International Journal of Pure and Applied Mathematics
Volume 83 No. 3 2013, 515-526
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v83i3.12
ijpam.eu

OSCILLATIONS OF OF FIRST ORDER IMPULSIVE DIFFERENTIAL NEUTRAL CONSTANT DELAY EQUATIONS WITH MIXED COEFFICIENTS

M.B. Dimitrova[1], V.I. Donev[2 §]
[1,2]Department of Mathematics
Technical University of Sliven
8800, Sliven, BULGARIA

Abstract: In this paper we consider first order impulsive neutral differential equation with constant delay, constant coefficient into the neutral term and variable coefficient into the rest. The asymptotic behavior of a non-oscillatory solution for such equations is investigated and sufficient conditions for oscillation of all the solutions are found.

AMS Subject Classification: 34A37, 34K11, 34K40, 34K45
Key Words: oscillation, impulsive effect, neutral differential equation

1. Introduction

The Neutral Impulsive Differential Equations (NIDE) are part of the Impulsive Differential Equations with Deviating Arguments (IDEDA). Generally speaking, IDEDA are very interesting mixture between the Impulsive Differential Equations (IDE) (see [1] and [10]) and the Differential Equations with Deviating Argument (DEDA) (see [4] and [9]). We note here that [5] is the first work where IDEDA were considered and where an oscillation theory of such equation was studied. Among the numerous publications concerning the oscillation

Received: December 29, 2012

§Correspondence author

properties of IDEDA – with delayed or advanced arguments, we choose to refer to [2], [6] and [8].

NIDE are characterized with neutral argument in which the highest-order derivative of the unknown function appears in the equation both with and without delay. Moreover, the unknown function in them, may have discontinuities of first kind at points, which we call jump points. Such equations can be used to model processes, that occur in the theory of optimal control, industrial robotics, biotechnologies, etc.

The authors investigated impulsive neutral constant delay differential equations with constant coefficients and found there necessary and sufficient conditions for existence of eventually positive solutions in [3] and established oscillation criteria in [7], as well. In the present paper we study the asymptotic behavior of the eventually non-oscillatory solutions of similar equations and obtain oscillation criteria when the delays are constant, the coefficient in the neutral term is constant, while the coefficient into the rest is h-periodic function.

2. Preliminary Notes

The object of investigation in the present work is the first order impulsive neutral differential equation with constant delays, constant coefficient into the neutral term and variable coefficient into the rest of the form

$$\frac{d}{dt}[y(t)-cy(t-h)]+p(t)y(t-\sigma)=0,\ t\neq\tau_k,\ k\in N \qquad (E_1)$$

$$\Delta[y(\tau_k)-cy(\tau_k-h)]+p_{\tau_k}y(\tau_k-\sigma)=0,\ t=\tau_k,\ k\in N,$$

as well as the corresponding to it inequalities

$$\frac{d}{dt}[y(t)-cy(t-h)]+p(t)y(t-\sigma)\leq 0,\ t\neq\tau_k,\ k\in N \qquad (N_{1,\leq})$$

$$\Delta[y(\tau_k)-cy(\tau_k-h)]+p_{\tau_k}y(\tau_k-\sigma)\leq 0,\ t=\tau_k,\ k\in N$$

and

$$\frac{d}{dt}[y(t)-cy(t-h)]+p(t)y(t-\sigma)\geq 0,\ t\neq\tau_k,\ k\in N \qquad (N_{1,\geq})$$

$$\Delta[y(\tau_k)-cy(\tau_k-h)]+p_{\tau_k}y(\tau_k-\sigma)\geq 0,\ t=\tau_k,\ k\in N.$$

The points $\tau_k\in(0,+\infty),\ k\in N$ are the moments of impulsive effect (let us call them jump points), where the unknown function reveals its discontinuities of first kind as jumps. In order to manifest these jumps of the unknown function $y(t)$, we use the notation

$$\Delta\{y(\tau_k) - cy(\tau_k - h)\} = \Delta y(\tau_k) - c\Delta y(\tau_k - h), \quad \Delta y(\tau_k) = y(\tau_k + 0) - y(\tau_k - 0).$$

Denote by $P_\tau C(R, R)$ the set of all functions $u\colon R \to R$, which satisfy the following conditions:

(i) u is piecewise continuous on $(\tau_k, \tau_{k+1}]$, $k \in N$,

(ii) u is continuous from the left at the points τ_k, i.e.

$$u(\tau_k - 0) = \lim_{t\to\tau_k-0} u(t) = u(\tau_k),$$

(iii) there exists a sequence of reals $\{u(\tau_k + 0)\}_{k=1}^{\infty}$, such that

$$u(\tau_k + 0) = \lim_{t\to\tau_k+0} u(t),$$

(iv) u may have discontinuities of first kind at the jump points τ_k, $k \in N$, that we qualify as down-jumps when $\Delta u(\tau_k) < 0$, or as up-jumps when $\Delta u(\tau_k) > 0$, $k \in N$.

Introduce the following hypotheses, where $R^+ = (0, +\infty)$ and $R_0^+ = [0, +\infty)$:

(H₁) $0 < \tau_1 < \tau_2 < \ldots < \tau_k < \ldots$, $\lim_{k\to+\infty} \tau_k = +\infty$, $\max\left\{\tau_{k+1} - \tau_k\right\} < +\infty$, $k \in N$;

(H₂) $h > 0$, $\sigma > 0$, $c \geq 1$;

(H₃) $p \in P_\tau C(R, R^+)$, $p(t)$ is not identically zero on any positive half-line;

(H₄) $p_{\tau_k} = p(\tau_k)$, $k \in N$, $\sum\limits_{k=1}^{+\infty} p_{\tau_k}^2 \neq 0$, $\int\limits_0^{+\infty} p(s)ds + \sum\limits_{k=1}^{+\infty} p_{\tau_k} = +\infty$;

(H₅) $p(t)$ is a h-periodic, $\tau_{k+1} - \tau_k = h > \sigma$, $k \in N$.

Let $\rho = \max\{h, \sigma\}$. We say that a real valued function $y(t)$ is a *solution* of the equation (E_1), if there exists a number $T_0 \in R$ such that $y \in P_\tau C([T_0 - \rho, +\infty), R)$, the function $z(t) = y(t) - cy(h(t))$ is continuously differentiable for $t \geq T_0 - \rho$, $t \neq \tau_k$, $k \in N$ and $y(t)$ satisfies (E_1) for all $t \geq T_0 - \rho$.

Without further mentioning we will assume throughout this paper, that every solution $y(t)$ of equation (E_1) that is under consideration here, is continuable to the right and is nontrivial. That is, $y(t)$ is defined on some ray of the form $[T_y, +\infty)$ and for each $T \geq T_y$ it is fulfilled $\sup\{|y(t)|: \ t \geq T\} > 0$. Such a solution is called a *regular solution* of (E_1).

We say that a real valued function u defined on an interval $[a, +\infty)$ has some property *eventually,* if there is a number $b \geq a$ such that u has this property on the interval $[b, +\infty)$.

A regular solution $y(t)$ of equation (E_1) is said to be *nonoscillatory,* if there exists a number $t_0 \geq 0$ such that $y(t)$ is of constant sign for every $t \geq t_0$. Otherwise, it is called *oscillatory*. Also, note that a *nonoscillatory* solution is called *eventually positive* (*eventually negative*), if the constant sign that determines its *nonoscillation* is positive (negative). Equation (E_1) is called *oscillatory* , if all its solutions are oscillatory.

Moreover, in this article, when we write a functional relation (or inequality), we will mean that it holds for all sufficiently large values of the argument.

However, let consider $y(t)$ as a solution of equation (E_1) and set

$$z(t) = y(t) - cy(t-h), \qquad \Delta z(\tau_k) = \Delta y(\tau_k) - c\Delta y(\tau_k - h), \quad k \in N, \quad (*)$$

$$w(t) = z(t) - cz(t-h), \qquad \Delta w(\tau_k) = \Delta z(\tau_k) - c\Delta z(\tau_k - h), \quad k \in N. \quad (**)$$

In order to assists our efforts on investigation of the oscillation of the solutions of equation (E_1), let introduce at the beginning two lemmas, which investigate the asymptotic behavior of the functions $z(t)$ and $w(t)$, when $y(t)$ is a nonoscillatory solution of (E_1). First of them is formulated and proved for eventually positive solution $y(t)$ of the equation (E_1).

