International Journal of Pure and Applied Mathematics
Volume 83 No. 2 2013, 261-270
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v83i2.5

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# A CANTOR $p-$ARY DECOMPOSITION ON THE HILBERT CUBE

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**Abstract:** Given a positive odd integer $p$ with $p \geq 3$, the Cantor $p-$ary $C_p$ set and the Cantor $p-$ary function $f_p$ are constructed. $C_p$ is a generalization of the Cantor set in the case that the measure of the set $C_p$ is still zero and $f_p^\infty$ defined on the Hilbert Cube $Q$ is a generalization of the Cantor function. Also, for any $s \in (0,1)$, let

$$G^s_{f_p^\infty} = \{\{s\} \times \left(f_p^\infty\right)^{-1}(c) \mid c \in Q_2\}$$

and $S$ is the set of all singletons in $\Big([0,s) \cup (s,1]\Big) \times Q_2$. Then $G = G^s_{f_p^\infty} \cup S$ is an upper semi continuous decomposition on the Hilbert Cube $Q$. Moreover, $Q/G$ is homeomorphic to $Q$.

**AMS Subject Classification:** 54C50
**Key Words:** decomposition, upper semi continuous, Cantor $p-$ary set

## 1. Introduction

A search through literatures about the cantor set and cantor function yields a well-known construction of the standard cantor set $C$ and the standard Cantor

Received: September 1, 2012

function $f$, see [2], [5], [6]. The purpose of this pater is to construct the Cantor $p-$ary set and the Cantor $p-$ary function where $p$ is an any odd integer with $p \geq 3$. Although the ideas of their constructions in the paper are predictable, it can rarely be found in any papers. It is thus obvious that the Cantor $p-$ary set $C_p$ and the Cantor $p-$ary function $f_p$ are the generalization of the standard cantor set $C$ and the standard Cantor function $f$, respectively.

In [7], Garity used the map $f^k$, where $f^k$ is the product of the standard Cantor function $f$, to construct the decomposition $G$ of $\mathbb{R}^n$ and then showed that $\mathbb{R}^n/G$ is homeomorphic to $\mathbb{R}^n$. An arising question is that " Is this still true for an infinite dimensional Hilbert Cube $Q$ ?" This paper will provide an answer to that question. That is, we will show that for any positive odd integer with $p \geq 3$, the decomposition $G$ is induced by the $f_p^\infty$ where $f_p^\infty$ is the infinite product of the Cantor $p-$ary function $f_p$ and show that $Q/G$ is homeomorphic to $Q$. For $p = 3$, see [10].

## 2. Notation and Terminology

For the readers who are not familiar with topology, all basic topological terminology, notation, and definitions can be found in [8], [9].

Let $I$ be the interval $[0, 1]$. For each $n \geq 1$, we write

$$I^n = \prod_{i=1}^{n} I_i, \quad Q_{n+1} = \prod_{i=n+1}^{\infty} I_i$$

where $I_i = I$. The **Hilbert Cube** is a countable product of $I_i$ where $I_i = [0, 1]$ for all $i \geq 1$, and is denoted by $Q$. That is,

$$Q = \prod_{i=1}^{\infty} I_i$$

Also, for each $n$ we can write the Hilbert Cube as

$$Q = I^n \times Q_{n+1}.$$

A **decomposition** $G$ of a space $Q$ is a partition of $Q$. Associated with any decomposition $G$ of a space $Q$ is the **decomposition space** denoted as $Q/G$. Its topology is described by means of the **decomposition map** $\pi : Q \to Q/G$ sending each $q \in Q$ to the unique element of $G$ containing $q$. The topology on $Q/G$ is the quotient space topology induced by $\pi$.

**Definition 1.** (see [4]) A decomposition $G$ of $Q$ is said to be **upper semicontinuous(usc)** if every $g \in G$ is compact and the quotient map

$$\pi : Q \to Q/G$$

is a closed map.

