A CANTOR p–ARY DECOMPOSITION
ON THE HILBERT CUBE

Aniruth Phon-On
Prince of Songkla University
Pattani Campus
Pattani, 94000, THAILAND
and
Centre of Excellence in Mathematics
CHE, Si Ayutthaya Rd., Bangkok, 10400, THAILAND

Abstract: Given a positive odd integer \( p \) with \( p \geq 3 \), the Cantor \( p \)-ary \( C_p \)
set and the Cantor \( p \)-ary function \( f_p \) are constructed. \( C_p \) is a generalization
of the Cantor set in the case that the measure of the set \( C_p \) is still zero and \( f_p^\infty \)
defined on the Hilbert Cube \( Q \) is a generalization of the Cantor function. Also,
for any \( s \in (0,1) \), let

\[
G_s^{f_p^\infty} = \{ \{ s \} \times \left( f_p^\infty \right)^{-1}(c) \mid c \in Q_2 \}
\]

and \( S \) is the set of all singletons in \( \left( [0,s) \cup (s,1] \right) \times Q_2 \). Then \( G = G_s^{f_p^\infty} \cup S \)
is an upper semi continuous decomposition on the Hilbert Cube \( Q \). Moreover,
\( Q/G \) is homeomorphic to \( Q \).

AMS Subject Classification: 54C50
Key Words: decomposition, upper semi continuous, Cantor \( p \)-ary set

1. Introduction

A search through literatures about the cantor set and cantor function yields a
well-known construction of the standard cantor set \( C \) and the standard Cantor
function $f$, see [2], [5], [6]. The purpose of this paper is to construct the Cantor $p-$ary set and the Cantor $p-$ary function where $p$ is an any odd integer with $p \geq 3$. Although the ideas of their constructions in the paper are predictable, it can rarely be found in any papers. It is thus obvious that the Cantor $p-$ary set $C_p$ and the Cantor $p-$ary function $f_p$ are the generalization of the standard cantor set $C$ and the standard Cantor function $f$, respectively.

In [7], Garity used the map $f^k$, where $f^k$ is the product of the standard Cantor function $f$, to construct the decomposition $G$ of $\mathbb{R}^n$ and then showed that $\mathbb{R}^n/G$ is homeomorphic to $\mathbb{R}^n$. An arising question is that ”Is this still true for an infinite dimensional Hilbert Cube $Q$?” This paper will provide an answer to that question. That is, we will show that for any positive odd integer with $p \geq 3$, the decomposition $G$ is induced by the $f^\infty_p$ where $f^\infty_p$ is the infinite product of the Cantor $p-$ary function $f_p$ and show that $Q/G$ is homeomorphic to $Q$. For $p = 3$, see [10].

2. Notation and Terminology

For the readers who are not familiar with topology, all basic topological terminology, notation, and definitions can be found in [8], [9].

Let $I$ be the interval $[0, 1]$. For each $n \geq 1$, we write

$$I^n = \prod_{i=1}^{n} I_i, \quad Q_{n+1} = \prod_{i=n+1}^{\infty} I_i$$

where $I_i = I$. The **Hilbert Cube** is a countable product of $I_i$ where $I_i = [0, 1]$ for all $i \geq 1$, and is denoted by $Q$. That is,

$$Q = \prod_{i=1}^{\infty} I_i$$

Also, for each $n$ we can write the Hilbert Cube as

$$Q = I^n \times Q_{n+1}.$$
Definition 1. (see [4]) A decomposition $G$ of $Q$ is said to be upper semicontinuous (usc) if every $g \in G$ is compact and the quotient map

$$
\pi : Q \to Q/G
$$

is a closed map.

