb{N}; a_k \geq 0) \quad (1.2)$$

which are p -valent in $\mathbb{U}$.

The Hadamard product of two power series

$$f(z) = z^p + \sum_{k=n+p} a_k z^k \text{ and } g(z) = z^p + \sum_{k=n+p} b_k z^k$$

is defined by

$$(f * g)(z) = z^p + \sum_{k=n+p} a_k b_k z^k. \quad (1.3)$$

Let $\phi_p(a, c; z)$ be the incomplete beta function defined by

$$\begin{aligned} \phi_p(a, c; z) &= z^p + \sum_{k=p+n} \frac{(a)_{k-p}}{(c)_{k-p}} z^k, \\ a &\in \mathbb{R}, \quad c \in \mathbb{R} \setminus \mathbb{Z}_0^- = \{\dots, -2, -1, 0\}, \quad z \in \mathcal{U}, \end{aligned} \quad (1.4)$$

where $(a)_k$ is the Pochhammer symbol defined by

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = \begin{cases} 1, & k = 0 \\ a(a+1)(a+2)\dots(a+k-1), & k \in \mathbb{N}. \end{cases}$$

Further, for $f \in A_p(n)$

$$\begin{aligned} L_p(a, c)f(z) &= \phi_p(a, c; z) * f(z), \\ &= z^p + \sum_{k=p+n} \frac{(a)_{k-p}}{(c)_{k-p}} a_k z^k, \quad z \in \mathbb{U}, \end{aligned}$$

where $L_p(a, c)$ leads to Saitoh operator [19] for $n = 1$ and yields Carlson - Shaffer operator [4] for $n = 1$ and $p = 1$.

Definition 1.1. For real numbers $\mu > 0$, γ and η , Saigo hypergeometric fractional integral operator $I_{0,z}^{\mu,\gamma,\eta}$ is defined by

$$I_{0,z}^{\mu,\gamma,\eta} f(z) = \frac{z^{-\mu-\gamma}}{\Gamma(\mu)} \int_0^z (z-t)^{\mu-1} {}_2F_1\left(\mu + \gamma, -\eta; \mu; 1 - \frac{t}{z}\right) f(t) dt, \quad (1.5)$$

where the function $f(z)$ is analytic in a simply-connected region of the complex z -plane containing the origin, with the order

$$f(z) = O(|z|^\xi) \quad (z \rightarrow 0; \xi > \max\{0, \gamma - \eta\} - 1)$$

and the multiplicity of $(z-t)^{\mu-1}$ is removed by requiring $\log(z-t)$ to be real when $(z-t) > 0$.

Definition 1.2. Under the hypotheses of Definition 1.1, Saigo hypergeometric fractional derivative operator $J_{0,z}^{\mu,\gamma,\eta}$ is defined by

$$J_{0,z}^{\mu,\gamma,\eta} f(z) = \begin{cases} \frac{\frac{d}{dz} \left\{ z^{\mu-\gamma} \int_0^z (z-t)^{-\mu} {}_2F_1\left(\gamma-\mu, 1-\eta; 1-\mu; 1-\frac{t}{z}\right) f(t) dt \right\}}{\Gamma(1-\mu)}, & 0 \leq \mu < 1, \\ \frac{d^k}{dz^k} J_{0,z}^{\mu-k,\gamma,\eta} f(z), & k \leq \mu < k+1; k \in \mathbb{N}, \end{cases} \quad (1.6)$$

where the multiplicity of $(z-t)^{-\mu}$ is removed as in Definition 1.1.

It may be remarked that

$$I_{0,z}^{\mu,-\mu,\eta} f(z) = D_z^{-\mu} f(z) \quad (\mu > 0)$$

and

$$J_{0,z}^{\mu,\mu,\eta} f(z) = D_z^\mu f(z) \quad (0 \leq \mu < 1),$$

where $D_z^{-\mu}$ denotes fractional integral operator and D_z^μ denotes fractional derivative operator considered by Owa [13] and subsequently by Owa and Srivastava [14].

Let $U_{0,z}^{\mu,\gamma,\eta} : A_p(n) \rightarrow A_p(n)$, be a generalized fractional differintegral operator defined by

$$U_{0,z}^{\mu,\gamma,\eta} f(z) = \begin{cases} \frac{\Gamma(1+p-\gamma)\Gamma(1+p+\eta-\mu)}{\Gamma(1+p)\Gamma(1+p+\eta-\gamma)} z^\gamma J_{0,z}^{\mu,\gamma,\eta} f(z), & 0 \leq \mu < \eta + p + 1, z \in \mathbb{U}, \\ \frac{\Gamma(1+p-\gamma)\Gamma(1+p+\eta-\mu)}{\Gamma(1+p)\Gamma(1+p+\eta-\gamma)} z^\gamma I_{0,z}^{-\mu,\gamma,\eta} f(z), & -\infty < \mu < 0, z \in \mathbb{U}. \end{cases} \quad (1.7)$$

For $n = 1$ the generalized fractional differintegral operator was considered by Goyal and prajapat [7] (see also [16]). It is easily seen from (1.7) that for a function f of the form (1.1), we have

$$\begin{aligned} U_{0,z}^{\mu,\gamma,\eta} f(z) &= z^p + \sum_{k=n+p} \frac{(1+p)_k (1+p+\eta-\gamma)_k}{(1+p-\gamma)_k (1+p+\eta-\mu)_k} a_k z^k \\ &= z^p {}_3F_1(1, 1+p, 1+p+\eta-\gamma; 1+p-\gamma, 1+p+\eta-\mu; z) * f(z) \\ &\quad (z \in \mathbb{U}; p \in \mathbb{N}; \gamma, \eta \in \mathbb{R}; \gamma < p+1; -\infty < \mu < \eta+p+1), \end{aligned} \quad (1.8)$$

where $*$ is given by (1.3) and ${}_pF_q$ is well known generalized hypergeometric function.

