PRIME GAMMA NEAR-RINGS WITH DERIVATIONS

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Abstract: Let N be a prime Γnear-ring with the center Z(N). The objective of this paper is to study derivations on N. We prove two results:

(a) Let N be 2-torsion free and let D₁ and D₂ be derivations on N such that $D₁D₂$ is also a derivation. Then $D₁ = 0$ or $D₂ = 0$ if and only if $[D₁(x), D₂(y)]_α = 0$ for all $x, y ∈ N, α ∈ Γ$;

(b) Let $n$ be an integer greater than 1, $N$ be $n!$-torsion free, and $D$ be a derivation with $D^n(N) = \{0\}$. Then $D(Z(N)) = \{0\}$.

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1. Introduction

The notion of derivations in near-rings has been introduced by Bell and Mason [2]. They obtained some basic properties of derivations in near-rings. Then Asci [1] investigated commutativity conditions for a Γ-near-ring with derivations. Cho and Jun [7] studied some characterizations of Γ-near-rings and some regularity conditions. In classical ring theory, Posner [14], Herstein [11], Bergen
[5], Bell and Daif [4] studied derivations in prime and semiprime rings and obtained commutativity results of prime and semiprime rings with derivations. In near ring theory, Bell and Mason [3], and also Cho [8] worked on derivations in prime and semiprime near-rings.

In this paper, we deal with the prime Γ-near-rings with derivations. Here we extend the results of Wang [20] on prime near-rings to Γ-near-rings.

2. Preliminaries

A Γ-near-ring is a triple \((N, +, \Gamma)\), where:

(i) \((N, +)\) is a group (not necessarily abelian),
(ii) \(\Gamma\) is a non-empty set of binary operations on \(N\) such that for each \(\alpha \in \Gamma\),

\((N, +, \alpha)\) is a left near-ring.

(iii) \(x\alpha(y\beta z) = (x\alpha y)\beta z\), for all \(x, y, z \in N\) and \(\alpha, \beta \in \Gamma\).

Examples of Γ-near-rings and motivations to study has been given in [16, 17].

Throughout this paper, \(N\) will denote a zero-symmetric left Γ-near-ring with multiplication center \(Z(N)\). A Γ-near-ring \(N\) is called a prime Γ-near-ring if \(N\) has the property that for \(x, y \in N\), \(x\Gamma N \Gamma y = \{0\}\) implies \(x = 0\) or \(y = 0\). A Γ-near-ring \(N\) is called a semiprime Γ-near-ring if \(N\) has the property that for \(x \in N\), \(x\Gamma N \Gamma x = \{0\}\) implies \(x = 0\). A derivation \(D\) on \(N\) is an additive endomorphism of \(N\) with the property that for all \(x, y \in N\) and \(\alpha \in \Gamma\), \(D(x\alpha y) = x\alpha D(y) + D(x)\alpha y\). An additive endomorphism \(D\) of \(N\) is called a derivation on \(N\) if \(D(x\alpha y) = x\alpha D(y) + D(x)\alpha y\) for all \(x, y \in N\), \(\alpha \in \Gamma\).

A Γ-near-ring \(N\) is called commutative if \((N, +)\) is abelian, and 2-torsion free if \(2x = 0\) implies \(x = 0\).

3. Derivations in Prime Γ-Near-Rings

We begin with the following lemmas on derivations on prime Γ-near-rings \(N\).

**Lemma 3.1.** Let \(D\) be an additive endomorphism of \(N\). Then:

\[ D(x\alpha y) = x\alpha D(y) + D(x)\alpha y \]

for all \(x, y \in N\) and \(\alpha \in \Gamma\), if and only if

\[ D(x\alpha y) = D(x)\alpha y + x\alpha D(y) \]
for all \( x, y \in N \) and \( \alpha \in \Gamma \).

**Proof.** We assume that \( D(x\alpha y) = x\alpha D(y) + D(x)\alpha y \) for all \( x, y \in N \) and \( \alpha \in \Gamma \).

