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ON A QUARTER SYMMETRIC NON-METRIC
CONNECTION IN A KENMOTSU MANIFOLDS

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Abstract: In this paper we study a quarter-symmetric non-metric connection in a Kenmotsu manifold.

1. Introduction

In 1975, Golab [11] introduced the notion of quarter-symmetric connection in a Riemannian manifold with affine connection. This was further developed by Jun, De and Pathak [4],Yano and Imai [6], Rastogi [13], [14], Mishra and Pandey [8], Mukhopadhyay, Roy and Barua [12], Biswas and De [10], Sengupta and Biswas [3], Singh and Pandey [9] and many other geometers.

In this paper we have studied a quarter-symmetric non-metric connection in a Kenmotsu manifold, as introduced by Prakash and Narain [1]. The curvature tensor and the Ricci tensor of the quarter-symmetric non-metric connection is found. A necessary and sufficient condition for the Ricci tensor of $\overline{D}$ to be skew-symmetric under certain conditions. A necessary and sufficient condition for

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the projective Ricci tensor of $\overline{D}$ to be skew-symmetric under certain condition. We also find the necessary and sufficient condition for the Einstein manifold of the connection D is equal to the Einstein manifold of the connection $\overline{D}$ under certain condition.

2. Preliminaries

Let M^n be an n- dimensional almost contact Riemannian manifold with an almost contact metric structure (ϕ, ξ, η, g), where ϕ is a (1,1)-tensor field, ξ is a vector field, η is a 1-form and g is the Riemannian metric, which satisfy (see [2],[5])

$$\phi^2 X = -X + \eta(X)\xi, \tag{1.1}$$

$$\eta(\xi) = 1, \tag{1.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{1.3}$$

$$g(X, \xi) = \eta(X), \tag{1.4}$$

$$(D_X \phi)(Y) = -g(X, \phi Y)\xi - \eta(Y)\phi X, \tag{1.5}$$

$$D_X \xi = X - \eta(X)\xi, \tag{1.6}$$

for any vector fields X and Y, where D denotes covariant differentiation with respect to g.

Also, in a Kenmotsu manifold M^n with structure (ϕ, ξ, η, g) the following relation hold: (see [4])

$$(a) \quad \phi\xi = 0 \quad (b) \quad \eta(\phi X) = 0 \quad (c) \quad \text{rank}(\phi) = (n-1). \tag{1.7}$$

$$(D_X \eta)(Y) = g(X, Y) - \eta(X)\eta(Y), \tag{1.8}$$

$$R(X, Y, Z, \xi) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X), \tag{1.9}$$

and

$$S(X, \xi) = -(n-1)\eta(X), \tag{1.10}$$

where R and S denotes curvature tensor of type (0, 4) and the Ricci tensor of type (0, 2) respectively.

A quarter-symmetric non-metric connection $\bar{D}$ in an LP-Sasakian manifold is defined by (Prakash and Narain [1])

$$\overline{D}_X Y = D_X Y + \eta(Y)\phi X + a(X)\phi Y. \tag{1.11}$$

3. The Curvature Tensor of a Kenmotsu Manifold with Respect to the Quarter-Symmetric Non-Metric Connection $\bar{D}$

Let $\bar{R}$ and R be the curvature tensor of the connection $\bar{D}$ and D respectively, then

$$\bar{R}(X,Y)Z = \bar{D}_X \bar{D}_Y Z - \bar{D}_Y \bar{D}_X Z - \bar{D}_{[X,Y]} Z. \tag{3.1}$$

From (1.11) and (3.1), we get

$$\bar{R}(X,Y)Z = \bar{D}_X(D_Y Z + \eta(Z)\phi Y + a(Y)\phi Z) - \bar{D}_Y(D_X Z + \eta(Z)\phi X + a(X)\phi Z)$$

$$- D_{[X,Y]}Z - \eta(Z)\phi([X,Y]) - a([X,Y])\phi Z. \tag{3.2}$$

Using (1.2), (1.5), (1.6) and (1.8) in (3.2), we get

$$\bar{R}(X,Y)Z$$
$$= R(X,Y)Z + g(X,Z)\phi Y - g(Y,Z)\phi X + 2g(\phi X,Y)\eta(Z)\xi - g(\phi Y,Z)a(X)\xi$$
$$+ g(\phi X,Z)a(Y)\xi + a(X)\eta(Z)\phi Y - a(Y)\eta(Z)\phi X + a(Y)\eta(Z)X - a(X)\eta(Z)Y$$
$$+ a(X)\eta(Y)\eta(Z)\xi - a(Y)\eta(X)\eta(Z)\xi + da(X,Y)\phi Z, \tag{3.3}$$

where $\qquad R(X,Y)Z = D_X\,D_Y Z - D_Y\,D_X Z - D_{[X,Y]}Z$ is the curvature tensor of D with respect to the Riemannian connection.