Note that

$$U_{0,z}^{0,0,0} f(z) = f(z), \quad U_{0,z}^{1,1,1} f(z) = U_{0,z}^{1,0,0} f(z) = \frac{zf(z)}{p}$$

and

$$U_{0,z}^{2,1,1} f(z) = \frac{zf(z) + z^2 f'(z)}{p^2}.$$

We also remark that

$$U_{0,z}^{\mu,\mu,\eta} f(z) = U_{0,z}^{\mu,\gamma,0} f(z) = \Omega_z^{\mu,p} f(z),$$

where $\Omega_z^{\mu,p}$ is an extended fractional differintegral operator studied very recently by [15](see also [17]). On the other hand, if we set

$$\mu = -\delta, \quad \gamma = 0 \text{ and } \eta = \nu - 1,$$

in (1.8) and using

$$I_{0,z}^{\delta,0,\nu-1} f(z) = \frac{1}{z^\nu \Gamma(\delta)} \int_0^z t^{\nu-1} \left(1 - \frac{t}{z}\right)^{\delta-1} f(t) dt,$$

we obtain following p -valent generalization of multiplier transformation operator [9, 10]:

$$Q_{\nu,p}^\delta f(z) = \binom{p+\delta+\nu-1}{p+\nu-1} \frac{\delta}{z^\nu} \int_0^z t^{\nu-1} \left(1 - \frac{t}{z}\right)^{\delta-1} f(t) dt \quad (1.9)$$

$$= z^p + \sum_{k=n+p} \frac{\Gamma(\nu+k)\Gamma(p+\delta+\nu)}{\Gamma(\delta+\nu+k)\Gamma(p+\nu)} a_k z^k \quad (\nu > -p, \delta+\nu > -p)$$

on the other hand, if we set

$$\mu = -1, \quad \gamma = 0 \text{ and } \eta = \nu - 1,$$

in (1.8), we obtain the generalized Bernardi-Libera integral operator $F_{\nu,p} : A_p(n) \rightarrow A_p(n)$ ($\nu > -p$) defined by

$$\begin{aligned} F_{\nu,p}f(z) &= \frac{p+\nu}{z^\nu} \int_0^z t^{\nu-1} f(t) dt \\ &= z^p + \sum_{k=n+p} \frac{p+\nu}{\nu+k} a_k z^k \quad (\nu > -p). \end{aligned} \quad (1.10)$$

For the choice $p = 1$, $n = 1$ and $\nu \in \mathbb{N}$, the operator defined by (1.10) reduces to the well-known Bernardi integral operator. Also the operators $Q_{\nu,p}^\delta$ and $F_{\nu,p}$ were studied recently in [12].

Further, For $f \in A_p(n)$

$$\begin{aligned} M_{0,z}^{\mu,\gamma,\eta} f(z) &= \phi_p(a, c, z) * U_{0,z}^{\mu,\gamma,\eta} f(z) \\ &= z^p + \sum_{k=p+n} \frac{(a)_{k-p}(1+p)_{k-p}(1+p+\eta-\gamma)_{k-p}}{(c)_{k-p}(1+p-\gamma)_{k-p}(1+p+\eta-\mu)_{k-p}} a_k z^k \\ &= z^p + \sum_{k=p+n} \Phi(k) a_k z^k \quad (z \in \mathbb{U}), \end{aligned} \quad (1.11)$$

where,

$$\Phi(k) = \frac{(a)_{k-p}(1+p)_{k-p}(1+p+\eta-\gamma)_{k-p}}{(c)_{k-p}(1+p-\gamma)_{k-p}(1+p+\eta-\mu)_{k-p}}. \quad (1.12)$$

A function $f \in A_p$ is said to be p -valently starlike of order α ($0 \leq \alpha < p$) in $\mathbb{U}$ if and only if,

$$Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \quad (z \in \mathbb{U}). \quad (1.13)$$

Similarly, a function $f(z)$ is said to be p -valently convex of order α ($0 \leq \alpha < p$) in $\mathbb{U}$ if and only if,

$$Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha \quad (z \in \mathbb{U}). \quad (1.14)$$

It follows from expression (1.13) and (1.14) that $f(z)$ is convex if and only if, $zf'(z)$ is starlike. A function $f(z) \in A_p$ is close-to-convex of order α if

$$Re \left\{ \frac{f'(z)}{z^{p-1}} \right\} > \alpha \quad (0 \leq \alpha < p). \quad (1.15)$$

Note that

$$\begin{aligned} L_p(m+1, 1)f(z) &= \frac{z^p}{(1-z)^{m+p}} * f(z) \\ &= z^p + \sum_{k=p+n} \frac{(m+1)_{k-p}}{(1)_{k-p}} a_k z^k \\ &= D^{m+p-1} f(z) \end{aligned} \quad (1.16)$$

is the Ruscheweyh derivative of order m .

Following the work of Goodman [6] and Ruscheweyh [18], we define the (n, δ) - neighborhood of a function $f \in A_p$ by

$$N_{n,\delta}(f) = \{g : g \in A_p, g(z) = z^p - \sum_{k=p+n} b_k z^k \text{ and } \sum_{k=p+n} k|a_k - b_k| \leq \delta\}. \quad (1.17)$$

In particular, for the function $e(z) = z^p$, $(p \in \mathbb{N})$

$$N_{n,\delta}(e) = \{g \in A_p, g(z) = z^p - \sum_{k=p+n} b_k z^k \text{ and } \sum_{k=p+n} k|b_k| \leq \delta\}. \quad (1.18)$$

A function $f \in A_p$ defined by (1.2) is said to be in the class $S^{\alpha_1}(\mu, \gamma, \eta, \alpha, \beta)$ if there exists a function $k(z) \in S(\mu, \gamma, \eta, \alpha, \beta)$, such that

$$\left| \frac{f(z)}{k(z)} - 1 \right| < p - \alpha_1 \quad (z \in U, 0 \leq \alpha_1 < p). \quad (1.19)$$

Inspired by the works of Altintas et al., [1, 2], Aouf [3], Darwish and Aouf [5], Khairnar and More [8], now by making use of the operator $M_{0,z}^{\mu,\gamma,\eta}$, we define a new subclass of functions in $A_p(n)$ as follows.

Definition 1.3. A function $f \in A_p$ is said to be in the class $S(\mu, \gamma, \eta, \alpha, \beta)$ if it satisfies the following inequality

$$Re \left\{ \frac{z(M_{0,z}^{\mu,\gamma,\eta} f(z))' + \lambda z^2 (M_{0,z}^{\mu,\gamma,\eta} f(z))''}{(1-\lambda)M_{0,z}^{\mu,\gamma,\eta} f(z) + \lambda z (M_{0,z}^{\mu,\gamma,\eta} f(z))'} \right\} > \alpha, \quad (1.20)$$

where $0 \leq \lambda \leq 1$, $0 \leq \alpha < 1$.

Definition 1.4. A function $f \in A_p$ is said to be in the class $M(\mu, \gamma, \eta, \alpha, \beta)$ if it satisfies the following inequality

$$Re \left\{ \frac{z(M_{0,z}^{\mu,\gamma,\eta} f(z))' + \lambda z^2 (M_{0,z}^{\mu,\gamma,\eta} f(z))''}{(1-\lambda)M_{0,z}^{\mu,\gamma,\eta} f(z) + \lambda z (M_{0,z}^{\mu,\gamma,\eta} f(z))'} \right\} < \beta, \quad (1.21)$$

where $0 \leq \lambda \leq 1$, $\beta > 1$.

2. Coefficient Estimates

Theorem 2.1. A function $f(z)$ defined by (1.2) is in $S(\mu, \gamma, \eta, \alpha, \beta)$ if

$$\sum_{k=p+n} [1 + \lambda(k-1)](k-\alpha)\Phi(k)|a_k| \leq (p-\alpha)[1 + \lambda(p-1)], \quad (2.1)$$

where $0 \leq \lambda \leq 1$, $0 \leq \alpha < 1$ and $\Phi(k)$ is given by (1.12).

