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ON NUMERICAL SOLUTION OF THE PROBLEM ABOUT TENSION OF THE CYLINDER WITH VARIABLE PROPERTIES

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Abstract: In this connection, it is interesting to develop a numerical method for solving the problem of the stress-strain state of multilayered cylindrical bodies. Here we consider the formulation of the problem, the main equation and the algorithm for solving the problem and the results of solving a particular problem.

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Analytical solution of the problem about stress-strain state of a composite cylinder with a large number of layers is a challenge. It is especially difficult if the temperature levels are such that it is necessary to take into account the dependence of the material properties on them. In this connection, it is interesting to develop a numerical method for solving the problem of the stress-strain state of multilayered cylindrical bodies. Here we consider the formulation of the problem, the main equation and the algorithm for solving the problem and the results of solving a particular problem.

Problem. Consider a multi-layer cylinder consisting of N layers, where R_0

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and $R_N$are, respectively, inner and outer radiuses $(R_0 \leq r \leq R_N)$. The composite cylinder is in the axisymmetric temperature field, $T = T\left(r\right)$. The layers are composed from the materials with different elastic and thermal properties. We assume, that the physical and mechanical properties of the material of each layer - the shear modulus μ^i, Poisson's ratio ν^i and coefficient of linear expansion α^i (where i - current, $i = 1, ..., N$) depend on the level of temperature, and are experimentally determined functions. Suppose that, all layers are deformed without sliding and separation, the stress components on planes, tangential to the contact surface, have no gaps.

The main equation: In the case of axisymmetric deformation of a cylinder

$$u^i\left(R_i\right) = u^{i+1}\left(R_i\right) \quad when \quad r = R_i, \tag{1}$$

$$\sigma_r^i\left(R_i\right) = \sigma_r^{i+1}\left(R_i\right) i = 1, 2, ..., N-1.$$

Here – radius of the contact surface of the i-th and $(i+1)$ i-th layers, u^i- the radial displacement, σ_r^i- radial stress in the i-th layer The condition of thermal contact of any layer with an adjacent layer has the following form

$$T^i\left(R_i\right) = T^{i+1}\left(R_i\right) \quad when \quad r = R_i, \tag{2}$$

$i = 1, 2, ..., N-1$. The equation of equilibrium for each i-th layer can be written as

$$\frac{d\sigma_r^i}{dr} + \frac{\sigma_r^i - \sigma_\varphi^i}{r} = 0. \tag{3}$$

Here σ_φ^i is a ring strain in the i-th layer. Deformations in the i-th layer are defined through the radial displacement u^i with dependencies

$$\varepsilon_r^i = \frac{du^i}{dr}, \quad \varepsilon_\varphi^i = \frac{u^i}{r}. \tag{4}$$

Futuremore, we use the following dimensionless parameters:

$$u_1^i = \frac{u^i}{R_N}, \quad \vec{r} = \frac{r}{R_N},$$

$$u_2^i = \frac{1-\nu^{\square}}{2\mu^{\square}}\sigma_r^i, \quad \vec{\sigma}_\varphi^i = \frac{1-\nu^{\square}}{2\mu^{\square}}\sigma_\varphi^i, \quad \vec{\sigma}_z^i = \frac{1-\nu^{\square}}{2\mu^{\square}}\sigma_z^i, \tag{5}$$

where $\mu^{\square} = \sup\left\{\mu^1, \mu^2, ..., \mu^N\right\}$, $\nu^{\square} = \inf\left\{\nu^1, \nu^2, ..., \nu^N\right\}$.