Lemma 1. *Let $y(t)$ be an eventually positive solution of (E_1) and the hypotheses $(H1) - (H4)$ are satisfied. Then:*

(a) $z(t)$, defined by $(*)$, is a decreasing function of t with down-jumps;

(b) $z(t)$, defined by $(*)$, is an eventually negative function, i.e. $z(t) < 0$ for enough large t

$$\text{and} \quad \lim_{t \to +\infty} z(t) = -\infty.$$

Proof. (a) Let $y(t)$ be an eventually positive solution of the equation (E_1), i.e. there exists a number $\tilde{t} > 0$ such that $y(t)$ is defined for $t \geq \tilde{t}$ and $y(t) > 0, y(t-\sigma) > 0, y(t-h) > 0$ for $t \geq \tilde{t} + \rho = t_0$. From (E_1) and $(*)$, it follows that $z'(t) = -p(t)y(t-\sigma) < 0, \ t \neq \tau_k, k \in N, t \geq t_0$ and $\Delta z(\tau_k) = -p_k y(\tau_k - \sigma) < 0, \ k \in N, \ \tau_k \geq t_0$. Therefore, $z(t)$ is a decreasing function for $t \geq t_0$ and $\Delta z(\tau_k) < 0$, i.e. $z(t)$ has "down-jumps" at the points of impulsive effect τ_k. The proof of (a) is complete.

(b) It follows from (a) that $\lim\limits_{t\to+\infty} z(t)$ does exist, with $z(t)$ as a strictly decreasing function with down-jumps. So, $\lim\limits_{t\to+\infty} z(t) = L$, where L could be positive number, negative number, zero, or $-\infty$.

Assume $\lim\limits_{t\to+\infty} z(t) = L \geq 0$, i.e. z(t) never becomes negative in the intervals $(\tau_k, \tau_{k+1}],\ k \in N$. That is, $z(t) = y(t) - cy(t-h) \geq 0, t \in (\tau_k, \tau_{k+1}],\ k \in N$, or more precisely $y(t+nh) \geq c^{n+1}y(t-h)$. Here, we can conclude $\lim\limits_{t\to+\infty} y(t) = +\infty$. Moreover, there exist a number $\hat{t} \geq t_0$ and a constant $M_{\hat{t}} > 0$ such that $\lim\limits_{t\to+\infty} y(t) \geq M_{\hat{t}}$. Let now integrate (E_1) from $\hat{t}$ to t. Then, we have

$$\int_{\hat{t}}^{t} z'(s)ds + \int_{\hat{t}}^{t} p(s)y(s-\sigma)ds = 0,$$

or

$$z(t) - z(\hat{t}) - \sum_{\hat{t}\leq\tau_k<t} \Delta z(\tau_k) + \int_{\hat{t}}^{t} p(s)y(s-\sigma)ds = 0. \tag{1}$$

But $\Delta z(\tau_k) = -p_k y(\tau_k - \sigma)$, hence

$$z(t) = z(\hat{t}) - \sum_{\hat{t}\leq\tau_k<t} p_{\tau_k} y(\tau_k - \sigma) - \int_{\hat{t}}^{t} p(s)y(s-\sigma)ds \tag{2}$$

Because $y(t)$ is a bounded function from below, then (2) reduces to

$$z(t) \leq z(\hat{t}) - M_{\hat{t}}[\sum_{\hat{t}\leq\tau_k<t} p_{\tau_k} + \int_{\hat{t}}^{t} p(s)ds], \tag{3}$$

which will imply $\lim\limits_{t\to+\infty} z(t) = -\infty$ and it will contradict our assumption.