**Definition 2.** (see [4]) A decomposition $G$ of a space $Q$ is **realized by a pseudo-isotopy** if there exists a pseudo-isotopy $\Psi_t$ of $Q$ to $Q$ such that $\Psi_0 = Id_Q$ and $G = \{\Psi_1^{-1}(x) \mid x \in Q\}$. By a **pseudo-isotopy** $\Psi_t$ of $Q$ to $Q$ we mean a homotopy $\Psi_t : Q \to Q$ such that $\Psi_t$ is a homeomorphism for each $t \in [0,1)$ and $\Psi_1$ is a closed surjection. Similarly, by an **isotopy** $\Psi_t$ of $Q$ to $Q$ we mean a homotopy $\Psi_t : Q \to Q$ such that $\Psi_t$ is a homeomorphism for each $t \in [0,1]$.

**Definition 3.** A closed set C in $\mathbb{R}^n$ or in an $n-$dimensional manifold is said to be **cellular** if there is a nested sequence $C_1, C_2, ...$ of $n$ cells with $C_{i+1}$ a subset of the interior of $C_i$ and $C = \bigcap_i^\infty C_i$.

Next we will define a cellularity in $Q$ which is quite similar to the definition in finite dimensional case. Here we replace the term $n-$cells by normal cubes. The definition of a normal cube and related definitions can be found in [1], [3].

**Definition 4.** (see [1]) Let $X$ be a closed subset of $Q$. $X$ is said to be a **cellular** subset of $Q$ if $X = \cap_{i=1}^\infty K_i$ where $K_{i+1} \subset int(K_i)$ and $K_i$ is a normal cube for all $i$.

## 3. Construction of a Cantor $p-$ary Set and a Cantor $p-$ary Function

### 3.1. Construction of a Cantor $p-$ary Set

Basically, the standard Cantor set $C$ is constructed by removing middle third interval in each step, see [6], [5], [2]. We will use this idea to construct the $p-$ary Cantor set $C_p$ so that $C_p \cong C$.

**Definition 5.** Let $n \in \mathbb{Z}^+$ and $p \in \mathbb{N}$ where $p$ is a positive odd integer $p$ with $p \geq 3$. Define an interval $\Theta(k_1, \ldots, k_n)$ by

$$\Theta(k_1, \ldots, k_n) = \Big[\sum_{i=1}^{i=n} \frac{k_i}{p^i}, \sum_{i=1}^{i=n} \frac{k_i}{p^i} + \frac{1}{p^n}\Big] \text{ for } k_i \in K_p$$

where $K_p = \Big\{0, 1, 2, \ldots, p-1\Big\}$.

It is clear by the definition that the length of $\Theta(k_1,\ldots,k_n)$ is $\frac{1}{p^n}$. Also, for each $n$,

$$\Theta(k_1,\ldots,k_n) = \cap_{i=1}^{n}\Theta(k_1,\ldots,k_i)$$

Let us denote the sets $K_p^e = \Big\{0,2,4,\ldots,p-1\Big\}$ and $K_p^o = \Big\{1,3,5,\ldots,p-2\Big\}$

$$C_n = \Big\{\Theta(k_1,k_2,\ldots,k_n) \mid k_i \in K_p^e\Big\}.$$

Let $C_1^c = \Big\{\Theta(1),\Theta(3),\Theta(5),\ldots,\Theta(p-2)\Big\}$, and for $n \geq 2$, define

$$C_n^c = \Big\{\Theta(k_1,\ldots,k_{n-1},k_n) \mid k_1,k_2,\ldots,k_{n-1} \in K_p^e, k_n \in K_p^o\Big\}.$$

Also, let $C^c = \bigcup_{n=1}^{\infty}(\bigcup C_n^c)$.

Now we will use the definition of $C_n$ to define the Cantor $p-$ary set. That is, the Cantor $p-$ary set $C_p$ is defined as:

$$C_p = \bigcap_{n=0}^{\infty}\bigcup C_n.$$

Let $\mathbf{M}$ be the Lebesque measure. Then we have the following lemma.