The equations of equilibrium of the kinematic relations and physical dependence of Duhamel-Neumann, using the dimensionless parameters, can be reduced to a system consisting of two differential equations

$$\bar{V}^{i'} = \left\{u_1^{i'}, u_2^i\right\} = A\bar{V}^i + \bar{b}T, \tag{6}$$

and two algebraic relations

$$\varepsilon_\varphi^i = \frac{u_1^i}{r}, \quad \sigma_\varphi^i = c_1 u_1^i + c_2 u_2^i + c_3 T. \tag{7}$$

Elements of the matrix , components of the vector $\vec{b} = \{b_1, b_2\}$ and scalar coefficients $C_k\,(k = 1, 2, 3)$ are defined by the following equalities

$$\begin{gathered}
a_{11} = -\frac{\nu^i}{(1-\nu^i)\,r}, \qquad a_{12} = \frac{\mu\left(1-2\nu^i\right)}{\mu_1^i\,(1-\nu^{\square})\,(1-\nu^i)}, \\
a_{21} = -\frac{\mu_1^i}{(1-\nu^i)\,r^2}, \qquad a_{22} = \frac{\left(1-2\nu^i\right)}{(1-\nu^i)\,r}, \\
b_1 = \frac{\left(1+\nu^i\right)}{(1-\nu^i)}, \qquad b_2 = -\frac{\mu_1^i\left(1+\nu^i\right)}{(1-\nu^i)\,r}, \\
c_1 = \frac{\mu_1^i}{(1-\nu^i)\,r}, \quad c_2 = \frac{\nu^i}{(1-\nu^i)}, \qquad c_3 = -\frac{\alpha^i\left(1+\nu^i\right)}{(1-\nu^i)}, \\
\sigma_z^i = \nu^i\left(u_2^i + \sigma_\varphi^i\right) - \mu_1^i\left(1+\nu^i\right)\alpha^i T, \varepsilon_z = 0
\end{gathered} \tag{8}$$

for plain strain and

$$\begin{gathered}
a_{11} = \frac{\nu^i}{r}, \qquad a_{12} = \frac{\left(1-\nu^i\right)}{r}, \\
a_{21} = \mu_1^i\left(1+\nu^i\right)r^2, \qquad a_{22} = \frac{\left(1-\nu^i\right)}{r}, \\
b_1 = \left(1+\nu^i\right)\alpha^i T, \qquad b_2 = \mu_1^i\left(1+\nu^i\right)\alpha^i r, \\
c_1 = \frac{\mu_1^i\left(1+\nu^i\right)}{r}, \qquad c_2 = \nu^i, \quad c_3 = -\alpha^i\left(1+\nu^i\right)\mu_1^i, \\
\sigma_z = 0, \qquad \varepsilon_z^i = -\frac{\nu^i\left(u_2+\sigma_\varphi^i\right)}{(1+\nu^i)\,\mu_1^i + \alpha^i T},
\end{gathered} \tag{9}$$

for plain stress, where $\mu_1^i = \frac{(1-\nu^*)\mu^{\mathbf{i}}}{\mu}$.

Therefore, to determine the status of each layer we have a linear system of differential equations (6) with variable coefficients $a_{ij} = (i, j = 1, 2)$.

The Algorithm of the Solution

Solutions of the system of equations (6) are numerically as a superposition of two linearly independent solutions of the Cauchy problem for each layer. Moreover, the thickness of the i-th layer is divided into $M_i\,(i = 1, 2, ..., N)$ equal intervals in the radial direction. At each of the segments the temperature can be linearly approximated between its values at the endpoints. From the experimental curve $\mu = \mu(T), \nu = \nu(T), \alpha = \alpha(T)$ for each level of temperature there are the current values of the parameters μ^i, ν^i, α^i at each point of the segment $[\rho_k, \rho_{k+1}]$ $(r_i \leq \rho_k < \rho_{k+1} \leq r_{k+1})$, where $k = 0, 1, 2, ..., M_{i-1}$. Then, by (8) and (9), the elements of the matrix A, the components of the vector $\vec{b}$ and the coefficients $_k(\mathrm{k} = 1, 2, 3)$ are completely determined. Thus, at each point of the segment $[\rho_k, \rho_{k+1}]$ the equation (6) and (7) can be defined. For each layer in the two linearly independent conditions on the inside its surfaces $r_{i-1} = \frac{R_{\mathbf{i}-1}}{R_{\mathbf{N}}}$ by the Runge-Kutt method with a modification of Merson we can numerically solve a system of linear differential equations (6) with variable coefficients (8) and (9) at a given temperature $T(r)$ distribution by r. It allows us to set two linearly independent solutions $\bar{V}_1^i, \bar{V}_2^i$ for each layer. Using the method of linear combinations of integrals, we can find general solution for the i-th layer in the form $\bar{V}^i = C_1^i\bar{V}_1^i + C_2^i\bar{V}_2^i$. Here C_j^i- constants $(j = 1, 2. i = 1, 2, ..., N)$, whose total number is $2 * N$. The conditions of contact between the layers (1) allow us to set $2 * N - 2$ algebraic equations relatively to the constants C_j^i. The missing equations are the conditions given on the inside $\left(r_0 = \frac{R_0}{R_{\mathbf{N}}}\right)$ and outer $(r_N = 1)$ surfaces of the laminated cylinder. In general, the various boundary conditions on the inner and outer surfaces can be written as

