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COEFFICIENT BOUND OF A GENERALIZED CLOSE-TO-CONVEX FUNCTION

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Abstract: We look at function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, which are analytic in the unit disc $E = \{z : |z| < 1\}$. For$|\alpha| < \pi$ and $\cos\alpha > \delta$, let $G(\alpha, \delta)$ denote the class of function f, $f(0) = f'(0) - 1 = 0$ for which Re $\{e^{i\alpha}\frac{2zf'(z)}{f(z)-f(-z)}\} > \delta$. In this paper, we determine the basic properties such as the representation theorem, extreme point and we obtain sharp bound for a_n of $G(\alpha, \delta)$.

AMS Subject Classification: 30C45
Key Words: analytic function, starlike function, representation theorem, extreme points, coefficient bound

1. Introduction

Let $\mathcal{A}$ denote be the class of functions given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ that are analytic in $E = \{z : |z| < 1\}$. We define $G(\alpha, \delta)$ as the class of the normalize function $f \in \mathcal{A}$ satifying the condition

$$Re\{e^{i\alpha}\frac{2zf'(z)}{g(z)}\} > \delta \quad (z \in E). \tag{1}$$

where $|\alpha| < \pi$, $\cos\alpha > \delta$ and $g(z) = f(z) - f(-z)$. These function are called starlike with respect to symmetric point and were intoduced by Sakaguchi[3] in 1959. We also define subclass of $\mathcal{A}$ consisting of function that are univalent,

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starlike with respect to origin, and convex denotes by S, St, and K, respectively. A paper by Silverman and Silvia[4] gave some result for $G(\alpha,0)$ when $g'(z)=1$, and for $G(\alpha,\delta)$ it has been done by Mohamad[2] when he considers the same $g(z)$ as Silverman and Silvia did. Afterward, Soh and Mohamad[5] studied the class $G(\alpha,\delta)$ where the function for this class satisfy

$$Re\{e^{i\alpha}\frac{f'(z)}{g'(z)}\}>\delta \quad (z\in E; |\alpha|<\pi; \cos\alpha>\delta; g'(z)=\frac{1+z}{1-z}) \tag{2}$$

Later, Akbarally, Mohamad, Soh and Kaharuddin[1] define $G_K(\alpha,\delta)$ as the class of functions that satisfy

$$Re\{e^{i\alpha}\frac{f'(z)}{g'(z)}\}>\delta \quad (z\in E; |\alpha|<\pi; \cos\alpha>\delta; g'(z)=\frac{1}{1-z}) \tag{3}$$

Our purpose on this paper is to obtain similar result for $G(\alpha,\delta)$, when we consider a different kind of $g(z)$ which is similar to the Sakaguchi class.

2. Representation Theorem

Let P be the set of function with positive real part in E . Function in P can be written as

$$p(z)=1+p_1z+p_2z^2+\ldots+p_nz^n+\ldots=1+\sum_{n=1}^{\infty}p_nz^n \tag{4}$$

that are regular in E and such that for z in E, Re $p(z)>0$. First we related function in $G(\alpha,\delta)$ to the functions in P. For any $f\in\mathcal{A}$, put

$$\frac{e^{i\alpha}(\frac{2zf'(z)}{f(z)-f(-z)})-isin\alpha-\delta}{\cos\alpha-\delta}=p(z) \quad (z\in E).$$

and from (1), based on Sakaguchi[3], we have

$$\frac{2zf'(z)}{f(z)-f(-z)}\ll\frac{(1+z)}{(1-z)}.$$

On the other hand, since $f(z)-f(-z)$ is odd and starlike with respect to the origin for $|z|<1$, Sakaguchi[3] generalized the following form

$$f(z)-f(-z)\ll\frac{2z}{1-z^2}.$$

We have $f \in G(\alpha, \delta)$ if and only if $p \in P$, and in particular the representation of any function $f \in G(\alpha, \delta)$ as

$$f'(z) = e^{i\alpha}(\frac{1}{1-z^2})(A_{\alpha\delta}p(z) + isin\alpha + \delta), \quad (z \in E). \tag{5}$$

where $A_{\alpha\delta} = \cos\alpha - \delta$.

