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FIXED POINT THEOREM IN FUZZY METRIC SPACE FOR NON COMPATIBLE MAPS

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Abstract: In this paper ,the concept of non compatible maps in fuzzy metric space has been applied to prove common fixed point theorem . A fixed point theorem for six self maps has been established using the concept of non compatible maps. These results are proved without exploiting the notion of continuity and without imposing any condition on t-norm.

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1. Introduction

Zadeh [3] introduced the concept of fuzzy sets in 1965 and Kramosil and Michalek [7] introduced the concept of fuzzy metric space in 1975. George and Veeramani [1] modified the concept of fuzzy metric space introduced by Kramosil and Michalek [7]. Later several authors Grabiec [5], G.Jungck [2], R.Vasuki [8] obtained fixed and common fixed point theorems satisfying various contractive conditions in fuzzy metric spaces.

In the study of fixed points of metric spaces, Pant [9, 10, 11] has initiated work using the concept of non-compatible maps in metric spaces. Recently

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Aamri and Moutawakil [1] introduced the property (E.A) and thus generalized the concept of non-compatible maps. The results obtained in the fuzzy metric fixed point theory by using the notion of non-compatible maps or the property (E.A) are very interesting. The aim of this paper is to obtain common fixed point of mappings satisfying generalized contractive type conditions without exploiting the notion of continuity in the setting of fuzzy metric spaces.

2. Preliminaries

Definition 2.1. (see [9]) A binary operation * : [0, 1] × [0, 1] → [0, 1] is called a t-norm if ([0, 1], *) is an abelian topological monoid with unit 1 such that a * b ? c *d whenever a ? c and b ? d for a, b, c, d ? [0, 1].

Examples of t-norms are a * b = ab and a * b = min{a, b}.

Definition 2.2. (see [9]) The 3-tuple (X, M, *) is said to be a *Fuzzy metric space* if X is an arbitrary set, * is a continuous t-norm and M is a Fuzzy set in $X^2 \times [0, \infty)$ satisfying the following conditions: for all $x, y, z \in X$ and $s, t > 0$:

$$
\begin{aligned}
&(FM-1) && M(x,y,0) = 0,\\
&(FM-2) && M(x,y,t) = 1 for all t < 0 if and only if x = y,\\
&(FM-3) && M(x,y,t) = M(y,x,t),\\
&(FM-4) && M(x,y,t) * M(y,z,s) \leq M(x,z,t+s),\\
&(FM-5) && M(x,y,.) : [0,\infty) \to [0,1] is left continuous,\\
&(FM-6) && \lim_{n \to \infty} M(x,y,t) = 1
\end{aligned}
$$

Definition 2.3. (see [10]) Self mappings A and S of a Fuzzy metric space (X, M, *) are said to be *compatible* if and only if $M(ASx_n, SAx_n, t) \to 1$ for all t > 0, whenever x_n is a sequence in X such that $Sx_n, Ax_n \to p$ for some p in X as $n \to \infty$.

Definition 2.4. Mappings f and g are non compatible maps, if there exists a sequence $\{x_n\}$ in X such that $\lim_{n \to \infty} fx_n = p = \lim_{n \to \infty} gx_n$ but either

$$\lim_{n \to \infty} M(fgx_n, gfx_n, t) \neq 1,$$

or the limit does not exists for all p ∈ X.

Definition 2.5. (see [7]) Two maps A and B from a Fuzzy metric space (X, M, *) into itself are said to be weakly compatible if they commute at their coincidence points, i.e. $Ax = Bx$ implies $ABx = BAx$ for some x ∈ X.

Lemma 2.1. (see [1]) Let $(X, M, *)$ be a fuzzy metric space. If there exists $k \in (0, 1)$ such that for all $x, y \in X, M(x, y, z, kt) \geq M(x, y, z, t) \forall t > 0$, then $x = y = z$.

Definition 2.6. Let f and g be self maps on a fuzzy metric space $(X, M, *)$. They are said to satisfy (EA) property if there exists a sequence xn in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n$$

for some $x \in X$.

Definition 2.7. Mappings A, B, C, S, T and U on a fuzzy metric space $(X, M, *)$ are said to satisfy common (EA) property if there exists sequences $\{x_n\}$ and $\{y_n\}$ in X such that:

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Ux_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} Cz_n = \lim_{n\to\infty} = Sz_n \ldots$$

for some $x \in X$.

For more on (EA) and common (EA) properties, we refer to [1] and [9].

Note that compatible, non-compatible, compatible of type (I) and compatible of type (II) satisfy (EA) property but converse is not true in general.

3. Main Result

3.1. Common Fixed Point Theorems

The following result provides necessary conditions for the existence of common fixed point of six non-compatible maps in a Fuzzy metric space.