$$\bar{\lambda}\bar{V}^1(r_0) = \rho, \qquad \bar{\beta}\bar{V}^N(1) = q. \tag{10}$$

Depending on the values of specified components of the vectors $\bar{\lambda}, \bar{\beta}$ we can obtain all the possible types of boundary conditions for multi-layer cylinder. For example, if the vectors $\bar{\lambda}, \bar{\beta}$ are given in the form $\bar{\lambda} = \{1, 0\}$,$\bar{\beta} = \{0, 1\}$, $\rho = q = 0$, then the inner surface $\left(r_0 = \frac{R_0}{R_{\mathbf{N}}}\right)$ is rigidly fixed, and the outer$(r_N = 1)$ surface is free from stress. Thus, to determine the constants of integration C_j^i we get a system of $2 * N$ algebraic equations

$$\begin{array}{l} \bar{\lambda}\bar{V}^1(r_0) = \rho \\ \bar{V}^i(r_i) = \bar{V}^{i+1}(r_i)\ , \qquad i = 1, 2, ..., N - 1. \\ \bar{\beta}\bar{V}^N(1) = q \end{array} \tag{11}$$

The solution of the system (11) allows us to determine the constants C_j^i

$(j = 1, 2.i = 1, 2, ..., N)$. Applying the method of linear combinations of integrals for each layer, it is possible to find a solution u_1^i, u_2^i. Other components of the stress and strain can be found by (7), (8) and (9) for each layer of a multilayer cylinder.

Numerical implementation of the problem is carried out as follows: The thickness of the cylinder (in the radial direction r) is divided into n equal segments. At each of the segments the temperature is linearly approximated between its values at the ends of the segments. From the experimentally constructed $\mu = \mu(T), \nu = \nu(T), \alpha = \alpha(T)$ for a given level of temperature there are the current values of the parameters μ, ν, α at each point of the interval $[r_i, r_{i+1}]$.

In the integration of the Cauchy problem it is convenient to use one-step methods, which are different versions of Runge - Kutt methods. The length of each segment $h = (R_2 - R_1)/n$ is a move method.

For the calculations we consider single-layered thick-walled cylinder with $R_1 = 4,5cm$ and $R_2 = 18cm$. Along the outer contour the cylinder is rigidly fixed. The inner surface is free from stress.

The following cases are especially considered:

1. Mechanical and thermal properties of the material do not depend on temperature. In this case, for the shear modulus μ, Poisson's ratio ν and the linear expansion coefficient α we take the values of these parameters, corresponding to the normal temperature of 15^0 C. Relevant to this case the calculated curves are denoted by 1 in Figures 1-2.

2. The shear modulus depends on temperature $\mu = \mu(T)$, and Poisson's ratio and coefficient of linear expansion does not depend on it, they are constant. The numerical results in this case are denoted by 2.

3. The shear modulus and Poisson's ratio depend on the temperature, but coefficient of linear expansion is constant. The curves, corresponding to this case, denoted by 3.

4. The shear modulus and coefficient of linear expansion depend on temperature, Poisson's ratio is constant. The curves, corresponding to this case, denoted by 4.

5. All characteristics of the material depend on temperature. For the calculations we have used the curves $\mu = \mu(T), \nu = \nu(T), \alpha = \alpha(T)$.