Since
\[
x\alpha(y + y) = x\alpha y + x\alpha y
\]
and
\[
D(x\alpha(y+y)) = x\alpha D(y+y) + D(x)\alpha(y+y) = x\alpha D(y) + x\alpha D(y) + D(x)\alpha y + D(x)\alpha y
\]
and
\[
D(x\alpha y + x\alpha y) = D(x\alpha y) + D(x\alpha y) = x\alpha D(y) + D(x)\alpha y + x\alpha D(y) + D(x)\alpha y
\]
we get
\[
x\alpha D(y) + D(x)\alpha y = D(x)\alpha y + x\alpha D(y),
\]
so
\[
D(x\alpha y) = D(x)\alpha y + x\alpha D(y),
\]
for all \( x, y \in N \alpha \in \Gamma \).

The converse is proved in a similar way.

Note that due to Lemma 3.1, \( D \) is a derivation if and only if
\[
D(x\alpha y) = D(x)\alpha y + x\alpha D(y), \text{ for all } x, y \in N \text{ and } \alpha \in \Gamma.
\]

We make use the following lemma from [15], Lemma 3.5.

**Lemma 3.2.** Suppose that \( N \) is a prime \( \Gamma \)-near-ring.

(i) any nonzero element of the center of \( N \) is not zero divisor.

(ii) If there exist a nonzero element of \( Z(N) \) such that \( x + x \in Z(N) \), then \( (N, +) \) is commutative.

(iii) Let \( d \) be a nonzero derivation on \( N \). If one of the \( x\Gamma d(N) = \{ 0 \} \) and \( d(N)\Gamma x = \{ 0 \} \) holds then \( x = 0 \).

**Lemma 3.3.** Let \( D \) be a derivation on \( N \). Then \( N \) satisfies the following partial distributive laws

(i) \( (x\alpha D(y) + D(x)\alpha y)\beta z = x\alpha D(y)\beta z + D(x)\alpha y\beta z \) for all \( x, y, z \in N \) and \( \alpha, \beta \in \Gamma \)

(ii) \( (D(x)\alpha y + x\alpha D(y))\beta z = D(x)\alpha y\beta z + x\alpha D(y)\beta z \) for all \( x, y, z \in N \) and \( \alpha, \beta \in \Gamma \)
Proof. (i) Consider \( D((x\alpha y)\beta z) = D(x\alpha(y\beta z)) \) for all \( x, y, z \in N \) and \( \alpha, \beta \in \Gamma \). Then by using Lemma 3.1, we obtain the required result.

(ii) Consider \( D((x\alpha y)\beta z) = D(x\alpha(y\beta z)) \). Then we make use Lemma 3.1 to obtain,

\[
D((x\alpha y)\beta z) = D(x\alpha y)\beta z + x\alpha y\beta D(z) = (D(x)\alpha y + x\alpha D(y))\beta z + x\alpha y\beta D(z)
\]

and

\[
D(x\alpha(y\beta z)) = D(x)\alpha y\beta z + x\alpha D(y)\beta z
\]

\[
= D(x)\alpha y\beta z + x\alpha(D(y)\beta z + y\beta D(z)) = D(x)\alpha y\beta z + x\alpha D(y)\beta z + x\alpha y\beta D(z),
\]

for all \( x, y, z \in N \) and \( \alpha, \beta \in \Gamma \).

Comparing the above two relations we get the required result:

\[
(D(x)\alpha y + x\alpha D(y))\beta z = D(x)\alpha y\beta z + x\alpha D(y)\beta z
\]

for all \( x, y, z \in N \) and \( \alpha, \beta \in \Gamma \).

Now we prove our main results.

**Theorem 3.4.** Let \( N \) be a 2-torsion-free prime \( \Gamma \)-near-ring, and let \( D_1 \) and \( D_2 \) be derivations on \( N \) such that \( D_1 D_2 \) is also a derivation. Then the following two conditions are equivalent:

1. either \( D_1 = 0 \) or \( D_2 = 0 \);
2. (ii) \( [D_1(x), D_2(y)]_\alpha = 0 \) for all \( xy \in N \), \( \alpha \in \Gamma \)

Proof. We need prove only the part (ii) \( \Rightarrow \) (i) since (i) \( \Rightarrow \) (ii) is obvious. Consider

\[
D_1 D_2(x\alpha y) = x\alpha D_1 D_2(y) + D_1 D_2(x)\alpha y
\]

for all \( x, y, z \in N \) and \( \alpha \in \Gamma \).