**Lemma 6.** *Let $C_p$ be the Cantor $p-$ary set where $p$ is a positive odd integer with $p \geq 3$. Then*

$$\mathbf{M}(C_p) = 0.$$

*In other words, $C_p$ has a zero measure.*

*Proof.* Note that for each $n$, the sum of the length of all elements $\Theta$ in $C_n^c$ is

$$\left(\frac{p-1}{2}\right)\left(\frac{p+1}{2}\right)^{n-1}(\frac{1}{p^n}).$$

Since $C_n^c \cap C_{n+1}^c = \emptyset$ for all $n$, we have

$$\mathbf{M}(C^c) = \sum_{n=1}^{\infty}\left(\frac{p-1}{2}\right)\left(\frac{p+1}{2}\right)^{n-1}(\frac{1}{p^n}) = \left(\frac{p-1}{2p}\right)\left(\frac{1}{1-\left(\frac{p+1}{2p}\right)}\right) = 1.$$

But we know that

$$C_p \subset \big([0,1]\setminus C^c\big)\bigcup \cup_{n=1}^{\infty}E_n$$

where $E_n$ is the set of all endpoints of each interval in $C_n^c$. It is obvious that each $E_n$ is countable and hence $\mathbf{M}(\cup_{n=1}^{\infty} E_n) = 0$. Thus,

$$0 \leq \mathbf{M}(C_p) \leq \big([0,1] \setminus C^c\big) \bigcup \cup_{n=1}^{\infty} E_n = (1 - \mathbf{M}(C^c)) + 1 = 1 - 1 + 0 = 0.$$

That is, $\mathbf{M}(C_p) = 0$. □

**Theorem 7.** (see [11]) *A compact set $X$ is homeomorphic to the standard Cantor set $C$ if and only if $X$ is totally disconnected and perfect.*

**Lemma 8.** *Let $C_p$ be the Cantor $p-$ary set where $p$ is an odd integer. Then $C_p$ is homeomorphic to $C$ where $C$ is the standard Cantor set. Moreover, each element in $C_p$ can be written as a $p-$ary representation consisting entirely of elements in $K_p^e$.*

*Proof.* Note that each element in $C_n$ for all $n$ is compact. Since the intersection of the collection of compact sets with the non-empty finite intersection property is compact, is clear that $C_p$ is compact. Since the size of the components of $C_n$ is going to zero, this implies the totally disconnectedness of $C_p$. Moreover, each element of $C_n$ has more than two elements of $C_{n+1}$, this implies every point in $C_p$ is a limit point. Hence $C_p$ is perfect. Therefore, by Theorem 7, $C_p$ is homeomorphic to $C$. Next, we can see that for each $c \in C_p$, $c \in \bigcup C_n$ for all $n$. Thus for each $i$, there exists $k_i$ in $K_p^e$ so that $c \in \cap_{n=1}^{\infty} \Theta(k_1, \dots, k_i)$. This implies that $c_p = \sum_{i=1}^{\infty} \frac{k_i}{p^i}$. □

**Remark 9.** If $p = 3$, then $C_p = C$ where $C$ is the standard Cantor set.

### 3.2. Construction of a Cantor $p-$ary Function

The Cantor $p-$ary map $f_p : I \to I$ is also defined as a constant on the closure of each component of $I \setminus C_p$ and on $C_p$ is defined by:

$$f_p(\sum_{i=1}^{\infty} \frac{a_i}{p^i}) = \sum_{i=1}^{\infty} \frac{a_i}{2(\frac{p+1}{2})^i},$$

where $a_i \in K_p^e$.

**Remark 10.** Let $P$ be the set of all $\frac{p+1}{2}$-adic rationals in the closed unit interval where $p$ is an odd integer. That is,

$$P = \{\frac{m}{(\frac{p+1}{2})^n} \in [0,1] \mid m, n \in \mathbb{Z}\}$$

1. If $c \in I$, the
$$f_p^{-1}(c) = \begin{cases} 1 - \text{cell} & \text{if } c \in P \\ \text{singleton} & \text{if } c \notin P. \end{cases}$$
2. $f_p|_{C_p}$ is two-to-one over the $\frac{p+1}{2}-$adic rationals in $P$;
3. $f_p|_{C_p}$ is one-to-one over the complement of $P$;
4. $f_p$ itself is one-to-one over the complement of $P$.