In Figure 1 the calculated values of displacement for the above mentioned cases of a long thick-walled cylinder have been shown.

Changing of the radial stresses over the thickness of thick-walled cylinder has been shown in Figure 2. Data of the curve, corresponding to the case 3, is

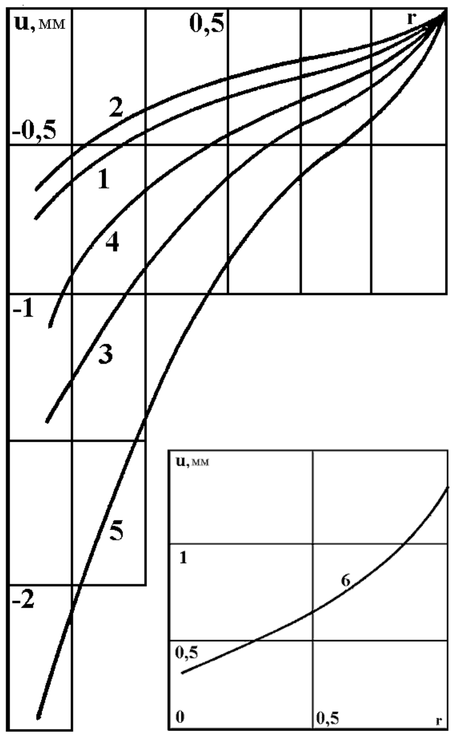

Figure 1: Distribution of radial displacements across the thickness of the cylinder

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In Figure 2 we also present the district voltage. The curves of the district stresses, corresponding to the cases 3 and 4, are located between the curves, corresponding to cases 1 and 2. The curve, corresponding to the case when the physical and mechanical properties are dependent on temperature, is lower than the curves 1, 2, 3, 4. Circuit voltage, as well as radial stresses are compressive everywhere. The appearance of compressive radial and circumferential stresses due to heating of the thick-walled cylinder and the condition of fixing it on the outer surface $r = 1$.

Figure 3 shows the axial stress for the above mentioned cases. Axial stress for all cases are compressive. Axial stress, in all cases, in two third parts of the thickness from the inner side is almost constant, for the other cases it increases. Such a distribution of axial stresses in thickness is connected with the nature of the temperature distribution along the radius of the cylinder.

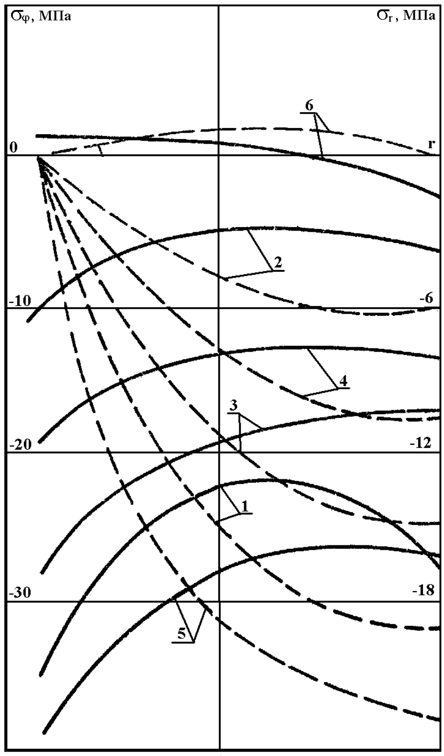

Figure 2: Distributions of radial and circumferential stresses through the thickness of the cylinder

The radial and axial strain components are shown in Figures 3 and 4 (respectively). The largest values are taken the components of the strain, corresponding to the case 5 (all characteristics depend on the temperature).

From the obtained numerical results we can see that in the trials dependence of the shear modulus on temperature has a negligible effect on the strain state, but has an appreciable effect on the level of stress. If we set on the degree of their influence on the stress - strain state, then after the shear modulus coefficient of linear expansion has the greatest influence. Since the curve, corresponding to case 4, is after the curve is 1. The curve 3 is in the next place. The top (Figure 4) and the lowest (Figure 3) curves correspond to the case when all the physical and mechanical properties depend on temperature.

