ON THE FINITENESS PROPERTIES OF GROUPS

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Abstract: For an automorphism \( \varphi \) of the group \( G \), the connection between the centralizer \( C_G(\varphi) \) and the commutator \([G, \varphi]\) is investigated and as a consequence of the Schur theorem it is shown that if \( G/C_G(\varphi) \) and \( G' \) are both finite, then so is \([G, \varphi]\).

AMS Subject Classification: 20D45, 20F14, 20B05

Key Words: autocommutator subgroup, Schur theorem, derived subgroup

1. Introduction

Schur proved that [5] if the centre of a group has finite index, then the derived subgroup of \( G \) is finite. The converse of this theorem has been proved under certain additional assumptions by many authors. Niroomand showed [3] that if \( G' \) is finite and \( G/Z(G) \) is finitely generated, then \( G/Z(G) \) is finite.

For \( \varphi \in Aut(G) \), the group \([G, \varphi] = \langle g^{-1}g^\varphi | g \in G \rangle\) is called the autocommutator subgroup. With the following definitions, we focus on the converse of the Schur theorem for autocommutators.

Received: August 28, 2012

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The centralizer $C_G(\varphi)$ of $\varphi$ in $G$ is defined by

$$C_G(\varphi) = \langle g \in G | g^\varphi = g \rangle$$

where $g^\varphi$ is the action of $\varphi$ on $g \in G$. In the other word $C_G(\varphi)$ is the set of all fixed points of $\varphi$ in $G$. Also define

$$[G, \varphi] = \langle x^{-1}x^\varphi | x \in G \rangle$$

For each $x$ and $y$ in $G$, we set $z = y^{\varphi-1}$, then we have

$$y^{-1}\{x^{-1}x^\varphi\}y = (z^{-1})^\varphi(x^{-1})x^\varphi z^\varphi = (z^{-1})^\varphi z(xz)^{-1}(xz)^\varphi$$

which is an element of $[G, \varphi]$ and so $[G, \varphi]$ is a normal subgroup of $G$.

In this paper we prove that if $C_G(\varphi)$ has a finite index in $G$ and $G'$ is finite, then so is $[G, \varphi]$.

Hilton [2] proved that for a finitely generated group $G$, if $G'$ is finite, then $G\frac{Z(G)}{Z(G)}$ is finite. Niroomand [3] proved that

**Theorem.** Let $G$ be any group such that $d\left(\frac{G}{Z(G)}\right)$ and $G'$ are both finite then $\left|\frac{G}{Z(G)}\right| \leq |G'|^{d(G/Z(G))}$, where $d(X)$ is the minimal number of generators of the group $X$.

**2. Main Results**

**Theorem.** If $\varphi$ is any automorphism of the group $G$ such that $C_G(\varphi)$ has finite index in $G$ and $G'$ is finite, then so is $[G, \varphi]$.

**Proof.** For simplicity, we denote by $C$ the centralizer $C_G(\varphi)$. By the assumption there exist a transversal $T = \{g_1, g_2, \ldots, g_n\}$ of $C$ in $G$ such that $G = \{Cg_1, Cg_2, \ldots, Cg_n\}$. consider the set

$$A = \{a^{-1}a^\varphi | a \in T\}.$$

For any element $g \in G$, there is an element $g_i$ in $T$ such that

$$Cg_i = Cg$$

which means

$$(g^{-1}g_i)^\varphi = g^{-1}g_i$$

and
\[ g^\varphi = g_i^{-1}g \] g_i^\varphi. \]

As \( g^{-1}g^\varphi \) is a generator of \([G, \varphi]\) we obtain
\[
g^{-1}g^\varphi = g^{-1}g_i^{-1}gg_i^\varphi = g^{-1}g_i^{-1}gg_i^{-1}g_i^\varphi = [g, g_i]g_i^{-1}g_i^\varphi \]
which is an element of \( G' \). But this is finite by assumption, so that \([G, \varphi]\) is finite.

Endiomoni and Moravec [1] state and prove the converse of the Schur theorem for autocommutators, which is proved for the sake of completeness as follows:

**Theorem.** Let \( \varphi \) be an automorphism of the group \( G \) such that \([G, \varphi]\) is finite. Then the centralizer \( C_G(\varphi) \) has a finite index in \( G \).

**Proof.** Let \( m \) be the order of \([G, \alpha]\) and consider \( m + 1 \) distinct elements \( x_1, x_2, \cdots, x_m \) of \( G \). In the elements
\[
x_1^{-1}x_1^\varphi, x_2^{-1}x_2^\varphi, \cdots, x_{m+1}^{-1}x_{m+1}^\alpha \]
there are two coincide elements. If for \( i, j \in \{1, 2, \cdots, m + 1\} \), and \( i \neq j \), we have
\[
x_i^{-1}x_i^\varphi = x_j^{-1}x_j^\varphi \]
we have
\[(x_i x_j^{-1})^\varphi = x_i x_j^{-1} \]
so \( x_i x_j^{-1} \) is in \( C_G(\varphi) \) and \([G : C_G(\varphi)] < m \) which complete the proof.

**References**