Assume $\lim\limits_{t\to+\infty} z(t) = L < 0,\ L = const$. Then, because $z(t)$ is a decreasing function with down-jumps, for some $t_1 \geq t_0$ there will exist $\delta_\nu > 0$ such that $z(t) < -\delta_\nu$, for every $t \geq t_1$, $t \neq \tau_k,\ k \in N$, i.e.

$$y(t) - cy(t-h) < -\delta_\nu,\ t \neq \tau_k,\ t \geq t_1.$$

Except that, because the sequence of eventually negative numbers $\{z(\tau_k)\}_{k=1}^{+\infty}$ is decreasing, for our $\delta_\nu > 0$, there will be such a term τ_ν in the sequence of the impulsive moments $\{\tau_k\}$, whereafter $z(\tau_k) < -\delta_\nu$, for every $\tau_k \geq \tau_\nu$, when $k \geq \nu$, $k \in N,\ \nu \in N$. Hence,

$$y(\tau_k) - cy(\tau_k - h) < -\delta_\nu, \qquad \tau_k \geq \tau_\nu,\ k \geq \nu,\ k \in N,\ \nu \in N.$$

Denote $t_\nu = \max\{t_1, \tau_\nu\}$ and combine the last two inequalities as

$$y(t) < -\delta_\nu + cy(t-h), \qquad t \geq t_\nu. \tag{4}$$

Now, it is obvious that the right side of (4) has to be positive, because of the positivity of $y(t)$. So, we obtain the inequality $0 < -\delta_\nu + cy(t-h)$, which shows clearly, that $y(t)$ is a bounded function from below. Hence, if integrate (E_1) from t_0 to t we can easily get (3), which will imply $\lim_{t\to+\infty} z(t) = -\infty$ and it will contradict our assumption again.

Finally, let we assume

$$\lim_{t\to+\infty} z(t) = \lim_{t\to+\infty} [y(t) - cy(t-h)] = L = 0. \tag{5}$$

It does mean that $z(t) > 0$ eventually, i.e. there exists a number $t_1 \geq t_0$, such that we have $y(t) > cy(t-h)$ for every $t \geq t_1$. Observe that the last inequality holds as well as for those moments of impulsive effect τ_k , for which $\tau_k > t_1,\ k \in N$. However, our assumption implies that there will exist a strictly increasing sequence $\{y(t_n)\}_{n=1}^{\infty}$ (where $t_n = t_{n-1} + h$), which is bounded by a positive number, i.e. $\lim_{n\to+\infty} y(t_n) = K,\ K > 0$ or which is unbounded, i.e. $\lim_{n\to+\infty} y(t_n) = +\infty$ and for which (5) has to be fulfilled. But, for this sequence we have

$$\lim_{t_n\to+\infty} z(t_n) = \lim_{t_n\to+\infty} y(t_n) - c \lim_{t_n\to+\infty} y(t_n - h) = (1-c) \lim_{t_n\to+\infty} y(t_n) < 0$$

and the contradiction with (5) is evident, because $c > 1$.

So, the above considerations approve $\lim_{t\to+\infty} z(t) = -\infty$. The proof of (b) and of the Lemma are completed.

The second lemma is only formulated for eventually negative solution $y(t)$ of the equation (E_1), but the prove is carried out analogously to the prove of Lemma 1.

Lemma 2. *Let $y(t)$ be an eventually negative solution of (E_1) and the hypotheses $(H1)$ – $(H4)$ are satisfied. Then:*

(a) $z(t)$, defined by $(*)$, is an increasing function of t with up-jumps;

(b) $z(t)$, defined by $(*)$, is an eventually positive function, i.e. $z(t) > 0$ for enough large t

$$\text{and} \qquad \lim_{t\to+\infty} z(t) = +\infty.$$

Lemma 1 and Lemma 2, applied to the functions $z(t)$ and $w(t)$, where $y(t)$ is a nonoscillatory solution of equation (E_1), lead to the following proposition which is useful for our purposes.

Lemma 3. *Let $y(t)$ be a solution of the equation (E_1) and the hypotheses $(H1)-(H5)$ are satisfied. Then:*

(a) the function $z(t)$, defined by $(*)$, and the function $w(t)$, defined by $(**)$, are also solutions of the equation (E_1);

(b) if $y(t)$ is an eventually positive function and $p(t)$ is a nondecreasing function, then $w''(t)$ is an eventually nonnegative function;

(c) if $y(t)$ is an eventually negative function and $p(t)$ is a nondecreasing function, then $w''(t)$ is an eventually nonpositive function.

Proof. (a) Direct substitution in the equation (E_1) of $z(t)$, defined by $(*)$, shows that $z(t)$ is a solution of (E_1), as well as the same holds for $w(t)$, defined by $(**)$. So, (a) is proved.