Theorem 3.1. *Let (X,M, *) be a fuzzy metric space. Let A, B, C, S, T and U be maps from X into itself with $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$ and $C(X) \subseteq U(X)$ and there exists a constant $k \in (0, 1)$ such that*

$$\begin{aligned} M(Ax, By, Cz, kt) \geq &\phi(M(Ux, Ty, Szt), M(Ax, Ux, Czt), M(By, Ty, Axt), \\ &M(Cz, Sz, Byt), M(Ay, Ty, Cz, \alpha, t), \\ &M(By, Sz, Ax, (2-\alpha)t), M(Cz, Ux, By(3-\alpha)t/2)), \quad (1) \end{aligned}$$

for all x, y, $z \in X$, $\alpha \in (0,3)$, $t > 0$ and $\phi \in \psi$. Then A, B, C, S, T and U have a unique common fixed point in X provided the pair $\{A,U\}$, $\{B,T\}$ or $\{C,S\}$ satisfies (EA) property, one of $A(X)$, $T(X)$, $B(X)$, $S(X)$, $U(X)$ and $C(X)$ is a closed subset of X and the pairs $\{C,S\}$, $\{B,T\}$ and $\{A,U\}$ are weakly compatible.

Proof. Suppose that a pair $\{B,T\}$ satisfies property (EA), therefore there exits a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Bx_n = \beta = \lim_{n\to\infty} Tx_n$.

Now $B(X) \subseteq S(X)$ implies that there exists a sequence $\{y_n\}$ in X such that $Bx_n = Sy_n$.

Again pair (C,S) satisfies EA property, then

$$\lim_{n\to\infty} Cy_n = \beta = \lim_{n\to\infty} Sy_n.$$

And we have $C(X) \subseteq U(X)$,then there exist a sequence $\{z_n\}$ in X such that $Cy_n = Uz_n$

For $\alpha = 1$, $x = z_n$, $y = x_n$, and $z = y_n$, the inequality (1) takes the form

$$\begin{aligned} M(Az_n, Bx_n, Cy_n, kt) \geq \phi(&M(Uz_n, Tx_n, Sy_n t), M(Az_n, Uz_n, Cy_n t),\\ &M(Bx_n, Tx_n, Az_n, t), M(Cy_n, Sy_n, Bx_n, t),\\ &M(Az_n, Tx_n, Cy_n, t), M(Bx_n, Sy_n, Az_n, t),\\ &M(Cy_n, Uz_n, Bx_n, t)\end{aligned}$$

Letting $n \to \infty$, we obtain

$$\begin{aligned} M(\lim_{n\to\infty} Az_n, \beta, \beta, kt) \geq \phi(&M(\beta,\beta,\beta t), M(\lim_{n\to\infty} Az_n, \beta, \beta t), M(\beta, \beta, \lim_{n\to\infty} Az_n, t),\\ &M(\beta,\beta,\beta,t), M(\lim_{n\to\infty} Az_n, \beta, \beta, t), M(\beta, \beta, \lim_{n\to\infty} Az_n, t),\\ &M(\beta,\beta,\beta,t)\end{aligned}$$

since ϕ is increasing in each of its co ordinate and $\phi(t,t,t,t) > t$ for all $t \in [0,2)$,

$$M(\lim_{n\to\infty} Azn, \beta, \beta, kt) > M(\lim_{n\to\infty} Azn, \beta, \beta, t),$$

which by Lemma 2.1 implies that $\lim_{n\to\infty} Azn = \beta$.

Suppose that $U(X)$ is a closed subspace of X ,then $\beta = U\mu$ for some $\mu \in X$, now replacing x by μ, y by x_{2n+1} and z by y_{2n+1} and $\alpha = 1$ in (1), we have

$$\begin{aligned} M(A\mu, Bx_{2n+1}, Cy_{2n+1}, kt) \geq \phi(&M(U\mu, Tx_{2n+1}, Sy_{2n+1}t), M(A\mu, U\mu, Cy_{2n+1}, t),\\ &M(Bx_{2n+1}, Tx_{2n+1}, A\mu, t),\\ &M(Cy_{2n+1}, Sy_{2n+1}, Bx_{2n+1}, t),\\ &M(A\mu, Tx_{2n+1}, Cy_{2n+1}, t), M(Bx_{2n+1}, Sy_{2n+1}, A\mu, t),\\ &M(Cy_{2n+1}, U\mu, Bx_{2n+1}, t)\end{aligned}$$

Taking limit $n \to \infty$ we obtain

$$\begin{aligned} M(A\mu, \beta, \beta, kt) \geq \phi(&M(\beta, \beta, \beta, t), M(A\mu, \beta, \beta, t), M(\beta, \beta, A\mu, t),\\ &M(\beta, \beta, \beta, t), M(A\mu, \beta, \beta, t), M(\beta, \beta, A\mu, t),\\ &M(\beta, \beta, \beta, t)\\ > M&(A\mu, \beta, \beta t)\end{aligned}$$

which implies that $A\mu = \beta$, hence $A\mu = \beta = U\mu$.

Since $A(X) \subseteq T(X)$, there exist $\gamma \in X$ and $\delta \in X$ such that $\beta = T\gamma$ and $\beta = S\delta$. Following the argument similar to those given above we obtain $\beta = B\gamma = T\gamma$ and $\beta = C\delta = S\delta$, since μ is coincidence point of the pair $\{A, U\}$, therefore $UA\mu = AU\mu$ and $A\beta = U\beta$.