Definition 2. (see [4]) A decomposition $G$ of a space $Q$ is realized by a pseudo-isotopy if there exists a pseudo-isotopy $\Psi_t$ of $Q$ to $Q$ such that $\Psi_0 = Id_Q$ and $G = \{\Psi_1^{-1}(x) \mid x \in Q\}$. By a pseudo-isotopy $\Psi_t$ of $Q$ to $Q$ we mean a homotopy $\Psi_t : Q \to Q$ such that $\Psi_t$ is a homeomorphism for each $t \in [0,1)$ and $\Psi_1$ is a closed surjection. Similarly, by an isotopy $\Psi_t$ of $Q$ to $Q$ we mean a homotopy $\Psi_t : Q \to Q$ such that $\Psi_t$ is a homeomorphism for each $t \in [0,1]$.

Definition 3. A closed set $C$ in $\mathbb{R}^n$ or in an $n$-dimensional manifold is said to be cellular if there is a nested sequence $C_1, C_2, \ldots$ of $n$ cells with $C_{i+1}$ a subset of the interior of $C_i$ and $C = \bigcap_{i=1}^{\infty} C_i$.

Next we will define a cellularity in $Q$ which is quite similar to the definition in finite dimensional case. Here we replace the term $n$-cells by normal cubes. The definition of a normal cube and related definitions can be found in [1], [3].

Definition 4. (see [1]) Let $X$ be a closed subset of $Q$. $X$ is said to be a cellular subset of $Q$ if $X = \bigcap_{i=1}^{\infty} K_i$ where $K_{i+1} \subset int(K_i)$ and $K_i$ is a normal cube for all $i$.

3. Construction of a Cantor $p$-ary Set and a Cantor $p$-ary Function

3.1. Construction of a Cantor $p$-ary Set

Basically, the standard Cantor set $C$ is constructed by removing middle third interval in each step, see [6], [5], [2]. We will use this idea to construct the $p$-ary Cantor set $C_p$ so that $C_p \cong C$.

Definition 5. Let $n \in \mathbb{Z}^+$ and $p \in \mathbb{N}$ where $p$ is a positive odd integer $p$ with $p \geq 3$. Define an interval $\Theta(k_1, \ldots, k_n)$ by

$$
\Theta(k_1, \ldots, k_n) = \left[ \sum_{i=1}^{i=n} \frac{k_i}{p^i}, \sum_{i=1}^{i=n} \frac{k_i}{p^i} + \frac{1}{p^n} \right] \text{ for } k_i \in K_p
$$

where $K_p = \{0, 1, 2, \ldots, p-1\}$. 
It is clear by the definition that the length of $\Theta(k_1, \ldots, k_n)$ is $\frac{1}{p^n}$. Also, for each $n$, 

$$\Theta(k_1, \ldots, k_n) = \cap_{i=1}^n \Theta(k_1, \ldots, k_i)$$

Let us denote the sets $K_p^e = \{0, 2, 4, \ldots, p-1\}$ and $K_p^o = \{1, 3, 5, \ldots, p-2\}$

$$C_n = \{\Theta(k_1, k_2, \ldots, k_n) \mid k_i \in K_p^e\}.$$ 

Let $C_1^c = \{\Theta(1), \Theta(3), \Theta(5), \ldots, \Theta(p-2)\}$, and for $n \geq 2$, define

$$C_n^c = \{\Theta(k_1, \ldots, k_{n-1}, k_n) \mid k_1, k_2, \ldots, k_{n-1} \in K_p^e, k_n \in K_p^o\}.$$ 

Also, let $C^c = \bigcup_{n=1}^{\infty} C_n^c$.

Now we will use the definition of $C_n$ to define the Cantor $p$–ary set. That is, the Cantor $p$–ary set $C_p$ is defined as:

$$C_p = \bigcap_{n=0}^{\infty} \bigcup C_n.$$ 

Let $M$ be the Lebesgue measure. Then we have the following lemma.

**Lemma 6.** Let $C_p$ be the Cantor $p$–ary set where $p$ is a positive odd integer with $p \geq 3$. Then

$$M(C_p) = 0.$$ 

In other words, $C_p$ has a zero measure.