By Remark 10, for $c \in C_p$,

$$c \text{ is p-adic rational if and only if } f_p(c) \text{ is } \frac{p+1}{2}-\text{adic rational.}$$

Thus, if $f_p(c) = \frac{m}{(\frac{p+1}{2})^n}$ for some $m, n$, then by Remark 10(2), $c = \frac{2k}{p^n}$ or $c = \frac{2k+1}{p^n}$ for some $k$.

Let $f_p^k : I^k \to I^k$ be defined by

$$f_p^k(x) = (f_p(x_1), f_p(x_2), \ldots, f_p(x_k)) \quad \text{for all } \ x \in I^k$$

Note that $f_p^k$ is continuous since each component is continuous.

**Lemma 11.** *Let $c \in I^k$. Then $(f_p^k)^{-1}(c)$ is either a point or a $l-$cell where $l$ corresponds to the number of $\frac{p+1}{2}$-adic rational coordinates that $c$ has, and hence $(f_p^k)^{-1}(c)$ is either a point or a $l-$cell.*

*Proof.* Let $c = (x_1, \ldots, x_k) \in I^k$. If $c$ has no $\frac{p+1}{2}$-adic rational coordinates, then $x_i \notin P$ for all $i = 1, 2, \ldots, k$. Thus, $(f_p^k)^{-1}(x_i)$ is just a point in $I$ which implies that $(f_p^k)^{-1}(c)$ is a point in $I^k$. Next assume that the number of $\frac{p+1}{2}$-adic rational coordinates of $c$ is $l$. Denote each $b_i$ the $\frac{p+1}{2}-$adic rational coordinates of $c$ for $i = 1, \ldots, l$. Then each $(f_p^k)^{-1}(b_i)$ is a 1-cell in $I$ so $(f_p^k)^{-1}(c)$ is a $l-$cell in $I^k$. □

Next, we will define the function $f_p^\infty : Q_2 \to Q_2$. First for each $k$ define $g_p^k : Q_2 \to Q_2$ by

$$g_p^k((x_2, \ldots, x_k, \ldots)) = f_p^k((x_2, \ldots, x_k)) \times Id_{Q_{k+1}}(x_{k+1}, \ldots).$$

Thus, the function $f_p^\infty : Q_2 \to Q_2$ is defined by

$$\begin{aligned} f_p^\infty(x) &= \lim_{k\to\infty} g_p^k(x) \\ &= (f_p(x_2), f_p(x_3), \ldots). \end{aligned}$$

Since $Q_2$ is compact, it is obvious that $f_p^\infty$ is a closed map.

**Lemma 12.** *For each point $c \in Q_2$, $(f_p^\infty)^{-1}(c)$ is either a point, a cell or a copy of $Q_2$ and the dimension of these sets corresponds to the number of $\frac{p+1}{2}$-adic rational coordinates that $c$ has.*

*Proof.* If $c$ has no $\frac{p+1}{2}$-adic rational coordinates, it is clear that $(f_p^\infty)^{-1}(c)$ is just a point in $Q_2$. If $p$ has $l$ $\frac{p+1}{2}$-adic rational coordinates, then $(f_p^\infty)^{-1}(c)$ is a $l-$cell in $Q^2$. If $c$ has infinitely many $\frac{p+1}{2}$-adic rational coordinates, then $(f_p^\infty)^{-1}(c)$ is a copy of $Q_2$. □

## 4. Construction of the Cantor $p-$ary Decomposition $G$

Recall that $f_p^\infty$ is the map from $Q_2 \to Q_2$. To construct a decomposition $G$ on $Q$, first we will use the function $f_p^\infty$ to define the decomposition $G^s_{f_p^\infty}$ on $Q^2$ where $Q^2 = \{0\} \times Q_2$. Fix $s \in (0,1)$. Let

$$G^s_{f_p^\infty} = \Big\{\Big\{s\Big\} \times (f_p^\infty)^{-1}(c) \mid c \in Q_2\Big\}.$$

**Lemma 13.** *The decomposition $G^s_{f_p^\infty}$ defined as above is upper semicontinuous.*

*Proof.* This follows from the fact that $\pi_{G^s_{f_p^\infty}} = \{s\} \times f_p^\infty$. Moreover, by the Lemma 12, $G^s_{f_p^\infty}$ is cellular. □

Next will show that the decomposition $G^s_{f_p^\infty}$ is realized by a pseudo-isotopy.