Proof. Let $f(z) \in S(\mu, \gamma, \eta, \alpha, \beta)$. Assume that inequality (2.1) holds true. Then we find that

$$\begin{aligned} & \left| \frac{z(M_{0,z}^{\mu,\gamma,\eta} f(z))' + \lambda z^2 (M_{0,z}^{\mu,\gamma,\eta} f(z))''}{(1-\lambda)M_{0,z}^{\mu,\gamma,\eta} f(z) + \lambda z (M_{0,z}^{\mu,\gamma,\eta} f(z))'} - p \right| \\ &= \left| \frac{\sum_{k=p+n} [1 + \lambda(k-1)](k-p)\Phi(k)a_k z^k}{z^p [1 + \lambda(p-1)] + \sum_{k=p+n} [1 + \lambda(k-1)]\Phi(k)a_k z^k} \right| \\ &\leq \frac{\sum_{k=p+n} [1 + \lambda(k-1)](k-p)\Phi(k)a_k |z|^{k-p}}{[1 + \lambda(p-1)] + \sum_{k=p+n} [1 + \lambda(k-1)]\Phi(k)a_k |z|^{k-p}}. \end{aligned}$$

Choosing values of z on real axis and letting $z \rightarrow 1^-$ we have

$$\leq \frac{\sum_{k=p+n} [1 + \lambda(k-1)](k-p)\Phi(k)|a_k|}{[1 + \lambda(p-1)] - \sum_{k=p+n} [1 + \lambda(k-1)]\Phi(k)|a_k|}.$$

This last expression is bounded above by $(p-\alpha)$ if

$$\sum_{k=p+n} [1 + \lambda(k-1)](k-\alpha)\Phi(k)|a_k| \leq (p-\alpha)[1 + \lambda(p-1)].$$

This completes the proof of the Theorem 2.1. □

The result is sharp for the function $f(z)$ given by

$$f(z) = z^p + \frac{(p - \alpha)[1 + \lambda(p - 1)]}{[1 + \lambda(k - 1)](k - \alpha)\Phi(k)} z^k.$$

Corollary 2.1. *Let the function f defined by (1.2) be in the class $S(\mu, \gamma, \eta, \alpha, \beta)$. Then*

$$a_k \leq \frac{(p - \alpha)[1 + \lambda(p - 1)]}{[1 + \lambda(k - 1)](k - \alpha)\Phi(k)}, (k \geq p + n, n \in N)$$

where $\Phi(k)$ is given by (1.12).

Theorem 2.2. *Let $\beta > 1$ and $0 \leq \gamma \leq 1$. A function $f(z)$ defined by (1.2) is in $M(\mu, \gamma, \eta, \alpha, \beta)$ if*

$$\sum_{k=p+n} [1 + \lambda(k - 1)](k - \beta)\Phi(k)|a_k| \leq \beta[1 + \lambda(p - 1)], \quad (2.2)$$

where $0 \leq \lambda \leq 1$, $\beta > 1$ and $\Phi(k)$ is given by (1.12).

Proof. Let $f(z) \in M(\mu, \gamma, \eta, \alpha, \beta)$. It is sufficient to show that

$$\begin{aligned} & \left| \frac{z(M_{0,z}^{\mu,\gamma,\eta} f(z))' + \lambda z^2 (M_{0,z}^{\mu,\gamma,\eta} f(z))''}{(1 - \lambda)M_{0,z}^{\mu,\gamma,\eta} f(z) + \lambda z (M_{0,z}^{\mu,\gamma,\eta} f(z))'} \right| \\ & < \left| \frac{z(M_{0,z}^{\mu,\gamma,\eta} f(z))' + \lambda z^2 (M_{0,z}^{\mu,\gamma,\eta} f(z))''}{(1 - \lambda)M_{0,z}^{\mu,\gamma,\eta} f(z) + \lambda z (M_{0,z}^{\mu,\gamma,\eta} f(z))'} - 2\beta \right| \end{aligned}$$

Now,

$$\begin{aligned} & \left| \frac{z(M_{0,z}^{\mu,\gamma,\eta} f(z))' + \lambda z^2 (M_{0,z}^{\mu,\gamma,\eta} f(z))''}{(1 - \lambda)M_{0,z}^{\mu,\gamma,\eta} f(z) + \lambda z (M_{0,z}^{\mu,\gamma,\eta} f(z))'} \right| \\ & - \left| \frac{z(M_{0,z}^{\mu,\gamma,\eta} f(z))' + \lambda z^2 (M_{0,z}^{\mu,\gamma,\eta} f(z))''}{(1 - \lambda)M_{0,z}^{\mu,\gamma,\eta} f(z) + \lambda z (M_{0,z}^{\mu,\gamma,\eta} f(z))'} - 2\beta \right| < 0, \end{aligned}$$

$$\left| pz^p[1 + \lambda(p - 1)] + \sum_{k=p+n} \Phi(k)k[1 + \lambda(k - 1)]a_k z^k \right|$$

$$\begin{aligned}
& - \left| z^p [1 + \lambda(p-1)](p-2\beta) + \sum_{k=p+n} \Phi(k)[1 + \lambda(k-1)](k-2\beta)a_k z^k \right| < 0, \\
& |z|^p p[1 + \lambda(p-1)] + \sum_{k=p+n} \Phi(k)k[1 + \lambda(k-1)]a_k |z|^k \\
& - \left\{ |z|^p [1 + \lambda(p-1)](2\beta-p) - \sum_{k=p+n} \Phi(k)[1 + \lambda(k-1)](k-2\beta)a_k |z|^k \right\} < 0, \\
& p[1 + \lambda(p-1)] + \sum_{k=p+n} \Phi(k)k[1 + \lambda(k-1)]a_k \\
& - [1 + \lambda(p-1)](2\beta-p) + \sum_{k=p+n} \Phi(k)[1 + \lambda(k-1)](k-2\beta)a_k < 0, \\
& \sum_{k=p+n} (k-\beta)[1 + \lambda(k-1)]\Phi(k)a_k < \beta[1 + \lambda(p-1)]. \tag{2.3}
\end{aligned}$$

This completes the proof of the Theorem 2.2. $\square$

The result is sharp for the function $f(z)$ given by

$$f(z) = z^p + \frac{\beta[1 + \lambda(p-1)]}{(k-\beta)[1 + \lambda(k-1)]\Phi(k)} z^k.$$

Corollary 2.2. *Let the function f defined by (1.2) be in the class $S(\mu, \gamma, \eta, \alpha, \beta)$. Then*

$$a_k \leq \frac{\beta[1 + \lambda(p-1)]}{(k-\beta)[1 + \lambda(k-1)]\Phi(k)}, \quad (k \geq p+n, n \in N)$$

where $\Phi(k)$ is given by (1.12).