**Proof.** Note that for each $n$, the sum of the length of all elements $\Theta$ in $C_n^c$ is

$$\left(\frac{p-1}{2}\right) \left(\frac{p+1}{2}\right)^{n-1} \left(\frac{1}{p^n}\right).$$

Since $C_n^c \cap C_{n+1}^c = \emptyset$ for all $n$, we have

$$M(C^c) = \sum_{n=1}^{\infty} \left(\frac{p-1}{2}\right) \left(\frac{p+1}{2}\right)^{n-1} \left(\frac{1}{p^n}\right) \left(\frac{p-1}{2p}\right) \left(\frac{1}{1 - \left(\frac{p+1}{2p}\right)}\right) = 1.$$ 

But we know that

$$C_p \subset ([0,1] \setminus C^c) \bigcup_{n=1}^{\infty} E_n.$$
where $E_n$ is the set of all endpoints of each interval in $C_n^c$. It is obvious that each $E_n$ is countable and hence $M(\bigcup_{n=1}^{\infty}E_n) = 0$. Thus,

$$0 \leq M(C_p) \leq ([0,1] \setminus C^c) \bigcup \bigcup_{n=1}^{\infty}E_n = (1 - M(C^c)) + 1 = 1 - 1 + 0 = 0.$$ 

That is, $M(C_p) = 0$. \hfill $\square$

**Theorem 7.** (see [11]) A compact set $X$ is homeomorphic to the standard Cantor set $C$ if and only if $X$ is totally disconnected and perfect.

**Lemma 8.** Let $C_p$ be the Cantor $p$-ary set where $p$ is an odd integer. Then $C_p$ is homeomorphic to $C$ where $C$ is the standard Cantor set. Moreover, each element in $C_p$ can be written as a $p$-ary representation consisting entirely of elements in $K_p^e$.

**Proof.** Note that each element in $C_n$ for all $n$ is compact. Since the intersection of the collection of compact sets with the non-empty finite intersection property is compact, it is clear that $C_p$ is compact. Since the size of the components of $C_n$ is going to zero, this implies the totally disconnectedness of $C_p$. Moreover, each element of $C_n$ has more than two elements of $C_{n+1}$, this implies every point in $C_p$ is a limit point. Hence $C_p$ is perfect. Therefore, by Theorem 7, $C_p$ is homeomorphic to $C$. Next, we can see that for each $c \in C_p$, $c \in \bigcup C_n$ for all $n$. Thus for each $i$, there exists $k_i$ in $K_p^e$ so that $c \in \cap_{n=1}^{\infty} \Theta(k_1, \ldots, k_i)$. This implies that $c_p = \sum_{i=1}^{\infty} \frac{k_i}{p^i}$. \hfill $\square$

**Remark 9.** If $p = 3$, then $C_p = C$ where $C$ is the standard Cantor set.

### 3.2. Construction of a Cantor $p$-ary Function

The Cantor $p$-ary map $f_p : I \to I$ is also defined as a constant on the closure of each component of $I \setminus C_p$ and on $C_p$ is defined by:

$$f_p(\sum_{i=1}^{\infty} \frac{a_i}{p^i}) = \sum_{i=1}^{\infty} \frac{a_i}{2(\frac{p+1}{2})^i},$$

where $a_i \in K_p^e$.

**Remark 10.** Let $P$ be the set of all $\frac{p+1}{2}$-adic rationals in the closed unit interval where $p$ is an odd integer. That is,

$$P = \left\{ \frac{m}{\left(\frac{p+1}{2}\right)^n} \in [0,1] \mid m,n \in \mathbb{Z} \right\}$$
1. If \( c \in I \), the
\[
    f_p^{-1}(c) = \begin{cases} 
        1 - \text{cell} & \text{if } c \in P \\
        \text{singleton} & \text{if } c \notin P.
    \end{cases}
\]

2. \( f_p|_{C_P} \) is two-to-one over the \( \frac{p+1}{2} \)-adic rational in \( P \);

3. \( f_p|_{C_P} \) is one-to-one over the complement of \( P \);

4. \( f_p \) itself is one-to-one over the complement of \( P \).