Contracting (3.3), we find

$$\bar{S}(Y,Z) = S(Y,Z) + [1 - a(\xi)]g(\phi Y,\,Z) + (n-2)a(Y)\eta(Z)$$

$$+ a(\xi)\eta(Y)\eta(Z) + a(\phi Y)\eta(Z) + da(\phi Z,\,Y). \tag{3.4}$$

and

$$\bar{r} = r + (n-1)a(\xi) + \lambda, \tag{3.5}$$

where $\bar{S}$ and $\bar{r}$ are the Ricci tensor and scalar curvature with respect to $\bar{D}$ and

$$\lambda = \text{trace}\, da\,(\phi Z, Y).$$

Hence, we can state the following theorem :

Theorem 3.1. *The curvature tensor $\bar{R}(X,Y)Z$, the Ricci tensor $\bar{S}(Y,Z)$ and the scalar curvature $\bar{r}$ of a Kenmotsu manifold with respect to quarter-symmetric non-metric connection is given by (3.3), (3.4) and (3.5) respectively.*

Let us assume that $\bar{R}(X,Y)Z = 0$ in (3.3) and contracting, we get

$$S(Y,Z) = -[1 - a(\xi)]g(\phi Y,\,Z) - (n-2)a(Y)\eta(Z)$$

$$-a(\xi)\eta(Y)\eta(Z) - a(\phi Y)\eta(Z) - da(\phi Z, Y).,$$

which gives

$$r = -(n-1)a(\xi) - \lambda.$$

Hence, we can state the following theorem.

Theorem 3.2. If a Kenmotsu manifold M^n admits a quarter-symmetric non-metric connection whose curvature tensor vanishes, then the scalar curvature r is given by $r = -(n-1)a(\xi) - \lambda$.

4. Skew-Symmetric Condition of Ricci Tensor of $\bar{D}$ in a Kenmotsu Manifold

From (3.4), we have

$$\overline{S}(Z,Y) = S(Z,Y) + [1 - a(\xi)]g(\phi Z, Y) + (n-2)a(Z)\eta(Y)$$

$$+a(\xi)\eta(Y)\eta(Z) + a(\phi Z)\eta(Y) + da(\phi Y, Z). \qquad (4.1)$$

From (3.4) and (4.1), we have

$$\overline{S}(Y,Z) + \overline{S}(Z,Y) = 2S(Y,Z) + (n-2)a(Y)\eta(Z) + a(Z)\eta(Y)$$

$$+2a(\xi)\eta(Y)\eta(Z) + a(\phi Z)\eta(Y) + a(\phi Y)\eta(Z) + da(\phi Y, Z) + da(\phi Z, Y). \quad (4.2)$$

If $\overline{S}(Y,Z)$ is skew-symmetric then the left hand side of (4.2) vanishes, and we get

$$S(Y,Z) = -a(\xi)\eta(Y)\eta(Z) - \frac{(n-2)}{2}[a(Y)\eta(Z) + a(Z)\eta(Y)]$$

$$-\frac{1}{2}[a(\phi Z)\eta(Y) + a(\phi Y)\eta(Z) + da(\phi Y, Z) + da(\phi Z, Y)]. \qquad (4.3)$$

Moreover if $S(Y,Z)$ is given by (4.3), then from (4.2), we get

$$\overline{S}(Y,Z) + \overline{S}(Z,Y) = 0.$$

Hence, we can state the following theorem :

Theorem 4.1. *If a Kenmotsu manifold admits a quarter-symmetric non-metric connection $\overline{D}$ then a necessary and sufficient condition for the Ricci tensor of $\overline{D}$ to be skew-symmetric is that the Ricci tensor of the Levi-Civita connection D is given by (4.3).*

5. Skew-Symmetric Properties of Projective Ricci Tensor with Respect to Quarter-Symmetric Non-Metric Connection $\bar{D}$ in a Kenmotsu Manifold

Projective Ricci tensor in a Riemannian manifold is defined by (Chaki and Saha [7]) as follows

$$P(X,Y) = \frac{n}{(n-1)}[S(X,Y) - \frac{r}{n}g(X,Y)]. \tag{5.1}$$

Analogous to this definition, we define projective Ricci tensor with respect to quarter-symmetric non-metric connection $\overline{D}$, given by

$$\overline{P}(X,Y) = \frac{n}{(n-1)}[\overline{S}(X,Y) - \frac{\overline{r}}{n}g(X,Y)]. \tag{5.2}$$