3. Distortion Theorem

In this section, we obtain bounds for functions in the class $S(\mu, \gamma, \eta, \alpha, \beta)$.

Theorem 3.1. *Let the function f defined by (1.2) be in the class $S(\mu, \gamma, \eta, \alpha, \beta)$. Then for $|z| = r$ we have*

$$\begin{aligned} r^p - r^{p+n} \frac{(p-\alpha)[1+\lambda(p-1)]}{[1+\lambda(p+n-1)][p+n-\alpha]\Phi(p+n)} &\leq |f(z)| \\ &\leq r^p + r^{p+n} \frac{(p-\alpha)[1+\lambda(p-1)]}{[1+\lambda(p+n-1)][p+n-\alpha]\Phi(p+n)} \end{aligned}$$

and the result are sharp for f given by

$$f(z) = z^p + z^{p+n} \frac{(p-\alpha)[1+\lambda(p-1)]}{[1+\lambda(p+n-1)][p+n-\alpha]\Phi(p+n)}.$$

Proof. Given that $f(z) \in S(\mu, \gamma, \eta, \alpha, \beta)$, from the equation (2.1), we have

$$\begin{aligned} &[1+\lambda(p+n-1)][p+n-\alpha]\Phi(p+n) \sum_{k=p+n} |a_k| \\ &\leq \sum_{k=p+n} [1+\lambda(k-1)](k-\alpha)\Phi(k)|a_k| \leq (p-\alpha)[1+\lambda(p-1)], \end{aligned}$$

which is equivalent to,

$$\sum_{k=p+n} |a_k| \leq \frac{(p-\alpha)[1+\lambda(p-1)]}{[1+\lambda(p+n-1)][p+n-\alpha]\Phi(p+n)} \quad (3.1)$$

Using (1.2) and (3.1), we obtain

$$\begin{aligned} f(z) &= z^p + \sum_{k=p+n} a_k z^k \\ |f(z)| &\leq |z|^p + \sum_{k=p+n} |a_k| |z|^k \\ &\leq r^p + \sum_{k=p+n} |a_k| r^k \\ &\leq r^p + r^{p+n} \sum_{k=p+n} |a_k| \\ &\leq r^p + r^{p+n} \frac{(p-\alpha)[1+\lambda(p-1)]}{[1+\lambda(p+n-1)][p+n-\alpha]\Phi(p+n)}. \end{aligned}$$

Also,

$$\begin{aligned}
 f(z) &= z^p + \sum_{k=p+n} a_k z^k \\
 |f(z)| &\geq |z|^p - \sum_{k=p+n} |a_k| |z|^k \\
 &\geq r^p - \sum_{k=p+n} |a_k| r^k \\
 &\geq r^p - r^{p+n} \sum_{k=p+n} |a_k| \\
 &\geq r^p - r^{p+n} \frac{(p-\alpha)[1+\lambda(p-1)]}{[1+\lambda(p+n-1)][p+n-\alpha]\Phi(p+n)}.
 \end{aligned}$$

Hence, the proof for Theorem 3.1 is complete. $\square$

Theorem 3.2. *Let the function f defined by (1.2) be in the class $S(\mu, \gamma, \eta, \alpha, \beta)$. Then for $|z| = r$ we have*

$$\begin{aligned}
 &pr^{p-1} - (p+n)r^{p+n-1} \frac{(p-\alpha)[1+\lambda(p-1)]}{[1+\lambda(p+n-1)][p+n-\alpha]\Phi(p+n)} \\
 \leq |f'(z)| &\leq pr^{p-1} + (p+n)r^{p+n-1} \frac{(p-\alpha)[1+\lambda(p-1)]}{[1+\lambda(p+n-1)][p+n-\alpha]\Phi(p+n)}
 \end{aligned}$$

and the result are sharp for f given by

$$f(z) = z^p + z^{p+n} \frac{(p-\alpha)[1+\lambda(p-1)]}{[1+\lambda(p+n-1)][p+n-\alpha]\Phi(p+n)}.$$

Proof. From (1.2) and (3.1)

$$\begin{aligned}
 f'(z) &= pz^{p-1} + \sum_{k=p+n} k a_k z^{k-1} \\
 |f'(z)| &\leq p|z|^{p-1} + \sum_{k=p+n} k |a_k| |z|^{k-1} \\
 &\leq pr^{p-1} + \sum_{k=p+n} k |a_k| r^{k-1}
 \end{aligned}$$

$$\begin{aligned}
&\leq pr^{p-1} + (p+n)r^{p+n-1} \sum_{k=p+n} |a_k| \\
&\leq pr^{p-1} + (p+n)r^{p+n-1} \frac{(p-\alpha)[1+\lambda(p-1)]}{[1+\lambda(p+n-1)][p+n-\alpha]\Phi(p+n)}.
\end{aligned}$$

Also,

$$\begin{aligned}
f'(z) &= pz^{p-1} + \sum_{k=p+n} ka_k z^{k-1} \\
|f'(z)| &\geq p|z|^{p-1} - \sum_{k=p+n} k|a_k||z|^{k-1} \\
&\geq pr^{p-1} - \sum_{k=p+n} k|a_k|r^{k-1} \\
&\geq pr^{p-1} - (p+n)r^{p+n-1} \sum_{k=p+n} |a_k| \\
&\geq pr^{p-1} - (p+n)r^{p+n-1} \frac{(p-\alpha)[1+\lambda(p-1)]}{[1+\lambda(p+n-1)][p+n-\alpha]\Phi(p+n)}.
\end{aligned}$$

Hence, the proof for Theorem 3.2 is complete. $\square$

4. Extreme Points

Theorem 4.1. Let $f_p(z) = z^p$ and

$$f_k(z) = z^p + \frac{(p-\alpha)[1+\lambda(p-1)]}{[1+\lambda(k-1)][k-\alpha]\Phi(k)} z^k, (k \geq p+n, n \in \mathbb{N}).$$

Then $f(z) \in S(\mu, \gamma, \eta, \alpha, \beta)$, if and only if, $f(z)$ can be expressed in the form

$$f(z) = \sum_{k=p+n} \lambda_k f_k(z),$$

where $\lambda_k \geq 0$ and $\sum_{k=p+n} \lambda_k = 1$.

Proof. Let $f(z)$ can be expressible in the form

$$\begin{aligned} f(z) &= \sum_{k=p+n} \lambda_k f_k(z) \\ &= z^p + \sum_{k=p+n} \frac{(p-\alpha)[1+\lambda(p-1)]}{[1+\lambda(k-1)](k-\alpha)\Phi(k)} \lambda_k z^k, (n \in \mathbb{N}) \end{aligned}$$

and we get

$$\begin{aligned} \sum_{k=p+n} \frac{(p-\alpha)[1+\lambda(p-1)]}{[1+\lambda(k-1)](k-\alpha)\Phi(k)} \lambda_k &= \frac{[1+\lambda(k-1)](k-\alpha)\Phi(k)}{(p-\alpha)[1+\lambda(p-1)]} \\ &= \sum_{k=p+n} \lambda_k = 1 - \lambda_p \leq 1, (n \in \mathbb{N}). \end{aligned}$$

By virtue of the Theorem 2.1, we can show that $f(z) \in S(\mu, \gamma, \eta, \alpha, \beta)$.