By Remark 10, for \( c \in C_p \),
\( c \) is \( p \)-adic rational if and only if \( f_p(c) \) is \( \frac{p+1}{2} \)-adic rational.

Thus, if \( f_p(c) = \frac{m}{(p+1)^n} \) for some \( m, n \), then by Remark 10(2), \( c = \frac{2k}{p^n} \) or \( c = \frac{2k+1}{p^n} \) for some \( k \).

Let \( f_p^k : I^k \to I^k \) be defined by
\[
    f_p^k(x) = (f_p(x_1), f_p(x_2), \ldots, f_p(x_k)) \quad \text{for all } x \in I^k
\]

Note that \( f_p^k \) is continuous since each component is continuous.

**Lemma 11.** Let \( c \in I^k \). Then \((f_p^k)^{-1}(c)\) is either a point or a \( l \)-cell where \( l \) corresponds to the number of \( \frac{p+1}{2} \)-adic rational coordinates that \( c \) has, and hence \((f_p^k)^{-1}(c)\) is either a point or a \( l \)-cell.

**Proof.** Let \( c = (x_1, \ldots, x_k) \in I^k \). If \( c \) has no \( \frac{p+1}{2} \)-adic rational coordinates, then \( x_i \notin P \) for all \( i = 1, 2, \ldots, k \). Thus, \((f_p^k)^{-1}(x_i)\) is just a point in \( I \) which implies that \((f_p^k)^{-1}(c)\) is a point in \( I^k \). Next assume that the number of \( \frac{p+1}{2} \)-adic rational coordinates of \( c \) is \( l \). Denote each \( b_i \) the \( \frac{p+1}{2} \)-adic rational coordinates of \( c \) for \( i = 1, \ldots, l \). Then each \((f_p^k)^{-1}(b_i)\) is a 1-cell in \( I \) so \((f_p^k)^{-1}(c)\) is a \( l \)-cell in \( I^k \).

Next, we will define the function \( f_p^\infty : Q_2 \to Q_2 \). First for each \( k \) define \( g_p^k : Q_2 \to Q_2 \) by
\[
    g_p^k((x_2, \ldots, x_k, \ldots)) = f_p^k((x_2, \ldots, x_k)) \times \text{Id}_{Q_{k+1}}(x_{k+1}, \ldots).
\]

Thus, the function \( f_p^\infty : Q_2 \to Q_2 \) is defined by
\[
    f_p^\infty(x) = \lim_{k \to \infty} g_p^k(x)
    = (f_p(x_2), f_p(x_3), \ldots).
\]

Since \( Q_2 \) is compact, it is obvious that \( f_p^\infty \) is a closed map.
Lemma 12. For each point \( c \in Q_2 \), \((f_p^\infty)^{-1}(c)\) is either a point, a cell or a copy of \( Q_2 \) and the dimension of these sets corresponds to the number of \( \frac{p+1}{2} \)-adic rational coordinates that \( c \) has.

Proof. If \( c \) has no \( \frac{p+1}{2} \)-adic rational coordinates, it is clear that \((f_p^\infty)^{-1}(c)\) is just a point in \( Q_2 \). If \( p \) has \( l \frac{p+1}{2} \)-adic rational coordinates, then \((f_p^\infty)^{-1}(c)\) is a \( l \)-cell in \( Q^2 \). If \( c \) has infinitely many \( \frac{p+1}{2} \)-adic rational coordinates, then \((f_p^\infty)^{-1}(c)\) is a copy of \( Q_2 \).

4. Construction of the Cantor \( p \)-ary Decomposition \( G \)

Recall that \( f_p^\infty \) is the map from \( Q_2 \rightarrow Q_2 \). To construct a decomposition \( G \) on \( Q \), first we will use the function \( f_p^\infty \) to define the decomposition \( G_{f_p^\infty}^s \) on \( Q^2 \) where \( Q^2 = \{0\} \times Q_2 \). Fix \( s \in (0, 1) \). Let

\[
G_{f_p^\infty}^s = \left\{ \left\{ s \right\} \times (f_p^\infty)^{-1}(c) \mid c \in Q_2 \right\}.
\]

Lemma 13. The decomposition \( G_{f_p^\infty}^s \) defined as above is upper semicontinuous.