The establishment of the representation theorem for $G(\alpha, \delta)$ will be carried on by the same approach of the Herglotz Representation Theorem for function in P.

Theorem 1. *Let $f \in G(\alpha, \delta)$. Then for some probability measure μ on the unit circle X, we have*

$$f(x) = \int_X [\frac{-e^{-i\alpha}(-e^{-i\alpha} - 2\delta)}{2x} \log(\frac{1+xz}{1-xz}) + \frac{A_{\alpha\delta e^{-i\alpha}}}{2x}(\frac{2}{xz-1} + \log(\frac{1+xz}{1-xz}) - 2)]d\mu(x).$$

Conversely, if f is given by the above equation, then $f \in G(\alpha, \delta)$.

Proof. We have

$$p \in P \Leftrightarrow p(z) = \int_X \frac{1+xz}{1-xz} d\mu(x)$$

for some probability measure μ on the unit circle X. Using (4), we have

$$e^{i\alpha}\frac{2zf'(z)}{f(z) - f(-z)} = A_{\alpha\delta}\int_X \frac{1+xz}{1-xz}d\mu(x) + i\sin\alpha + \delta$$

or

$$f'(z) = \int_X (\frac{1}{1-z^2})[\frac{1 + (e^{-2i\alpha} - 2\delta e^{-i\alpha})xz}{1-xz}]$$

so that

$$f(z) = \int_0^z \int_X (\frac{1}{1-\varphi^2})[-e^{-i\alpha}(e^{-i\alpha} - 2\delta) + \frac{2A_{\alpha\delta}e^{-i\alpha}}{1-x\varphi}]d\mu(x)d\varphi \tag{6}$$

Upon reversing the order of integration we have

$$f(z) = \int_X [\int_0^z \frac{1}{1-\varphi^2}([-e^{-i\alpha}(e^{-i\alpha} - 2\delta) + \frac{2A_{\alpha\delta}e^{-i\alpha}}{1-x\varphi}])d\varphi]d\mu(x) \tag{7}$$

and integrating with respect to φ, we have

$$f(x) = \int_X [\frac{-e^{-i\alpha}(e^{-i\alpha} - 2\delta)}{2x} \log(\frac{1+xz}{1-xz}) + \frac{A_{\alpha\delta} e^{-i\alpha}}{2x} (\frac{2}{xz-1} + \log(\frac{1+xz}{1-xz}) - 2)] d\mu(x).$$

and this desired representation. □

Corollary 2. *We note from this result that the extreme points of* $G(\alpha, \delta)$ *are the unit points masses*

$$f(x)_x = \frac{-e^{-i\alpha}(e^{-i\alpha} - 2\delta)}{2x} \log(\frac{1+xz}{1-xz}) + \frac{A_{\alpha\delta} e^{-i\alpha}}{2x} (\frac{2}{xz-1} + \log(\frac{1+xz}{1-xz}) - 2)$$

with $|x| = 1$*. Furthermore, the derivatives of the extreme points of* $G(\alpha, \delta)$ *are the point masses*

$$f_x'(z) = (\frac{1 + (e^{-2i\alpha} - 2\delta e^{-i\alpha})xz}{1-xz})(\frac{1}{1-z^2}), \quad |x| = 1. \tag{8}$$

3. Results

We obtain the coefficient bound by using the representation theorem which establish before.