**Lemma 14.** *The decomposition $G^s_{f_p^\infty}$ is realized by a pseudo-isotopy.*

*Proof.* Recall $f_p^\infty : Q_2 \to Q_2$ is a generalized Cantor $p-$ary function in which each component is the Cantor $p-$ary function $f_p : \Big[0,1\Big] \to \Big[0,1\Big]$. To show that the decomposition $G^s_{f_p^\infty}$ is realized by a pseudo-isotopy, it suffices to show that there exists a pseudo-isotopy $\Psi_t$ of $Q^2_s \to Q^2_s$ such that $\Psi_0$ is the identity $Id_{Q^2_s}$ and $G^s_{f_p^\infty} = \Big\{\Psi_1^{-1}(c) \mid c \in Q^2\Big\}$. For $t \in \Big[0,1\Big]$, define $\Psi_t : Q^2_s \to Q^2_s$ by

$$\Psi_t(s,x) = \big(s, (1-t)x + tf_p^\infty(x)\big)\,.$$

It is clear that $\Psi_1 = (s, f_p^\infty)$ which is a closed surjection. For $t < 1$, $\Psi_t$ is onto since each component is onto by the Intermediate Value Theorem. Also, it is continuous, and hence $\Psi_t^{-1}$ is continuous since $\Psi_t$ is a closed map. It remains to show that for $t < 1$, $\Psi_t$ is one-to-one. Let $(s,x),(s,y) \in Q^2$ be such that $\Psi_t(x) = \Psi_t(y)$. Then

$$\big(s,(1-t)x + tf_p^\infty(x)\big) = \big(s,(1-t)y + tf_p^\infty(y)\big)$$

and so

$$(1-t)x + tf_p^\infty(x) = (1-t)y + tf_p^\infty(y).$$

It implies that $(1-t)(x-y) = t(f_p^\infty(y) - f_p^\infty(x))$. If $x \neq y$, then there is $i$ such that $x_i \neq y_i$. Without loss of generality, assume that $x_i < y_i$. We know that $(1-t)(x_i - y_i) = t(f_p(y_i) - f_p(x_i))$ Then the left hand side of equation is negative but the right hand side of equation is non-negative since the Cantor function $f_p$ is non-decreasing function. This leads to a contradiction. Thus $\Psi_t$ is one-to-one. Also, we can see that

$$G^s_{f_p^\infty} = \Big\{\Psi_1^{-1}(c) \mid c \in Q^2\Big\}. \qquad \square$$

The next lemma follows from the fact that $G^s_{f_p^\infty}$ is realized by a pseudo-isotopy.

**Lemma 15.** *Let $G^s_{f_p^\infty}$ be the decomposition of $Q_s^2 = \Big\{s\Big\} \times Q_2$ induced by the map $f_p^\infty$. Then $\pi_{G_p^\infty}$ from $\Big\{s\Big\} \times Q_2$ to $(\{s\} \times Q_2^s)/G^s_{f_p^\infty}$ is approximable by homeomorphisms.*