Proof. This follows from the fact that \( \pi_{G_{f_p^\infty}^s} = \{s\} \times f_p^\infty \). Moreover, by the Lemma 12, \( G_{f_p^\infty}^s \) is cellular.

Next will show that the decomposition \( G_{f_p^\infty}^s \) is realized by a pseudo-isotopy.

Lemma 14. The decomposition \( G_{f_p^\infty}^s \) is realized by a pseudo-isotopy.

Proof. Recall \( f_p^\infty : Q_2 \rightarrow Q_2 \) is a generalized Cantor \( p \)-ary function in which each component is the Cantor \( p \)-ary function \( f_p : [0, 1] \rightarrow [0, 1] \). To show that the decomposition \( G_{f_p^\infty}^s \) is realized by a pseudo-isotopy, it suffices to show that there exists a pseudo-isotopy \( \Psi_t \) of \( Q^2_s \rightarrow Q^2_s \) such that \( \Psi_0 \) is the identity \( Id_{Q^2_s} \) and \( G_{f_p^\infty}^s = \left\{ \Psi_1^{-1}(c) \mid c \in Q^2 \right\} \). For \( t \in [0, 1] \), define \( \Psi_t : Q^2_s \rightarrow Q^2_s \) by

\[
\Psi_t(s, x) = (s, (1 - t)x + tf_p^\infty(x)).
\]

It is clear that \( \Psi_1 = (s, f_p^\infty) \) which is a closed surjection. For \( t < 1 \), \( \Psi_t \) is onto since each component is onto by the Intermediate Value Theorem. Also, it is
continuous, and hence $\Psi_t^{-1}$ is continuous since $\Psi_t$ is a closed map. It remains to show that for $t < 1$, $\Psi_t$ is one-to-one. Let $(s, x), (s, y) \in Q^2$ be such that $\Psi_t(x) = \Psi_t(y)$. Then

$$(s, (1 - t)x + tf_p^\infty(x)) = (s, (1 - t)y + tf_p^\infty(y))$$

and so

$$(1 - t)x + tf_p^\infty(x) = (1 - t)y + tf_p^\infty(y).$$

It implies that $(1 - t)(x - y) = t(f_p^\infty(y) - f_p^\infty(x))$. If $x \neq y$, then there is $i$ such that $x_i \neq y_i$. Without loss of generality, assume that $x_i < y_i$. We know that $(1 - t)(x_i - y_i) = t(f_p(y_i) - f_p(x_i))$ Then the left hand side of equation is negative but the right hand side of equation is non-negative since the Cantor function $f_p$ is non-decreasing function. This leads to a contradiction. Thus $\Psi_t$ is one-to-one. Also, we can see that

$$G_{f_p^\infty}^s = \left\{ \Psi_1^{-1}(c) \mid c \in Q^2 \right\}.$$ 

The next lemma follows from the fact that $G_{f_p^\infty}^s$ is realized by a pseudo-isotopy.

**Lemma 15.** Let $G_{f_p^\infty}^s$ be the decomposition of $Q_s^2 = \{s\} \times Q_2$ induced by the map $f_p^\infty$. Then $\pi_{G_p^\infty}$ from $\{s\} \times Q_2$ to $(\{s\} \times Q_s^2) / G_{f_p^\infty}^s$ is approximable by homeomorphisms.