Theorem 3. *Let* $f \in S$ *and* $f \in G(\alpha, \delta)$*, then*

$$|a_n| = \frac{2}{n}\Big[\frac{1}{2} + \frac{(n-1)}{2} A_{\alpha\delta}\Big], \quad n = 2, 3, 4, \ldots$$

Equality is attained for each n *when* f *is an extreme point of* $G(\alpha, \delta)$*.*

Proof. From (5) we have

$$f(z) = \int_0^z [\int_X (\frac{-e^{-i\alpha}(e^{-i\alpha} - 2\delta)}{1 - x\varphi^2} + \frac{2A_{\alpha\delta} e^{-i\alpha}}{(1-x\varphi)^2(1+x\varphi)}) d\mu(x)] d\varphi$$

and

$$f(z) = \frac{1}{2} \int_0^z \int_X (\frac{-e^{-i\alpha}(e^{-i\alpha} - 2\delta)}{1 - x\varphi} + \frac{(3 - x\varphi)A_{\alpha\delta} e^{-i\alpha}}{(1-x\varphi)^2} + \frac{-e^{-i\alpha}(e^{-i\alpha} - 2\delta)}{1 + x\varphi} + \frac{A_{\alpha\delta} e^{-i\alpha}}{(1+x\varphi)}) d\mu(x) d\varphi$$

and since

$$\frac{1}{1-x\varphi}=\sum_0^{\infty}(x\varphi)^n,\ \frac{1}{(1-x\varphi)^2}=\sum_0^{\infty}(n+1)(x\varphi)^n,\ \text{and}\ \frac{1}{1+x\varphi}=\sum_0^{\infty}(-1)^n(x\varphi)^n$$

$$\begin{aligned}
f(z) &= \int_0^z\int_X\frac{1}{2}\Big(-e^{-i\alpha}(e^{-i\alpha}-2\delta)\sum_0^{\infty}(x\varphi)^n\Big)d\mu(x)d\varphi\\
&+ \int_0^z\int_X\frac{1}{2}\Big((3-x\varphi)A_{\alpha\delta}e^{-i\alpha}\sum_0^{\infty}(n+1)(x\varphi)^n\Big)d\mu(x)d\varphi\\
&+ \int_0^z\int_X\frac{1}{2}\Big(-e^{-i\alpha}(e^{-i\alpha}-2\delta)\sum_0^{\infty}(-1)^n(x\varphi)^n\Big)d\mu(x)d\varphi\\
&+ \int_0^z\int_X\frac{1}{2}\Big(A_{\alpha\delta}e^{-i\alpha}\sum_0^{\infty}(-1)^n(x\varphi)^n\Big)d\mu(x)d\varphi\\
&= \frac{1}{2}\int_0^z -e^{-i\alpha}(e^{-i\alpha}-2\delta)\int_X\Big(\sum_0^{\infty}(x\varphi)^n\Big)d\mu(x)d\varphi\\
&+ \frac{1}{2}\int_0^z(3-x\varphi)A_{\alpha\delta}e^{-i\alpha}\int_X\Big(\sum_0^{\infty}(n+1)(x\varphi)^n\Big)d\mu(x)d\varphi\\
&+ \frac{1}{2}\int_0^z -e^{-i\alpha}(e^{-i\alpha}-2\delta)\int_X\Big(\sum_0^{\infty}(-1)^n(x\varphi)^n\Big)d\mu(x)d\varphi\\
&+ \frac{1}{2}\int_0^z A_{\alpha\delta}e^{-i\alpha}\int_X\Big(\sum_0^{\infty}(-1)^n(x\varphi)^n\Big)d\mu(x)d\varphi
\end{aligned}$$

Upon simplification of $f(z)$, we achived

$$\begin{aligned}
f(z) &= \int_0^z\Big(1+\frac{-e^{-i\alpha}(e^{-i\alpha}-2\delta)}{2}\Big)\Big(\int_X\sum_1^{\infty}(x)^n d\mu(x)\Big)\varphi^n\\
&+ \int_0^z\Big(\frac{3A_{\alpha\delta}e^{-i\alpha}}{2}\Big)\Big(\int_X\sum_1^{\infty}(n+1)(x)^n d\mu(x)\Big)\varphi^n\\
&- \int_0^z\Big(\frac{A_{\alpha\delta}e^{-i\alpha}}{2}\Big)\Big(\int_X\sum_1^{\infty}n(x)^n d\mu(x)\Big)\varphi^n
\end{aligned}$$