(b) From (a), it follows that $z(t)$, defined by $(*)$, is a solution of (E_1), i.e.

$$[z(t) - cz(t-h)]' = -[p(t)z(t-\sigma)], \ t \neq \tau_k.$$

From here, it is easy to see that

$$w(t)'' = [z(t) - cz(t-h)]'' = -p'(t)z(t-\sigma) - p(t)z'(t-\sigma), \ t \neq \tau_k. \tag{6}$$

From (6) and Lemma 1 one can easily derive (b).

(c) From (6) and Lemma 2 one can easily derive (c). The proof of the lemma is complete.

3. Main Results

Utilizing the conclusions of the previous section, we establish some results, obtaining sufficient conditions under which the equation (E_1) is oscillatory.

Theorem 1. *Assume that the hypotheses $(H1)$ - $(H5)$ are satisfied. Let also $p'(t) > 0$ and:*

$$(i)\ \liminf_{t\to\infty} \Big[\prod_{t\le\tau_k<t+h-\sigma} \Big(1 + \frac{p_{\tau_k}}{c-1}\Big)\Big] \int_t^{t+h-\sigma} p(s)ds \ge \frac{c-1}{e}.$$

Then the equation (E_1) is oscillatory.

Proof. Let assume, for the sake of contradiction, that equation (E_1) has an eventually positive solution $y(t)$. Then, there exists a $\tilde{t} > 0$, such that $y(t)$ is defined for $t \geq \tilde{t}$, $y(t) > 0$ for $t \geq \tilde{t}$ and $y(t-h) > 0$, $y(t-\sigma) > 0$ for $t \geq \tilde{t} + \rho = t_0$.

Recalling $(*)$ and $(**)$, we conclude by Lemma 3(a) and Lemma 1, that for every $t \geq t_0$ the function $z(t)$ will be eventually negative decreasing solution to the equation (E_1), whereas by Lemma 3(a) and Lemma 2, $w(t)$ will be eventually positive increasing solution to the equation (E_1). That is, $w(t)$ satisfies as an increasing positive solution the equation

$$\frac{d}{dt}[w(t) - cw(t-h)] + p(t)w(t-\sigma) = 0, \ t \neq \tau_k, \tag{7}$$

$$\Delta[w(\tau_k) - cw(\tau_k - h)] + p_{\tau_k} w(\tau_k - \sigma) = 0, \ k \in N$$

Note that, by Lemma 3(b), $w'(t)$ is an increasing function. Therefore, from (7) it is easy to see that

$$\begin{aligned} w'(t-h) - cw'(t-h) + p(t)w(t-\sigma) \leq \\ \leq w'(t) - cw'(t-h) + p(t)w(t-\sigma) = \\ = \frac{d}{dt}[w(t) - cw(t-h)] + p(t)w(t-\sigma) = 0 \end{aligned} \tag{8}$$

Moreover, since $z(t)$ is a decreasing function, we see that $z(\tau_k - \sigma) < z(\tau_k - \sigma - h)$ and using the definitions of the functions $z(t)$ and $w(t)$, it is easy to conclude that

$$\Delta w(\tau_k) = -p_{\tau_k} z(\tau_k - \sigma) > -p_{\tau_k} z(\tau_k - \sigma - h) = \Delta w(\tau_k - h), \ k \in N$$

So, in view of the above observation, again from (7) it follows that for each $k \in N$

$$\begin{aligned} \Delta w(\tau_k - h) - c\Delta w(\tau_k - h) + p_{\tau_k} w(\tau_k - \sigma) \leq \\ \leq \Delta w(\tau_k) - c\Delta w(\tau_k - h) + p_{\tau_k} w(\tau_k - \sigma) = \\ = \Delta[w(\tau_k) - cw(\tau_k - h)] + p_{\tau_k} w(\tau_k - \sigma) = 0 \end{aligned} \tag{9}$$

Now, from (8) and (9), it follows that $w(t)$ is an eventually positive function for which

$$(1-c)w'(t-h) + p(t)w(t-\sigma) \leq 0, \ t \neq \tau_k$$

$$(1-c)\Delta w(\tau_k - h) + p_{\tau_k} w(\tau_k - \sigma) \leq 0, \ k \in N$$

Divide the last two inequalities by $1-c<0$ and denote

$$s=h-\sigma,\quad Q(t)=\frac{p(t)}{c-1},\quad q_{\tau_k}=\frac{p_{\tau_k}}{c-1}.$$