Conversely, assume that $f(z) \in S(\mu, \gamma, \eta, \alpha, \beta)$.

Then

$$a_k \leq \frac{(p-\alpha)[1+\lambda(p-1)]}{[1+\lambda(k-1)](k-\alpha)\Phi(k)}, (n \in \mathbb{N}, k \geq p+n).$$

Setting

$$\lambda_k = \frac{[1+\lambda(k-1)](k-\alpha)\Phi(k)}{(p-\alpha)[1+\lambda(p-1)]} a_k$$

and

$$\lambda_p = 1 - \sum_{k=p+n} \lambda_k.$$

Then we obtain

$$f(z) = \sum_{k=p+n} \lambda_k f_k(z).$$

This completes the proof. □

5. Radius of Close-to-Convexity, Starlikeness and Convexity

Theorem 5.1. *Let the function $f(z)$ defined by (1.2) be in the class $S(\mu, \gamma, \eta, \alpha, \beta)$. Then $f(z)$ is close-to-convex of order δ for $0 \leq \delta < p$ in $|z| = r_1$, where*

$$r_1 = \inf_{k \geq n+p} \left\{ \left(\frac{p-\delta}{k} \right) \frac{[1 + \lambda(k-1)](k-\alpha)(a)_{k-p}(1+p)_{k-p}(1+p+\eta-\gamma)_{k-p}}{(p-\alpha)[1 + \lambda(p-1)](c)_{k-p}(1+p-\gamma)_{k-p}(1+p+\eta-\mu)_{k-p}} \right\}^{\frac{1}{k-p}}.$$

Proof. It is sufficient to prove that

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| \leq p - \delta, |z| \leq r_1.$$

We find that

$$\begin{aligned} \left| \frac{f'(z)}{z^{p-1}} - p \right| &= \left| \frac{pz^{p-1} + \sum_{k=p+n} ka_k z^{k-1}}{z^{p-1}} - p \right| \\ &\leq \sum_{k=p+n} ka_k |z|^{k-p}. \end{aligned}$$

The above expression is less than $p - \delta$ if

$$\sum_{k=p+n} \frac{k}{p-\delta} a_k |z|^{k-p} < 1.$$

Using the fact, that $f(z) \in S(\mu, \gamma, \eta, \alpha, \beta)$ if and only if

$$\sum_{k=p+n} \frac{[1 + \lambda(k-1)](k-\alpha)\Phi(k)}{(p-\alpha)[1 + \lambda(p-1)]} |a_k| < 1. \quad (5.1)$$

But, by Theorem 2.1, (5.1) will be true if

$$\frac{k}{p-\delta} |z|^{k-p} < \frac{[1 + \lambda(k-1)](k-\alpha)\Phi(k)}{(p-\alpha)[1 + \lambda(p-1)]}. \quad (5.2)$$

Solving (5.2) for $|z|$ we obtain

$$|z| < \left\{ \frac{p-\delta}{k} \frac{[1 + \lambda(k-1)](k-\alpha)\Phi(k)}{(p-\alpha)[1 + \lambda(p-1)]} \right\}^{\frac{1}{k-p}}$$

which yields the close-to-convexity of the family. $\square$

Theorem 5.2. *Let the function $f(z)$ defined by (1.2) be in the class $S(\mu, \gamma, \eta, \alpha, \beta)$. Then $f(z)$ is starlike of order δ for $0 \leq \delta < p$ in $|z| = r_2$, where*

$$r_2 = \inf_{k \geq n+p} \left\{ \frac{(p-\delta)[1+\lambda(k-1)](k-\alpha)(a)_{k-p}(1+p)_{k-p}(1+p+\eta-\gamma)_{k-p}}{(k-\delta)[1+\lambda(p-1)](p-\alpha)(c)_{k-p}(1+p-\gamma)_{k-p}(1+p+\eta-\mu)_{k-p}} \right\}^{\frac{1}{k-p}}.$$

Proof. To find the required result it is sufficient to prove that

$$\left| \frac{zf'(z)}{f(z)} - p \right| \leq p - \delta, |z| \leq r_2.$$

We find that

$$\begin{aligned} \left| \frac{zf'(z)}{f(z)} - p \right| &= \left| \frac{pz^p + \sum_{k=p+n}^{\infty} k a_k z^k}{z^p + \sum_{k=p+n}^{\infty} a_k z^k} - p \right| \\ &= \left| \frac{\sum_{k=p+n}^{\infty} (k-p) a_k z^k}{z^p + \sum_{k=p+n}^{\infty} a_k z^k} \right| \\ &\leq \frac{\sum_{k=p+n}^{\infty} (k-p) a_k |z|^{k-p}}{1 - \sum_{k=p+n}^{\infty} a_k |z|^{k-p}}. \end{aligned}$$

The above expression is less than $p - \delta$ if

$$\sum_{k=p+n}^{\infty} \left(\frac{k-\delta}{p-\delta} \right) a_k |z|^{k-p} < 1.$$

Using the fact, that $f(z) \in S(\mu, \gamma, \eta, \alpha, \beta)$ if and only if

$$\sum_{k=p+n}^{\infty} \frac{[1+\lambda(k-1)](k-\alpha)\Phi(k)}{(p-\alpha)[1+\lambda(p-1)]} |a_k| < 1. \quad (5.3)$$

But, by Theorem 2.1, (5.3) will be true if

$$\left(\frac{k-\delta}{p-\delta} \right) |z|^{k-p} < \frac{[1+\lambda(k-1)](k-\alpha)\Phi(k)}{(p-\alpha)[1+\lambda(p-1)]}. \quad (5.4)$$

Solving (5.4) for $|z|$ we obtain

$$|z| < \left\{ \frac{(p-\delta)}{(k-\delta)} \frac{[1+\lambda(k-1)](k-\alpha)\Phi(k)}{(p-\alpha)[1+\lambda(p-1)]} \right\}^{\frac{1}{k-p}}.$$

which yields the starlikeness of the family. $\square$

Theorem 5.3. *Let the function $f(z)$ defined by (1.2) be in the class $S(\mu, \gamma, \eta, \alpha, \beta)$. Then $f(z)$ is convex of order δ for $0 \leq \delta < p$ in $|z| = r_3$, where*

$$r_3 = \inf_{k \geq n+p} \left\{ \left(\frac{p(p-\delta)}{k(k-\delta)} \right) \frac{[1 + \lambda(k-1)](k-\alpha)(a)_{k-p}(1+p)_{k-p}(1+p+\eta-\gamma)_{k-p}}{(p-\alpha)[1 + \lambda(p-1)](c)_{k-p}(1+p-\gamma)_{k-p}(1+p+\eta-\mu)_{k-p}} \right\}^{\frac{1}{k-p}}.$$