**Lemma 16.** The decomposition $G_{f_p^\infty}^s$ is cellular.

**Proof.** This follows from Lemma 12.

Next we will define a decomposition $G$ on $Q$. Given $s \in (0, 1)$. Let $G$ be the partition consisting of $G_{f_p^\infty}^s = \{\{s\} \times (f_p^\infty)^{-1}(c) \mid c \in Q_2\}$ and all singletons in $Q - \{s\} \times Q_2$. It is clear that $G$ is a usc decomposition of $Q$ by a similar idea as shown in Lemma 13. Next we also show that $G$ is realized by a pseudo-isotopy. First, for convenience, denote $\Psi_t^s(x) = (1 - t)x + tf^\infty(x)$ the second component of $\phi_t(s, x)$ defined in the previous section. Also, $\phi_t(x)$ is one-to-one since $\Psi_t(s, x)$ is one-to-one. Then define $K_t^s : [0, 1] \times Q_2 \to [0, 1] \times Q_2$ by

$$K_t^s(r, x) = \begin{cases} (r, (s - r)x + (1 + r - s)\Psi_t^s(x)) & \text{if } 0 \leq r \leq s \\ (r, (r - s)x + (1 - r + s)\Psi_t^s(x)) & \text{if } s \leq r \leq 1 \end{cases}$$

Claim that for $t < 1$, $K_t^s$ is homeomorphism. Clearly, $K_t^s$ is onto, continuous and $(K_t^s)^{-1}$ is continuous. It remains to show that $K_t^s$ is one-to-one. Suppose
that \( K_t^s(a, x) = K_t^s(b, y) \) for some \((a, x), (b, y) \in [0, 1] \times Q_2\). Then by the definition of \( K_t^s \) we have \( a = b \). If \( a, b \in [0, s] \), then

\[
(s - a)x + (1 + a - s)\Psi_t^o(x) = (s - a)y + (1 + a - s)\Psi_t^o(y).
\]

Consider

\[
\Psi_t^{o(1+a-s)}(x) = (1 - t(1 + a - s))x + t(1 + a - s)f^\infty(x)
= (s - a)x + (1 + a - s)\Psi_t^o(x)
= (s - a)y + (1 + a - s)\Psi_t^o(y)
= (1 - t(1 + a - s))y + t(1 + a - s)f^\infty(y)
= \Psi_t^{o(1+a-s)}(y).
\]

Since \( t < 1 \) and \( 1 + a - s \leq 1 \), it forces \( t(1 + a - s) \neq 1 \). This yields \( \Psi_t^{o(1+a-s)} \) is one-to-one and hence \( x = y \). Similarly, for \( a, b \in [s, 1] \), \( K_t^s \) is one-to-one. Therefore, \( K_t^s \) is one-to-one for all \( t < 1 \). Moreover, we can see that for each \((r, x) \in ([0, s] \cup (s, 1]) \times Q_2\),

\[
K_t^s(r, x) = \begin{cases} 
(r, (s - r)x + (1 + r - s)f_p^\infty(x)) & \text{if } 0 \leq r < s \\
(r, (r - s)x + (1 - r + s)f_p^\infty(x)) & \text{if } s < r \leq 1
\end{cases}
\]

is one-to-one since \( r \neq s \). Thus \((K_t^s)^{-1}(r, x)\) is singleton and if \((r, x) \in \{s\} \times Q_2\), \( K_1(s, x) = \{s\} \times f_p^\infty(x) \in G_{f_p}^s \). Thus,

\[
G = \left\{ K_1^{-1}(c) \mid c \in [0, 1] \times Q_2 \right\} = S \cup G_{f_p}^s
\]

where \( S \) is the set of all singleton in \(([0, s] \cup (s, 1]) \times Q_2\). Therefore, we see that \( G = G_{f_p}^s \cup S \) is realized by pseudo-isotopies \( K_t^s \). Thus, we have the following lemma.

**Lemma 17.** The decomposition \( G \) is realized by pseudo-isotopy.

The result of Lemma 17 gives the following main results.

### 5. Main Results

**Theorem 18.** Let \( G \) be the decomposition defined as above. Then \( \pi_G \) from \( Q \) to \( Q/G \) is approximable by homeomorphisms, and hence \( Q/G \cong Q \).

**Proof.** This follows from Lemma 17. \( \square \)
Lemma 19. The decomposition $G$ is cellular.

Proof. This follows from Lemma 12.

Acknowledgment

This research is supported by the Centre of Excellence in Mathematics, the Commission on Higher Education, Thailand.

References