Now we claim that $A\beta = \beta$, if not, then using (1) with $\alpha = 1$, we arrive at

$$\begin{aligned} M(A\beta, \beta, \beta kt) &= M(A\beta, B\gamma, C\delta, kt)\\ &\geq \phi(M(U\beta, T\gamma, S\delta t), M(A\beta, U\beta, C\delta, t), (B\gamma, T\gamma, A\beta, t),\\ &\quad M(C\delta, S\delta, B\delta, t), M(A\beta, T\gamma, C\delta, t), M(B\gamma, S\delta, A\beta, t),\\ &\quad M(C\delta, U\beta, B\gamma, t)\\ &> M(A\beta, \beta, \beta, t)\end{aligned}$$

a contradiction. Hence $\beta = A\beta = U\beta$ similarly $\beta = B\beta = T\beta$ and $\beta = C\beta = S\beta$. Uniqueness of β follows from (1).

Theorem 3.2. *Let $(X, M, *_{\_})$ be a fuzzy metric space. Let A, B, C, S, T and U be maps from X into itself such that*

$$\begin{aligned} M(Ax, By, Cz, kt) \geq \phi(&M(Ux, Ty, Szt),\\ &M(Ax, Ux, Czt), M(By, Ty, Ax, t),\\ &M(Cz, Sz, By, t), M(Ax, Ty, Cz, \alpha t),\\ &M(By, Sz, Ax, (2-\alpha)t),\\ &M(Cz, Ux, By, (3-\alpha)t/2)\end{aligned} \tag{2}$$

for all x, y, $z \in X$, $\alpha \in (0, 3)$, $t > 0$ and $\Phi \in \Psi$. Then A, B, C, S, T and U have a unique common fixed point in X provided the pair $\{A, U\}$, $\{B, T\}$ or $\{C, S\}$ satisfies (EA) property, $U(X)$, $T(X)$ and $S(X)$ is a closed subset of X and the pairs $\{C, S\}$, $\{B, T\}$ and $\{A, U\}$ are weakly compatible.

Proof. Suppose that a pair $\{A,U\}$, $\{B,T\}$,$\{C,S\}$ satisfies property (EA), therefore there exits three sequence $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Ux_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} Czn = \lim_{n\to\infty} Szn = \beta,$$

for some $\beta \in X$.

Now we claim that $A\mu = \beta$.For this replace x by μ, y by y_n and z by z_n in (2) with $\alpha = 1$,

$$\begin{aligned} M(A\mu, By_n, Cz_n, kt) \geq & \phi(M(U\mu, Ty_n, Sz_n t), \\ & M(A\mu, U\mu, Cz_n t), M(Bx_n, Ty_n, A\mu, t), \\ & M(Cz_n, Sz_n, By_n, t), \\ & M(A\mu, Ty_n, Cz_n, t), M(By_n, Sz_n, A\mu, t), \\ & M(Cz_n, U\mu, By_n, t) \end{aligned}$$

Taking limit $n \to \infty$ we obtain

$$M(A\mu, \beta, \beta, kt) > M(A\mu, \beta, \beta, t)$$

Hence $A\mu = \beta = U\mu$. Again using (2) with $\alpha = 1$

$$\begin{aligned} M(T\gamma, B\gamma, C\delta, kt) &= M(A\mu, B\gamma, C\delta, kt) \\ &\geq \phi(M(U\mu, T\gamma, S\delta t), M(A\mu, U\mu, C\delta, t), \\ &\quad M(B\gamma, T\gamma, A\mu, t), M(C\delta, S\delta, B\gamma, t), \\ &\quad M(A\mu, T\gamma, C\delta, t), M(B\gamma, S\delta, A\mu, t), \\ &\quad M(C\delta, U\mu, B\gamma, t) \\ &> M(U\mu, B\gamma, C\delta, t) \\ & M(T\gamma, B\gamma, C\delta, t) \end{aligned}$$

which implies that $T\gamma = B\gamma = \beta$. Again using (2) with $\alpha = 1$

$$\begin{aligned} M(A\mu, S\delta, C\delta, kt) &= M(A\mu, B\gamma, C\delta, kt) \\ &\geq \phi(M(U\mu, T\gamma, S\delta t), M(A\mu, U\mu, C\delta, t), \\ &\quad M(B\gamma, T\gamma, A\mu, t), M(C\delta, S\delta, B\gamma, t), \\ &\quad M(A\mu, T\gamma, C\delta, t), M(B\gamma, S\delta, A\mu, t), \\ &\quad M(C\delta, U\mu, B\gamma, t) \\ &> M(C\delta, S\delta, B\gamma, t) \\ &> M(A\mu, S\delta, C\delta, t) \end{aligned}$$

which implies that $C\delta = S\delta = \beta$. Hence $A\mu = U\mu = B\gamma = T\gamma = C\delta = S\delta = \beta$.

The rest of proof follows as in the proof of Theorem 1.

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