Then, one can realize that our assumption in the beginning leads us to the conclusion that $w(t)$ is an eventually positive solution to the inequality

$$w'(t)-Q(t)w(t+s)\geq 0,\ t\neq\tau_k, \tag{10}$$

$$\Delta w(\tau_k)-q_{\tau_k}w(\tau_k+s)\geq 0,\ k\in N,$$

for every $t\geq t_0$. From (10) we can see that

$$w'(t)=Q(t)w(t+s)>0,\ t\neq\tau_k,\ k\in N,$$

$$\Delta w(\tau_k)=q_{\tau_k}w(\tau_k+s)>0,\ k\in N$$

and confirm that $w(t)$ is a strictly increasing function for $t\geq t_0$ with "up-jumps" at the points of impulsive effect $(\Delta w(\tau_k)>0)$. Note that, dividing by $w(t)$ and using its increasing nature, we can rearrange (10) in order to obtain

$$\frac{w'(t)}{w(t)}>Q(t)\frac{w(t+s)}{w(t)},\qquad t\neq\tau_k,\ k\in N, \tag{11}$$

$$\Delta w(\tau_k)>q_{\tau_k}w(\tau_k+s)>q_{\tau_k}w(\tau_k),\qquad k\in N$$

Now, we integrate (11) from t to $t+s$, i.e.

$$\int_t^{t+s}\frac{w(r)'}{w(r)}dr>\int_t^{t+s}Q(r)\frac{w(r+s)}{w(r)}dr$$

and obtain

$$\ln\frac{w(t+s)}{w(t)}+\sum_{t\leq\tau_k<t+s}\ln\frac{w(\tau_k)}{w(\tau_k+0)}>\int_t^{t+s}Q(r)\frac{w(r+s)}{w(r)}dr. \tag{12}$$

Except that, $w(\tau_k+0)-w(\tau_k)=q_{\tau_k}w(\tau_k+s)>q_{\tau_k}w(\tau_k)$. Hence, $w(\tau_k+0)>(1+q_{\tau_k})w(\tau_k)$, i.e.

$$\frac{1}{1+q_k}>\frac{w(\tau_k)}{w(\tau_k+0)}.$$

So, we have

$$\ln\frac{1}{1+q_{\tau_k}}>\ln\frac{w(\tau_k)}{w(\tau_k+0)}$$

and from (11) and (12) we get

$$\ln\left[\frac{w(t+s)}{w(t)}\prod_{t\le\tau_k<t+s}\frac{1}{1+q_{\tau_k}}\right]>\int\limits_t^{t+s}Q(r)\frac{w(r+s)}{w(r)}dr, \tag{13}$$

Bring to the mind, that $w(t)$ is a strictly increasing function. Therefore, the function $\frac{w(t+s)}{w(t)}$ is bounded from below. That is why, we may denote $\liminf\limits_{t\to+\infty}\frac{w(t+s)}{w(t)}=w_0$, where $1<w_0<+\infty$.

Then, (13) implies

$$\ln\left[w_0\prod_{t\le\tau_k<t+s}\frac{1}{1+q_{\tau_k}}\right]>w_0\int\limits_t^{t+s}Q(r)dr,$$

from where, using the inequality $e^x>ex$, we obtain

$$\prod_{t\le\tau_k<t+s}(1+q_{\tau_k})\int\limits_t^{t+s}Q(r)dr<\frac{1}{e}.$$

Now, resuming $s=h-\sigma$, $Q(t)=\dfrac{p(t)}{c-1}$, $q_{\tau_k}=\dfrac{p_{\tau_k}}{c-1}$, from the last inequality we conclude

$$\prod_{t\le\tau_k<t+h-\sigma}\left(1+\frac{p_{\tau_k}}{c-1}\right)\int\limits_t^{t+h-\sigma}p(r)dr<\frac{c-1}{e}. \tag{14}$$

But the last conclusion contradicts the condition (i) of the theorem. The proof is complete.