On the other hand, both \( D_1 \) and \( D_2 \) are derivations, therefore

\[
D_1 D_2(x\alpha y) = D_1(D_2(x\alpha y))
\]

\[
= D_1(x\alpha D_2(y) + D_2(x)\alpha y)
\]

\[
= D_1(x\alpha D_2(y)) + D_1(D_2(x)\alpha y)
\]

\[
= x\alpha D_1 D_2(y) + D_1(x)\alpha D_2(y) + D_2(x)\alpha D_1(y) + D_1 D_2(x)\alpha y,
\]

for all \( x, y \in N \) and \( \alpha \in \Gamma \).
The above two relations for $D_1 D_2(x \alpha y)$ give

$$D_1(x)\alpha D_2(y) + D_2(x)\alpha D_1(y) = 0 \text{ for all } x, y \in N \text{ and } \alpha \in \Gamma.$$  \hfill (1)

Replacing $x$ by $x \beta D_2(z), \ z \in N, \ \beta \in \Gamma$ in (1), by using Lemma 31 and Lemma 3.3 we get

$$0 = D_1(x \beta D_2(z))\alpha D_2(y) + D_2(x \beta D_2(z))\alpha D_1(y)$$
$$= (D_1(x)\beta D_2(z) + x \beta D_1 D_2(z))\alpha D_2(y) + (x \beta D_2^2(z) + D_2(x)\beta D_2(z))\alpha D_1(y)$$
$$= D_1(x)\beta D_2(z)\alpha D_2(y) + x \beta D_1 D_2(z)\alpha D_2(y) + x \beta D_2^2(z)\alpha D_1(y)$$
$$+ D_2(x)\beta D_2(z)\alpha D_1(y)$$

for all $x, y, z \in N$ and $\alpha, \beta \in \Gamma$.

Then by using the equation (1) we obtain

$$x \beta (D_1 D_2(z)\alpha D_2(y) + D_2^2(z)\alpha D_1(y)) = 0.$$

If we replace $x$ by $D_2(z)$ in (1) then we get

$$D_1 D_2(z)\alpha D_2(y) + D_2^2(z)\alpha D_1(y) = 0.$$

Therefore

$$D_1(x)\beta D_2(z)\alpha D_2(y) + D_2(x)\beta D_2(z)\alpha D_1(y) = 0$$

for all $x, y, z \in N$ and $\alpha, \beta \in \Gamma$.  \hfill (2)

Replacing $x$ and $y$ by $z$ in (1), respectively, we obtain

$$D_2(z)\alpha D_1(y) = -D_1(z)\alpha D_2(y) \text{ for all } y, z \in N \text{ and } \alpha \in \Gamma,$$

and

$$D_1(x)\alpha D_2(z) = -D_2(x)\alpha D_1(z) \text{ for all } x, z \in N \text{ and } \alpha \in \Gamma.$$

Since $N$ is a zero-symmetric left $\Gamma$-near-ring, then due to (2) we obtain

$$0 = (-D_2(x)\beta D_1(z))\alpha D_2(y) + D_2(x)\beta (-D_1(z)\alpha D_2(y))$$
$$= D_2(x)\beta (-D_1(z))\alpha D_2(y) + D_2(x)\beta (-D_1(z)\alpha D_2(y))$$
$$= D_2(x)\beta [(-D_1(z))\alpha D_2(y) - D_1(z)\alpha D_2(y)].$$
for all $x, y, z \in N$ and $\alpha, \beta \in \Gamma$.

If $D_2 \neq 0$ then thanks to Lemma 3.2 we have

$$(-D_1(z))\alpha D_2(y) - D_1(z)\alpha D_2(y) = 0.$$  

That is

$$D_1(z)\alpha D_2(y) = (-D_1(z))\alpha D_2(y) \text{ for all } y, z \in N \text{ and } \alpha \in \Gamma. \quad (3)$$

The condition (ii) provides

$$(-D_1(z))\alpha D_2(y) = D_1(-z)\alpha D_2(y) = D_2(y)\alpha D_1(-z)$$

$$= D_2(y)\alpha(-D_1(z)) = -D_2(y)\alpha D_1(z) = -D_1(z)\alpha D_2(y).$$

Therefore

$$(-D_1(z))\alpha D_2(y) = -D_1(z)\alpha D_2(y) \text{ for all } y, z \in N \text{ and } \alpha \in \Gamma. \quad (4)$$

From (3) and (4) we obtain $2D_1(z)\alpha D_2(y) = 0$ for all $y, z \in N$ and $\alpha \in \Gamma$.