Despite the fact that there is discussion of the results of calculations given for only one type of fastening of the cylinder, we can make a general conclusion - changing the shear modulus, Poisson's ratio and coefficient of linear expansion

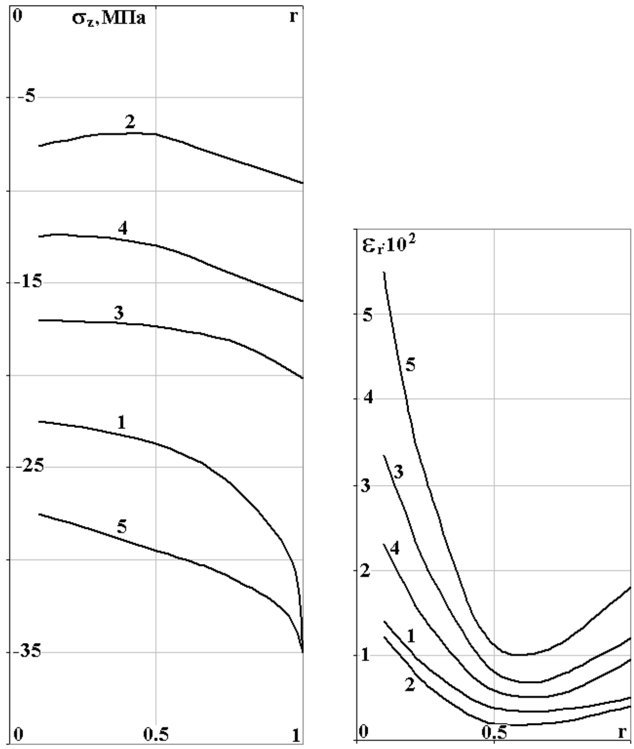

Figure 3: Distributions of axial σ_z stresses and radial ε_r strain through the thickness of the cylinder

of the temperature we can change the nature of the distribution and levels of stress - strain state.

The above analysis corresponds to the case when the inner surface is free from stress, and the outer surface is rigidly fixed. Circumferential stress at points, adjacent to the inner radius, is expanding, that as we move away from it gradually pass into a state of compression.

A similar analysis can be carried out for materials with different properties.

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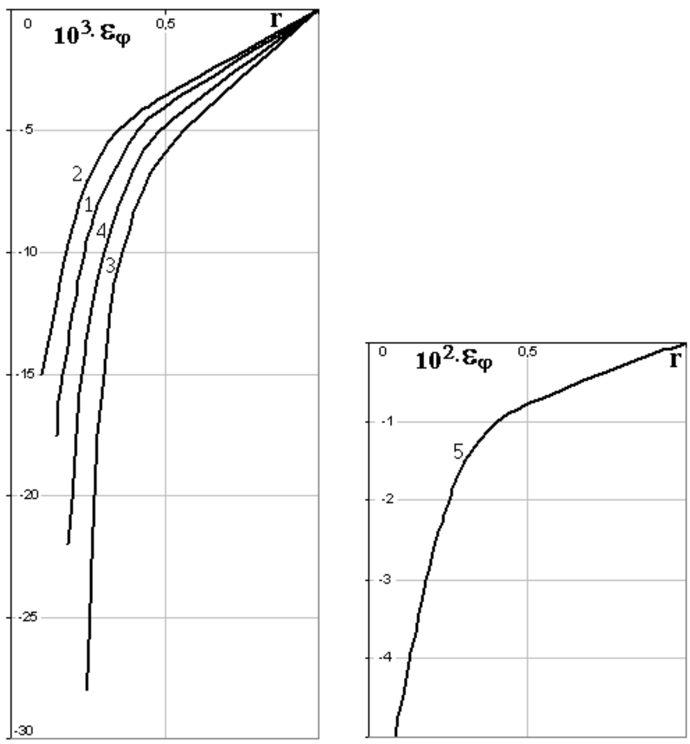

Figure 4: Distribution of circumferential strain over the thickness of the cylinder

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