From (3.4), (3.5) and (5.2), we have

$$
\begin{aligned}
\overline{P}(X,Y) =& \frac{n}{(n-1)}[S(X,Y) + \{1 - a(\xi)\}g(\phi X, Y) \\
&+ (n-2)a(X)\eta(Y) + a(\xi)\ \eta(X)\eta(Y) \\
&+ a(\phi X)\eta(Y) + da(\phi Y, X) \\
&- \left\{\frac{r + (n-1)\,a(\xi) + \lambda}{n}\right\}g(X,Y)].
\end{aligned}
\tag{5.3}
$$

From (5.3), we have

$$
\begin{aligned}
\overline{P}(Y,X) =& \frac{n}{(n-1)}[S(Y,X) + \{1 - a(\xi)\}g(\phi Y, X) \\
&+ (n-2)a(Y)\eta(X) + a(\xi)\ \eta(X)\eta(Y) \\
&+ a(\phi Y)\eta(X) + da(\phi X, Y) \\
&- \left\{\frac{r + (n-1)\,a(\xi) + \lambda}{n}\right\}g(Y,X)].
\end{aligned}
\tag{5.4}
$$

From equation (5.3) and (5.4), we have

$$
\overline{P}(X,Y) + \overline{P}(Y,X) = \frac{n}{(n-1)}[2S(X,Y) + (n-2)\{a(Y)\eta(X) + a(X)\eta(Y)\}
$$

$$
+ 2a(\xi)\ \eta(X)\eta(Y) - 2\left\{\frac{r + (n-1)\,a(\xi) + \lambda}{n}\right\}g(X,Y)
$$

$$
+ a(\phi X)\eta(Y) + a(\phi Y)\eta(X) + da(\phi X, Y) + da(\phi Y, X)]. \tag{5.5}
$$

If $\overline{P}(X,Y)$ is skew-symmetric then the left hand side of (5.5) vanishes, and we get

$$S(X,Y) = [-a(\xi)\,\eta(X)\eta(Y) - \frac{(n-2)}{2}\{a(Y)\eta(X) + a(X)\eta(Y)\}$$

$$-\frac{1}{2}\{a(\phi X)\eta(Y) + a(\phi Y)\eta(X) + da(\phi X,Y) + da(\phi Y,X)\}$$

$$+\left\{\frac{r + (n-1)\,a(\xi) + \lambda}{n}\right\} g(X,Y)]. \tag{5.6}$$

Moreover if $S(X,Y)$ is given by (5.6), then from (5.5), we get

$$\overline{P}(X,Y) + \overline{P}(Y,X) = 0,$$

i.e. Projective Ricci tensor of $\overline{D}$ is skew-symmetric.

Hence, we can state the following theorem :

Theorem 5.1. *If a Kenmotsu manifold admits a quarter-symmetric non-metric connection $\overline{D}$ then a necessary and sufficient condition for the Projective Ricci tensor of $\overline{D}$ to be skew-symmetric is that the Ricci tensor of the Levi-Civita connection D is given by (5.6).*

6. Einstein Manifold with Respect to Quarter-Symmetric Non-Metric Connection D̄ in a Kenmotsu Manifold

A Riemannian manifold M^n is called an Einstein manifold with respect to Riemannian connection if

$$S(X,Y) = \frac{r}{n}g(X,Y). \tag{6.1}$$

Analogous to this definition, we define Einstein manifold with respect to quarter-symmetric non-metric connection $\overline{D}$ by

$$\overline{S}(X,Y) = \frac{\overline{r}}{n}\,g(X,Y). \tag{6.2}$$

From (3.4), (3.5) and (6.2), we have

$$\overline{S}(X,Y) - \frac{\overline{r}}{n}\,g(X,Y) = S(X,Y) - \frac{r}{n}\,g(X,Y) + \{1 - a(\xi)\}g(\phi X,Y)$$

$$+(n-2)a(X)\eta(Y) + a(\xi)\,\eta(X)\eta(Y) + a(\phi X)\eta(Y) + da(\phi Y,X)$$

$$-\left[\frac{(n-1)a(\xi)+\lambda}{n}\right]g(X,Y). \tag{6.3}$$

If

$$\{1-a(\xi)\}g(\phi X,Y)+da(\phi Y,X)=\left[\frac{(n-1)a(\xi)+\lambda}{n}\right]g(X,Y)$$

$$-(n-2)a(X)\eta(Y)-a(\xi)\ \eta(X)\eta(Y)-a(\phi X)\eta(Y). \tag{6.4}$$

Then from (6.3), we get

$$\overline{S}(X,Y)-\frac{\overline{r}}{n}\,g(X,Y)=S(X,Y)-\frac{r}{n}\,g(X,Y). \tag{6.5}$$

Hence, we can state the following theorem.

Theorem 6.1. *In a Kenmotsu manifold M^n with quarter-symmetric non-metric connection if the relation (6.4) holds, then the manifold is an Einstein manifold for the Riemannian connection if and only if it is an Einstein manifold for the connection $\overline{D}$.*

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