Since $N$ is 2-torsion-free, this gives $D_1(z)\alpha D_2(y) = 0$ for all $y, z \in N$ and $\alpha \in \Gamma$.

Therefore $D_1(z)\alpha D_2(N) = \{0\}$. But $D_2 \neq 0$, so $D_1(z) = 0$ for all $z \in N$, that is $D_1 = 0$.

Note that Lemma 3.5. from [15] can be derived now as the following corollary from the theorem.

**Corollary 3.5** Let $N$ be a 2-torsion free prime $\Gamma$ near-ring, and let $D$ be a derivation on $N$ such that $D^2 = 0$. Then $D = 0$.

**Proof.** It is clear that $D^2 = 0$ is a derivation on $N$, and we have

$$0 = D^2(x\alpha y) = D(x\alpha D(y) + D(x)\alpha y) = D(x\alpha D(y)) + D(D(x)\alpha y)$$

$$= x\alpha D^2(y) + D(x)\alpha D(y) + D(x)\alpha D(y) + D^2(x)\alpha y = 2D(x)\alpha D(y)$$

for all $x, y \in N$ and $\alpha \in \Gamma$. Since $N$ is 2-torsion free we obtain $D(x)\alpha D(y) = 0$ for all $x, y \in N$ and $\alpha \in \Gamma$. Similarly we get $D(y)\alpha D(x) = 0$ for all $x, y \in N$ and $\alpha \in \Gamma$. Therefore $[D(x), D(y)] = 0$ for all $x, y \in N$ and $\alpha \in \Gamma$. Hence by Theorem 3.4, we get $D = 0$.

Another consequence of Theorem 3.4 is the following.

**Corollary 3.6.** Let $N$ be a $\Gamma$ near-ring and $D_1$ and $D_2$ be derivations on $N$ such that $D_1D_2$ is a derivation. Then $D_2D_1$ is also a derivation.
Proof. Obviously $D_2D_1$ is an additive endomorphism of $N$. By Theorem 3.4 we have

$$D_2D_1(x\alpha y) = D_2(D_1(x)\alpha y + x\alpha D_1(y)) = D_2(D_1(x)\alpha y) + D_2(x\alpha D_1(y))$$
$$= D_2D_1(x)\alpha y + (D_1(x)\alpha D_2(y) + D_2(x)\alpha D_1(y) + x\alpha D_2D_1(y)$$

for all $xy \in N$ and $\alpha \in \Gamma$.

This completes the proof.

The following is an extension of Wang [20] on Leibniz’s rule for derivations of rings to $\Gamma$-near-rings.

**Theorem 3.7.** Let $N$ be a $n!$-torsion free $\Gamma$-near-ring Let $n$ be an integer $n \geq 2$ and $D$ be a derivation on $N$. Then

$$D^n(x\alpha y) = D^n(x)\alpha y + \binom{n}{1} D^{n-1}(x)\alpha D(y) + \cdots + \binom{n}{i} D^i(x)\alpha D^i(y)$$
$$\quad + \cdots + \binom{n}{n-1} D(x)\alpha D^{n-1}(y) + x\alpha D^n(y),$$

for all $x, y \in N$ and $\alpha \in \Gamma$.

Proof. By Theorem 3.4 it can easily seen that

$$D(x)\alpha y + nx\alpha D(y) = nx\alpha D(y) + D(x)\alpha y$$

for all $x, y \in N$, $\alpha \in \Gamma$ and $n$ be an integer. The same observation gives

$$nD(x)\alpha y + nx\alpha D(y) = n(D(x)\alpha y + x\alpha D(y))$$

for all $x, y \in N$, $\alpha \in \Gamma$, and $n$ be an integer. (5)

We proceed the proof of Leibniz’s rule by induction on $n$. Let $n = 2$. Then

$$D^2(x\alpha y) = D(D(x)\alpha y + x\alpha D(y))$$
$$= D(D(x)\alpha y) + D(x\alpha D(y))$$
$$= D^2(x)\alpha y + D(x)\alpha D(y) + x\alpha D^2(y)$$
$$= D^2(x)\alpha y + 2D(x)\alpha D(y) + x\alpha D^2(y).$$