$$
\begin{aligned}
&+ \quad \int_0^z \Big(\frac{-e^{-i\alpha}(e^{-i\alpha}-2\delta)+A_{\alpha\delta}e^{-i\alpha}}{2}\Big)\Big(\int_X \sum_1^{\infty}(-1)^{n-1}(x)^n d\mu(x)\Big)\varphi^n \\
&= \quad z+\Big\{\frac{-e^{-i\alpha}(e^{-i\alpha}-2\delta)}{2}\Big(\int_X \sum_2^{\infty}(x)^{n-1}d\mu(x)\Big)\Big\}\frac{z^n}{n} \\
&+ \quad \Big\{\frac{3A_{\alpha\delta}e^{-i\alpha}}{2}\Big(\int_X \sum_2^{\infty} n(x)^{n-1}d\mu(x)\Big)\Big\}\frac{z^n}{n} \\
&- \quad \Big\{\frac{A_{\alpha\delta}e^{-i\alpha}}{2}\Big(\int_X \sum_2^{\infty}(n-1)(x)^{n-1}d\mu(x)\Big)\Big\}\frac{z^n}{n} \\
&+ \quad \Big\{\frac{-e^{-i\alpha}(e^{-i\alpha}-2\delta)+A_{\alpha\delta}e^{-i\alpha}}{2}\Big(\int_X \sum_2^{\infty}(-1)^{n-1}(x)^{n-1}d\mu(x)\Big)\Big\}\frac{z^n}{n}
\end{aligned}
$$

Now, since $f(z)=z+\sum_{n=2}^{\infty}a_n z^n$, then,

$$
\begin{aligned}
a_n \quad &= \quad \frac{-e^{-i\alpha}(e^{-i\alpha}-2\delta)}{2n}\Big[\int_X (x)^{n-1}d\mu(x)\Big]+\frac{3A_{\alpha\delta}e^{-i\alpha}}{2n}\Big[\int_X n(x)^{n-1}d\mu(x)\Big] \\
&- \quad \frac{A_{\alpha\delta}e^{-i\alpha}}{2n}\Big[\int_X (n-1)(x)^{n-1}d\mu(x)\Big] \\
&+ \quad \frac{-e^{-i\alpha}(e^{-i\alpha}-2\delta)+A_{\alpha\delta}e^{-i\alpha}}{2n}\Big[\int_X (-1)^{n-1}(x)^{n-1}d\mu(x)\Big]
\end{aligned}
$$

which gives

$$
\begin{aligned}
|a_n| \quad &= \quad \frac{|-e^{-i\alpha}(e^{-i\alpha}-2\delta)+2nA_{\alpha\delta}e^{-i\alpha}+(-e^{-i\alpha}(e^{-i\alpha}-2\delta)+2A_{\alpha\delta}e^{-i\alpha})|}{2n} \\
&\quad \times\Big|\int_X (x)^{n-1}d\mu(x)\Big| \\
&= \quad \frac{|-e^{-i2\alpha}-2\delta e^{-i\alpha}+(nA_{\alpha\delta}+A_{\alpha\delta})e^{-i\alpha}|}{n}\Big|\int_X (x)^{n-1}d\mu(x)\Big| \\
&= \quad \frac{|1-2A_{\alpha\delta}e^{-i\alpha}+(nA_{\alpha\delta}+A_{\alpha\delta})e^{-i\alpha}|}{n}\int_X \Big|(x)^{n-1}\Big|d\mu(x) \\
&= \quad \frac{|1+A_{\alpha\delta}e^{-i\alpha}(n-1)|}{n}\int_X \Big|(x)^{n-1}\Big|d\mu(x) \\
&\leq \quad \frac{1+A_{\alpha\delta}e^{-i\alpha}(n-1)}{n}\int_X \Big|(x)^{n-1}\Big|d\mu(x) \\
&= \quad \frac{2}{n}\Big[\frac{1}{2}+\frac{(n-1)}{2}A_{\alpha\delta}\Big], \quad \text{as required.}
\end{aligned}
$$

□

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