Proof. It is sufficient to prove that

$$\left| \frac{zf''(z)}{f'(z)} + 1 - p \right| \leq p - \delta, |z| \leq r_3.$$

We find that

$$\begin{aligned} \left| \frac{zf'(z)}{f(z)} + 1 - p \right| &= \left| \frac{p(p-1)z^{p-1} + \sum_{k=p+n} k(k-1)a_k z^{k-1}}{pz^{p-1} + \sum_{k=p+n} ka_k z^{k-1}} + 1 - p \right| \\ &= \left| \frac{\sum_{k=p+n} k(k-p)a_k z^{k-1}}{pz^{p-1} + \sum_{k=p+n} ka_k z^{k-1}} \right| \\ &\leq \frac{\sum_{k=p+n} k(k-p)a_k |z|^{k-p}}{p - \sum_{k=p+n} ka_k |z|^{k-p}}. \end{aligned}$$

The above expression is less than $p - \delta$ if

$$\sum_{k=p+n} \frac{k(k-\delta)}{p(p-\delta)} a_k |z|^{k-p} < 1.$$

Using the fact, that $f(z) \in S(\mu, \gamma, \eta, \alpha, \beta)$ if and only if

$$\sum_{k=p+n} \frac{[1 + \lambda(k-1)](k-\alpha)\Phi(k)}{(p-\alpha)[1 + \lambda(p-1)]} |a_k| < 1. \quad (5.5)$$

But, by Theorem 2.1, (5.5) will be true if

$$\frac{k(k-\delta)}{p(p-\delta)} |z|^{k-p} < \frac{[1 + \lambda(k-1)](k-\alpha)\Phi(k)}{(p-\alpha)[1 + \lambda(p-1)]}. \quad (5.6)$$

Solving (5.6) for $|z|$ we obtain

$$|z| < \left\{ \frac{p(p-\delta)}{k(k-\delta)} \frac{[1 + \lambda(k-1)](k-\alpha)\Phi(k)}{(p-\alpha)[1 + \lambda(p-1)]} \right\}^{\frac{1}{k-p}}.$$

which yields the convexity of the family. $\square$

6. Modified Hadamard Product

Theorem 6.1. Let the function $f(z)$ and $g(z)$ defined by $f(z) = z^p + \sum_{k=p+n} a_k z^k$ and $g(z) = z^p + \sum_{k=p+n} b_k z^k$ belongs to the class $S(\mu, \gamma, \eta, \alpha, \beta)$ and $S(\mu, \gamma, \eta, \xi, \beta)$ respectively. Then $(f * g)(z) \in S(\mu, \gamma, \eta, \sigma, \beta)$ where

$$\sigma = p - \frac{(k-p)[1+\lambda(p-1)](p-\alpha)(p-\xi)}{(k-\alpha)(k-\xi)[1+\lambda(k-1)]\Phi(k) - (p-\alpha)(p-\xi)[1+\lambda(p-1)]} \quad (6.1)$$

and the result is sharp for

$$\begin{aligned} f(z) &= z^p + \frac{(p-\alpha)[1+\lambda(p-1)]}{(1+\lambda p)(p-\alpha+1)\Phi(p+1)} z^{p+1} \\ g(z) &= z^p + \frac{(p-\xi)[1+\lambda(p-1)]}{(1+\lambda p)(p-\xi+1)\Phi(p+1)} z^{p+1}. \end{aligned}$$

Proof. To prove this theorem, it is sufficient to show that

$$\sum_{k=p+n} \frac{[1+\lambda(k-1)](k-\sigma)\Phi(k)}{(p-\sigma)[1+\lambda(p-1)]} a_k b_k \leq 1, \quad (6.2)$$

where $\Phi(k)$ is defined by (1.12) and σ is defined by (6.1). Now $f(z) \in S(\mu, \gamma, \eta, \alpha, \beta)$ and $g(z) \in S(\mu, \gamma, \eta, \xi, \beta)$ and thus we have

$$\begin{aligned} \sum_{k=p+n} \frac{[1+\lambda(k-1)](k-\alpha)\Phi(k)}{(p-\alpha)[1+\lambda(p-1)]} a_k &\leq 1 \\ \sum_{k=p+n} \frac{[1+\lambda(k-1)](k-\xi)\Phi(k)}{(p-\xi)[1+\lambda(p-1)]} b_k &\leq 1, \end{aligned}$$

by applying cauchy-schwarz inequality, we have

$$\sum_{k=p+n} \frac{[1+\lambda(k-1)]\sqrt{(k-\alpha)(k-\xi)}}{[1+\lambda(p-1)]\sqrt{(p-\alpha)(p-\xi)}} \Phi(k) \sqrt{a_k b_k} \leq 1. \quad (6.3)$$

In view of 6.2 it is suffice to show that

$$\sum_{k=p+n} \frac{[1+\lambda(k-1)](k-\sigma)}{(p-\sigma)[1+\lambda(p-1)]} \Phi(k) a_k b_k$$

$$\leq \sum_{k=p+n} \frac{[1 + \lambda(k-1)]\sqrt{(k-\alpha)(k-\xi)}}{[1 + \lambda(p-1)]\sqrt{(p-\alpha)(p-\xi)}} \Phi(k) \sqrt{a_k b_k},$$

that is

$$\sqrt{a_k b_k} \leq \frac{\sqrt{(k-\alpha)(k-\xi)}}{\sqrt{(p-\alpha)(p-\xi)}} \frac{(p-\sigma)}{(k-\sigma)}, \text{ for } (k \geq p+1). \quad (6.4)$$

In the view of (6.3) and (6.4) it is enough to show that

$$\frac{[1 + \lambda(p-1)]\sqrt{(p-\alpha)(p-\xi)}}{[1 + \lambda(k-1)]\sqrt{(k-\alpha)(k-\xi)}\Phi(k)} \leq \frac{\sqrt{(k-\alpha)(k-\xi)}}{\sqrt{(p-\alpha)(p-\xi)}} \frac{(p-\sigma)}{(k-\sigma)}$$