Assume that Leibniz’s rule holds for $n - 1$. That is, if $N$ is $(n - 1)!$-torsion-free. Then
\[ D^{n-1}(x\alpha y) = D^{n-1}(x\alpha y) + \cdots + \left( \frac{n-1}{i-1} \right) D^{ni}(x)\alpha D^{i-1}(y) \]

\[ + \left( \frac{n-1}{i} \right) D^{ni-1}(x)\alpha D^i(y) + \cdots + x\alpha D^{n-1}(y). \]

Since \( n! \)-torsion-freeness implies \( (n-l)! \)-torsion-freeness, by (5) we have

\[ D^n(x\alpha y) = D(D^{n-1}(x\alpha y)) \]

\[ = D(D^{n-1}(x\alpha y) + \cdots + \left( \frac{n-1}{i-1} \right) D^{ni}(x)\alpha D^{i-1}(y) + \left( \frac{n-1}{i} \right) D^{ni-1}(x)\alpha D^i(y) + \cdots + x\alpha D^{n-1}(y)) \]

\[ = D^n(x\alpha y) + \cdots + \left( \frac{n-1}{i-1} \right) D^{ni+1}(x)\alpha D^{i-1}(y) + \left( \frac{n-1}{i} \right) D^{ni}(x)\alpha D^i(y) + \cdots + x\alpha D^n(y) \]

The proof is complete.

**Lemma 3.8.** Let \( N \) be a \( \Gamma \)near-ring with center \( Z(N) \), and let \( D \) be a derivation on \( N \). Then \( D(Z(N)) \subseteq Z(N) \).

**Proof.** By Theorem 3.4, we have
\[ x\alpha D(z) + z\alpha D(x) = x\alpha D(z) + D(x)\alpha z = D(x\alpha z) = D(z\alpha x) = D(z)\alpha x + z\alpha D(x) \]

for all \( z \in Z(N), \ x \in N \) and \( \alpha \in \Gamma \).

Therefore \( x\alpha D(z) = D(z)\alpha x \) for all \( x, z \in N \) and \( \alpha \in \Gamma \). Thus \( D(z) \in Z(N) \)

**Lemma 3.9.** Let \( n \geq 2 \), and let \( N \) be an \( n! \)-torsion free \( \Gamma \) near-ring and \( D \) be a derivation with \( D^n(N) = \{0\} \). Then for each \( y \in N \), either \( D(y) = 0 \) or there exists \( k \) (\( 0 < k < n \)) such that \( D^k(y) \) is a nonzero divisor of zero.

**Proof.** Since \( n! \)-torsion-freeness implies \((n - 1)!\)-torsion-freeness, we may assume that \( D^{n-1}(N) \neq \{0\} \). Choose \( x \) such that \( D^{n-1}(x) \neq 0 \). Assume that \( D(y) \neq 0 \). Then there exists \( k \) with \( 0 < k < n \) such that \( D^k(y) \neq 0 \) and \( D^{k+1}(y) = 0 \). Then due to Theorem 3.7 we obtain

\[
0 = D^n(x\alpha D^k(y)) = D^n(x)\alpha D^{k-1}(y) + \binom{n}{1} D^{n-1}(x)\alpha D^k(y) + \binom{n}{2} D^{n-2}(x)\alpha D^{k+1}(y) + \cdots = \binom{n}{1} D^{n-1}(x)\alpha D^k(y) = nD^{n-1}(x)\alpha D^k(y)
\]

for all \( y \in N \) and \( \alpha \in \Gamma \).

Since \( N \) is \( n! \)-torsion-free, then we get \( D^{n-1}(x)D^k(y) = 0 \) for all \( y \in N \) and \( \alpha \in \Gamma \). By Lemma 3.2(i) \( D^k(y) \) is a nonzero divisor of zero.

We finalize the paper by the following two theorems which can be easily proven by using the previous results

**Theorem 3.10.** Let \( n \) be an integer \( \geq 1 \) and \( N \) be a prime \( \Gamma \) near-ring with center \( Z(N) \), and let \( N \) be \( n! \)-torsion free and \( D \) be a derivation with \( D^n(N) = \{0\} \). Then \( D(Z(N)) = \{0\} \).

**Theorem 3.11.** Let \( n \) be a positive integer and \( N \) be an \( n! \)-torsion free \( \Gamma \) near-ring with no divisor of zero, then \( N \) admits no nonzero derivation \( D \) with \( D^n = 0 \).

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