**Lemma 16.** *The decomposition $G^s_{f_p^\infty}$ is cellular.*

*Proof.* This follows from Lemma 12. $\square$

Next we will define a decomposition $G$ on $Q$. Given $s \in (0,1)$. Let $G$ be the partition consisting of $G^s_{f_p^\infty} = \big\{\{s\} \times (f_p^\infty)^{-1}(c) \mid c \in Q_2\big\}$ and all singletons in $Q - \{s\} \times Q_2$. It is clear that $G$ is a usc decomposition of $Q$ by a similar idea as shown in Lemma 13. Next we also show that $G$ is realized by a pseudo-isotopy. First, for convenience, denote $\Psi_t^o(x) = (1-t)x + tf^\infty(x)$ the second component of $\phi_t(s,x)$ defined in the previous section. Also, $\Phi_t^o(x)$ is one-to-one since $\Psi_t(s,x)$ is one-to-one. Then define $K_t^s : \Big[0,1\Big] \times Q_2 \to \Big[0,1\Big] \times Q_2$ by

$$K_t^s(r,x) = \begin{cases} (r,(s-r)x + (1+r-s)\Psi_t^o(x)) & \text{if } \ 0 \le r \le s \\ (r,(r-s)x + (1-r+s)\Psi_t^o(x)) & \text{if } \ s \le r \le 1 \end{cases}$$

Claim that for $t < 1$, $K_t^s$ is homeomorphism. Clearly, $K_t^s$ is onto, continuous and $(K_t^s)^{-1}$ is continuous. It remains to show that $K_t^s$ is one-to-one. Suppose that $K_t^s(a,x) = K_t^s(b,y)$ for some $(a,x),(b,y) \in \Big[0,1\Big] \times Q_2$. Then by the definition of $K_t^s$ we have $a=b$. If $a,b \in [0,s]$, then

$$(s-a)x + (1+a-s)\Psi_t^o(x) = (s-a)y + (1+a-s)\Psi_t^o(y).$$

Consider

$$\begin{aligned} \Psi^o_{t(1+a-s)}(x) &= (1-t(1+a-s))x + t(1+a-s)f^\infty(x) \\ &= (s-a)x + (1+a-s)\Psi_t^o(x) \\ &= (s-a)y + (1+a-s)\Psi_t^o(y) \\ &= (1-t(1+a-s))y + t(1+a-s)f^\infty(y) \\ &= \Psi^o_{t(1+a-s)}(y). \end{aligned}$$

Since $t<1$ and $1+a-s \leq 1$, it forces $t(1+a-s) \neq 1$. This yields $\Psi^o_{t(1+a-s)}$ is one-to-one and hence $x=y$. Similarly, for $a,b \in [s,1]$, $K_t^s$ is one-to-one. Therefore, $K_t^s$ is one-to-one for all $t<1$. Moreover, we can see that for each $(r,x) \in ([0,s) \cup (s,1]) \times Q_2$,

$$K_1^s(r,x) = \begin{cases} \big(r,(s-r)x + (1+r-s)f_p^\infty(x)\big) & \text{if } \ 0 \leq r < s \\ \big(r,(r-s)x + (1-r+s)f_p^\infty(x)\big) & \text{if } \ s < r \leq 1 \end{cases}$$

is one-to-one since $r \neq s$. Thus $(K_1^s)^{-1}(r,x)$ is singleton and if $(r,x) \in \{s\} \times Q_2$, $K_1(s,x) = \{s\} \times (f_p^\infty)^{-1}(x) \in G^s_{f_p^\infty}$. Thus,

$$G = \Big\{K_1^{-1}(c) \mid c \in [0,1] \times Q_2\Big\} = S \cup G^s_{f_p^\infty}$$

where $S$ is the set of all singleton in $([0,s) \cup (s,1]) \times Q_2$. Therefore, we see that $G = G^s_{f_p^\infty} \cup S$ is realized by pseudo-isotopies $K_t^s$. Thus, we have the following lemma.

**Lemma 17.** *The decomposition $G$ is realized by pseudo-isotopy.*

The result of Lemma 17 gives the following main results.

## 5. Main Results

**Theorem 18.** *Let $G$ be the decomposition defined as above. Then $\pi_G$ from $Q$ to $Q/G$ is approximable by homeomorphisms, and hence $Q/G \cong Q$.*

*Proof.* This follows from Lemma 17. □

**Lemma 19.** *The decomposition $G$ is cellular.*

*Proof.* This follows from Lemma 12. □

## Acknowledgment

This research is supported by the Centre of Excellence in Mathematics, the Commission on Higher Education, Thailand.

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