Or, equivalently,

$$\sigma \leq p - \frac{(k-p)(p-\alpha)(p-\xi)[1 + \lambda(p-1)]}{(k-\alpha)(k-\xi)[1 + \lambda(k-1)]\Phi(k) - (p-\alpha)(p-\xi)[1 + \lambda(p-1)]}$$

which completes the proof. $\square$

7. Quasi Hadamard Product

Definition 7.1. If the function $f_j(z)$, ($j = 1, 2, \dots, m$) be in the class $S(\mu, \gamma, \eta, \alpha, \beta)$ defined by

$$f_j(z) = z^p + \sum_{k=p+n} a_{k,j} z^k, (j = 1, 2, \dots, m; n, p \in \mathbb{N} = \{1, 2, \dots\}),$$

then the Quasi-Hadamard product of the functions $f_j(z)$ denoted by $(f_1 * f_2 * \dots * f_m)(z)$, define by

$$(f_1 * f_2 * \dots * f_m)(z) = z^p + \sum_{k=p+n} (a_{k,1}, a_{k,2}, \dots, a_{k,m}) z^k.$$

Theorem 7.2. Let $f_j(z) \in S(\mu, \gamma, \eta, \alpha_j, \beta)$. Then $(f_1 * f_2 * \dots * f_m)(z) \in S(\mu, \gamma, \eta, \omega, \beta)$ where

$$0 < \omega \leq p - \frac{n}{H(n+p, m) - 1} \quad (7.1)$$

and

$$H(n+p, m) = \left[\frac{[1 + \lambda(n+p-1)]\Phi(n+p)(n+p-\omega)}{[1 + \lambda(p-1)](p-\omega)} \right]^{m-1} \prod_{j=1}^m \left(\frac{n+p-\alpha_j}{p-\alpha_j} \right).$$

This result is sharp for the functions $f_j(z)$, $(j = 1, 2, \dots, m)$ given by

$$f_j(z) = z^p + \frac{[1 + \lambda(p-1)](p-\alpha_j)}{[1 + \lambda p](p-\alpha_j+1)\Phi(p+1)} z^{n+p}.$$

Proof. We prove this theorem by induction. For $m = 1$, we see that $\omega = \alpha_1$. For $m = 2$, the inequality (2.1) gives

$$\sum_{k=p+n} \frac{[1 + \lambda(k-1)](k-\alpha_j)\Phi(k)}{[1 + \lambda(p-1)](p-\alpha_j)} a_{k,j} \leq 1, (j = 1, 2; n, p \in \mathbb{N}),$$

thus

$$\sum_{k=p+n} \frac{[1 + \lambda(k-1)]}{[1 + \lambda(p-1)]} \Phi(k) \sqrt{\prod_{j=1}^2 \frac{(k-\alpha_j)}{(p-\alpha_j)}} a_{k,j} \leq 1.$$

Now we must find the largest ω such that

$$\sum_{k=p+n} \frac{[1 + \lambda(k-1)]}{[1 + \lambda(p-1)]} \Phi(k) a_{k,1} a_{k,2} \leq 1$$

or such that

$$\frac{k-\omega}{p-\omega} \sqrt{a_{k,1} a_{k,2}} \leq \sqrt{\prod_{j=1}^2 \frac{(k-\alpha_j)}{(p-\alpha_j)}}, (k \geq n+p)$$

we need to find the largest ω such that

$$\frac{k-\omega}{p-\omega} \leq \frac{[1 + \lambda(k-1)]}{[1 + \lambda(p-1)]} \Phi(k) \prod_{j=1}^2 \frac{(k-\alpha_j)}{(p-\alpha_j)}$$

or such that

$$\frac{k-\omega}{p-\omega} \leq A(k),$$

where

$$A(k) = \frac{[1 + \lambda(k-1)]}{[1 + \lambda(p-1)]} \Phi(k) \prod_{j=1}^2 \frac{(k-\alpha_j)}{(p-\alpha_j)}$$

or

$$(k - \omega) \leq (p - \omega)A(k)$$

or

$$0 < \omega \leq p - \frac{n}{A(k) - 1}.$$

Let

$$\phi(k) = p - \frac{n}{A(k) - 1}, (k \geq n + p)$$

and so,

$$0 < \omega \leq \phi(n + p) \leq p - \frac{n}{A(n + p) - 1}$$

where

$$A(n + p) = \frac{[1 + \lambda(n + p - 1)]}{[1 + \lambda(p - 1)]} \Phi(n + p) \prod_{j=1}^2 \frac{(n + p - \alpha_j)}{(p - \alpha_j)}$$

The result is true for $m = 2$.

Now, assume the result is true for fixed natural number m . We must show that the result is true for $m + 1$ natural number, that is $(f_1 * f_2 * \dots * f_m * f_{m+1})(z) \in S(\mu, \gamma, \eta, \xi, \beta)$, where ξ satisfies the condition

$$0 < \xi \leq p - \frac{n}{H - 1} \quad (7.2)$$

and $H = \frac{[1 + \lambda(n + p - 1)]\Phi(n + p)(n + p - \omega)(n + p - \alpha_{m+1})}{[1 + \lambda(p - 1)](p - \omega)(p - \alpha_{m+1})}$ also ω is given by (7.1)

From (7.2), we have $0 < \xi < p - \frac{n}{H(n + p, m + 1) - 1}$. This shows that the results holds true for $m + 1$. Therefore, using the induction, we obtain that for any positive integer m the result is true. $\square$

8. Inclusion Property

Theorem 8.1. *If $\Phi(k) \geq \Phi(p + n)$ for $k \geq p + n$, $n \in \mathbb{N}$ and*

$$\delta = \frac{(p + n)(p - \alpha)[1 + \lambda(p - 1)]}{(p + n - \alpha)[1 + \lambda(p + n - 1)]\Phi(p + n)} \quad (8.1)$$

then

$$S(\mu, \gamma, \eta, \alpha, \beta) \subseteq N_{n, \delta}(e). \quad (8.2)$$

Proof. Let $f(z) \in S(\mu, \gamma, \eta, \alpha, \beta)$. Then in view of assertion (2.1) of Theorem (2.1) and the condition $\Phi(k) \geq \Phi(p+n)$ for $k \geq p+n$, $n \in \mathbb{N}$, we get

$$\begin{aligned} & \Phi(p+n)[1+\lambda(p+n-1)](p+n-\alpha) \sum_{k=p+n} a_k \\ & \leq \sum_{k=p+n} [1+\lambda(k-1)](k-\alpha)\Phi(k)a_k \\ & \leq (p-\alpha)[1+\lambda(p-1)], \end{aligned} \quad (8.3)$$

which implies

$$\sum_{k=p+n} a_k \leq \frac{(p-\alpha)[1+\lambda(p-1)]}{(p+n-\alpha)[1+\lambda(p+n-1)]\Phi(p+n)}. \quad (8.4)$$

