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ON THE FINITENESS PROPERTIES OF GROUPS

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Abstract: For an automorphism φ of the group G, the connection between the centralizer $C_G(\varphi)$ and the commutator $[G, \varphi]$ is investigated and as a consequence of the Schur theorem it is shown that if $G/C_G(\varphi)$ and G' are both finite, then so is $[G, \varphi]$.

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1. Introduction

Schur proved that [5] if the centre of a group has finite index, then the derived subgroup of G is finite. The converse of this theorem has been proved under certain additional assumptions by many authors. Niroomand showed [3] that if G' is finite and $\frac{G}{Z(G)}$ is finitely generated, then $\frac{G}{Z(G)}$ is finite.

For $\varphi \in Aut(G)$, the group $[G, \varphi] =< g^{-1}g^{\varphi} | g \in G >$ is called the autocommutator subgroup. With the following definitions, we focus on the converse of the Schur theorem for autocommutators.

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The centralizer $C_G(\varphi)$ of φ in G is defined by

$$C_G(\varphi) =< g \in G | g^{\varphi} = g >$$

where g^{φ} is the action of φ on $g \in G$. In the other word $C_G(\varphi)$ is the set of all fixed points of φ in G. Also define

$$[G, \varphi] =< x^{-1}x^{\varphi} | x \in G >$$

For each x and y in G, we set $z = y^{\varphi^{-1}}$, then we have

$$\begin{aligned} y^{-1}\{x^{-1}x^{\varphi}\}y &= (z^{-1})^{\varphi}(x^{-1})x^{\varphi}z^{\varphi} \\ &= (z^{-1})^{\varphi}z(xz)^{-1}(xz)^{\varphi} \end{aligned}$$

which is an element of $[G, \varphi]$ and so $[G, \varphi]$ is a normal subgroup of G.

In this paper we prove that if $C_G(\varphi)$ has a finite index in G and G' is finite, then so is $[G, \varphi]$.

Hilton [2] proved that for a finitely generated group G, if G' is finite, then $\frac{G}{Z(G)}$ is finite. Niroomand [3] proved that

Theorem. *Let G be any group such that $d(\frac{G}{Z(G)})$ and G' are both finite then $\left|\frac{G}{Z(G)}\right| \leq |G'|^{d(G/Z(G))}$, where $d(X)$ is the minimal number of generators of the group X.*

2. Main Results

Theorem. *If φ is any automorphism of the group G such that $C_G(\varphi)$ has finite index in G and G' is finite, then so is $[G, \varphi]$.*

Proof. For simplicity, we denote by C the centralizer $C_G(\varphi)$. By the assumption there exist a transversal $T = \{g_1, g_2, \cdots, g_n\}$ of C in G such that $\frac{G}{C} = \{Cg_1, Cg_2, \cdots, Cg_n\}$. consider the set

$$A = \{a^{-1}a^{\varphi} | a \in T\}.$$

For any element $g \in G$, there is an element g_i in T such taht

$$Cg_i = Cg$$

which means

$$(g^{-1}g_i)^{\varphi} = g^{-1}g_i \qquad \text{and}$$

$$g^{\varphi} = g_i^{-1} g \, g_i^{\varphi}.$$

As $g^{-1}g^{\varphi}$ is a generator of $[G, \varphi]$ we obtain

$$\begin{aligned} g^{-1}g^{\varphi} &= g^{-1}g_i^{-1}gg_i^{\varphi} \\ &= g^{-1}g_i^{-1}gg_ig_i^{-1}g_i^{\varphi} \\ &= [g, g_i]g_i^{-1}g_i^{\varphi} \end{aligned}$$

which is an element of $G^{\square}A$. But this is finite by assumption, so that $[G, \varphi]$ is finite.

Endiomoni and Moravec [1] state and prove the converse of the Schur theorem for autocommuators, which is proved for the sake of completeness as follows:

Theorem. *Let φ be an automorphism of the group G such that $[G, \varphi]$ is finite. Then the centralizer $C_G(\varphi)$ has a finite index in G.*

Proof. Let m be the order of $[G, \alpha]$ and consider $m + 1$ distinct elements $x_1, x_2, \cdots, x_m$ of G. In the elements

$$x_1^{-1}x_1^{\varphi}, x_2^{-1}x_2^{\varphi}, \cdots, x_{m+1}^{-1}x_{m+1}^{\alpha}$$

there are two coinside elements. If for $i, j \in \{1, 2, \cdots, m + 1\}$, and $i \neq j$, we have

$$x_i^{-1}x_i^{\varphi} = x_j^{-1}x_j^{\varphi}$$

we have

$$(x_ix_j^{-1})^{\varphi} = x_ix_j^{-1}$$

so $x_ix_j^{-1}$ is in $C_G(\varphi)$ and $[G : C_G(\varphi)] < m$ which complete the proof.

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