For the needs of the next theorem, which can be considered as a consequence of the Theorem 1, we denote by $i[\tau_0,t)$ the number of fixed jump points τ_k that are situated in the interval $[\tau_0,t)$, $k\in N$, $t\in(\tau_0,+\infty)$. We clarify that

$$i[\tau_0,t)=\begin{cases}0, & \text{for } t\in(\tau_0,\tau_1],\\ 1, & \text{for } t\in(\tau_1,\tau_2],\\ \vdots & \\ k, & \text{for } t\in(\tau_k,\tau_{k+1}],\ k\in N.\\ \vdots & \end{cases}$$

Theorem 2. *Assume that the hypotheses* $(H1)$ - $(H5)$ *are satisfied. Suppose also that there exists* $p_0 = \min_{k\in N} p_{\tau_k}$, $m = \liminf_{t\in[t_0,+\infty)} i[t, t+h-\sigma)$ *and*

$$\liminf_{t\to\infty} \int_t^{t+h-\sigma} p(s)ds \geq \frac{(c-1)^{m+1}}{e(p_0+c-1)^m}.$$

Then the equation (E_1) *is oscillatory.*

Proof. Proceeding as in the proof of Theorem 1, we can get (14). But, $p_0 = \min_{k\in N} p_k$ and $m = \liminf_{t\in[t_0,+\infty)} i[t, t+h-\sigma)$. Therefore,

$$(1+\frac{p_0}{c-1})^m \leq \prod_{t\leq\tau_k<t+h-\sigma} (1+\frac{p_0}{c-1}) \leq \prod_{t\leq\tau_k<t+h-\sigma} (1+\frac{p_k}{c-1}).$$

It follows from (14) and from the last inequality, that

$$(1+\frac{p_0}{c-1})^m \int_t^{t+h-\sigma} p(s)ds < \frac{c-1}{e},$$

which contradicts the conditions of the theorem. The proof is complete.

As a natural consequence of the previous two theorems, we have the following result:

Corollary 1. *Let the conditions of Theorem 1, or Theorem 2 are satisfied. Then:*

(a) the inequality $(N_{1,\leq})$ *has no eventually positive solutions*

(b) the inequality $(N_{1,\geq})$ *has no eventually negative solutions*

The proof of the corollary is similar to that of Theorem 1 and that is why, it is omitted.

References

[1] D.D. Bainov, V. Lakshmikantham, P.S. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific, Singapore (1989).

[2] D.D. Bainov, M.B. Dimitrova, A.B. Dishliev, Oscillations of the solutions of nonlinear impulsive differential equations of first order with advanced argument, *Journ. of Appl. Anal.*, **5**, No. 2 (1999), 261-275

[3] M.B. Dimitrova, V.I. Donev, Existence and asymptotic behavior of the positive solutions of neutral impulsive differential equations, *Nonlinear Oscillations*, **8**, No. 3 (2005), 304-318.

[4] L.H. Erbe, Q. Kong, B.G. Zhang, *Oscillation Theory for Functional Differential Equations*, Marcel Decker Inc., New York (1995).

[5] K. Gopalsamy, B.G. Zhang, On delay differential equations with impulses, *J. Math. Anal. Appl.*, **139**, No. 1 (1989), 110-122

[6] M.K. Grammatikopoulos, M.B. Dimitrova, V.I. Donev, Oscillations of first order delay impulsive differential equations, *Technical Report*, University of Ioannina, Greece, **16** (2007), 171-182.

[7] M.K. Grammatikopoulos, M.B. Dimitrova, V.I. Donev, Oscillations of first order neutral impulsive delay differential equations with constant coefficients, *Technical Report*, University of Ioannina, Greece, **16** (2007), 183-193.

[8] M.K. Grammatikopoulos, M.B. Dimitrova, V.I. Donev, Oscillations of first order impulsive differential equations with variable coefficients and advanced argument, In: *Proceedings of 33-th International Conference AMEE'33* (Ed. M.D. Todorov), American Institute of Physics CP946 (2007), 206-214.

[9] I. Györi, G. Ladas *Oscillation Theory of Delay Differential Equations with Applications*, Clarendon Press, Oxford (1991).

[10] A.M. Samoilenko, N.A. Perestyuk, *Impulsive Differential Equations*, World Sci. Ser. A: Monograph and Treatises, **14** (1995).