Applying assertion (2.1) of Theorem (2.1) in conjunction with (8.4), we obtain

$$\begin{aligned} & [1+\lambda(p+n-1)]\Phi(p+n) \sum_{k=p+n} k a_k \\ & \leq (p-\alpha)[1+\lambda(p-1)] + \alpha[1+\lambda(p+n-1)]\Phi(p+n) \sum_{k=p+n} a_k \\ & \leq \frac{(p+n)(p-\alpha)[1+\lambda(p-1)]}{(p+n-\alpha)}, \end{aligned} \quad (8.5)$$

$$\sum_{k=p+n} k a_k \leq \frac{(p+n)(p-\alpha)[1+\lambda(p-1)]}{(p+n-\alpha)[1+\lambda(p+n-1)]\Phi(p+n)} = \delta$$

which by virtue of (1.17) establishes the inclusion relation(8.2). $\square$

9. Neighborhood Property

Theorem 9.1. *If $k(z) \in S(\mu, \gamma, \eta, \alpha, \beta)$ and*

$$\alpha_1 = p$$

$$- \frac{\delta[1 + \lambda(p + n - 1)](p + n - \alpha)\Phi(p + n)}{(p + n) \{ [1 + \lambda(p + n - 1)](p + n - \alpha)\Phi(p + n) - (p - \alpha)[1 + \lambda(p - 1)] \}}, \quad (9.1)$$

then $N_{n,\delta}(k) \subset S^{\alpha_1}(\mu, \gamma, \eta, \alpha, \beta)$.

Proof. Suppose that $f(z) \in N_{n,\delta}(k)$, we then find from (1.17) that

$$\sum_{k=p+n} k|a_k - b_k| \leq \delta,$$

which readily implies the following co-efficient inequality

$$\sum_{k=p+n} |a_k - b_k| \leq \frac{\delta}{p + n} \quad (n \in \mathbb{N}). \quad (9.2)$$

Next, since $k(z) \in S(\mu, \gamma, \eta, \alpha, \beta)$ in the view of (8.4), we have

$$\sum_{k=p+n} b_k \leq \frac{(p - \alpha)[1 + \lambda(p - 1)]}{(p + n - \alpha)[1 + \lambda(p + n - 1)]\Phi(p + n)}. \quad (9.3)$$

Using (9.2) and (9.3), we get

$$\begin{aligned} \left| \frac{f(z)}{k(z)} - 1 \right| &\leq \frac{\sum_{k=p+n} |a_k - b_k|}{1 - \sum_{k=p+n} b_k} \\ &\leq \frac{\delta}{(p + n) \left[1 - \frac{(p - \alpha)[1 + \lambda(p - 1)]}{[1 + \lambda(p + n - 1)](p + n - \alpha)\Phi(p + n)} \right]} \\ &\leq \frac{\delta[1 + \lambda(p + n - 1)](p + n - \alpha)\Phi(p + n)}{(p + n) \{ [1 + \lambda(p + n - 1)](p + n - \alpha)\Phi(p + n) - (p - \alpha)[1 + \lambda(p - 1)] \}} \\ &= p - \alpha_1, \end{aligned} \quad (9.4)$$

provided that α_1 is given by (9.1), thus by condition (1.19) $f(z) \in S^{\alpha_1}(\mu, \gamma, \eta, \alpha, \beta)$, where α_1 is given by (9.1). $\square$

10. Class Preserving Integral Operator

In this section we give a class preserving integral operator due to Jung-Kim-Srivastava, please refer([?]).

$$I(z) = Q_{d,p}^b f(z) = \left(\frac{b+d+p-1}{d+p-1} \right) \frac{b}{z^d} \int_0^z t^{d-1} \left(1 - \frac{t}{z}\right)^{b-1} f(t) dt \quad (10.1)$$

$(b > 0, d > -p, z \in \mathbb{U}).$

It can be easily verified that

$$I(z) = Q_{d,p}^b f(z) = z^p - \sum_{k=p+1}^{\infty} \frac{\Gamma(d+k)\Gamma(b+d+k)}{\Gamma(b+d+k)\Gamma(d+p)} a_k z^k. \quad (10.2)$$

A function $I(z)$ is said to close-to-convex and p -valent in the disc $|z| < r_4$ if

$$\left| \frac{I'(z)}{z^{p-1}} - p \right| \leq p \quad \text{in } |z| < r_4 \quad (10.3)$$

and $(b > 0, d > -p, z \in U).$

Theorem 10.1. *Let $b > 0, d > -p$ and $f(z)$ belong to the class $S(\mu, \gamma, \eta, \alpha, \beta)$. Then the function $I(z)$ defined by (10.1) is close-to-convex and p -valent in the disc $|z| < r_4$, where*

$$r_4 = \inf_k \left\{ \frac{p\Gamma(b+d+k)\Gamma(d+p)[1+\lambda(k-1)](k-\alpha)\Phi(k)}{k\Gamma(d+k)\Gamma(b+d+k)(p-\alpha)[1+\lambda(p-1)]} \right\}^{\frac{1}{k-p}}, \quad (10.4)$$

where $\Phi(k)$ is given by (1.12).

Proof. We show that

$$\left| \frac{I'(z)}{z^{p-1}} - p \right| \leq p \quad \text{in } |z| < r_4, \quad (10.5)$$

r_4 is given by (10.4).

In view of (10.2), we have

$$\left| \frac{I'(z)}{z^{p-1}} - p \right| = \left| - \sum_{k=p+1}^{\infty} \frac{k\Gamma(d+k)\Gamma(b+d+k)}{\Gamma(b+d+k)\Gamma(d+p)} a_k z^{k-p} \right|$$

$$\leq \sum_{k=p+n} \frac{k\Gamma(d+k)\Gamma(b+d+k)}{\Gamma(b+d+k)\Gamma(d+p)} a_k |z|^{k-p}.$$

The last inequality is bounded above by p if

$$\sum_{k=p+n} \frac{k\Gamma(d+k)\Gamma(b+d+k)}{p\Gamma(b+d+k)\Gamma(d+p)} a_k |z|^{k-p} \leq 1. \quad (10.6)$$

Also, since $f(z) \in S(\mu, \gamma, \eta, \alpha, \beta)$ by Theorem (2.1), we have

$$\sum_{k=p+n} \frac{[1 + \lambda(k-1)](k-\alpha)\Phi(k)a_k}{(p-\alpha)[1 + \lambda(p-1)]} \leq 1. \quad (10.7)$$

Thus (10.6) and consequently (10.5) will hold if

$$\frac{k\Gamma(d+k)\Gamma(b+d+k)}{p\Gamma(b+d+k)\Gamma(d+p)} a_k |z|^{k-p} \leq \frac{[1 + \lambda(k-1)](k-\alpha)\Phi(k)}{(p-\alpha)[1 + \lambda(p-1)]} a_k.$$

Or, equivalently,

$$|z| \leq \left\{ \frac{p\Gamma(b+d+k)\Gamma(d+p)[1 + \lambda(k-1)](k-\alpha)\Phi(k)}{k\Gamma(d+k)\Gamma(b+d+k)(p-\alpha)[1 + \lambda(p-1)]} \right\}^{\frac{1}{k-p}}$$

for $k \geq p+n$, $n \in \mathbb{N}$. The result follows by setting $|z| < r_4$. $\square$

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