CONTINUOUS BLOCK METHOD FOR THE SOLUTION OF
SECOND ORDER INITIAL VALUE PROBLEMS OF
ORDINARY DIFFERENTIAL EQUATION

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Abstract: We proposed a continuous blocks method for the solution of second order initial value problems with constant step size in this paper. The method was developed by interpolation and collocation of power series approximate solution to generate a continuous linear multistep method; this is evaluated for the independent solution to give a continuous block method which is evaluated at selected grid point to give discrete block method. The basic properties of the method were investigated and was found to be zero stable, consistent and convergent. The efficiency of the method was tested on some numerical examples and found to give better approximation than the existing methods.

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1. Introduction

This paper considers a numerical solution to the second order Initial value problems of the form

$$y'' = f(x, y, y'), \quad y(x_0) = y_0, \quad y'(x_0) = y'_0, \quad (1)$$

where x_0 is the initial point and f is continuous within the interval of integration.

Scholars have established that direct methods of solving higher order ordinary differential equations are better than the method of reduction to systems of first order ordinary differential equation (Adesanya et al [2], Awoyemi [6]), hence the study of direct method for the solution of higher order ordinary differential equation is important.

Adoption of method of collocation and interpolation of power series approximate solution to developed an implicit continuous linear multistep method have been studied by many scholars, among them are Fatunla [8], Awoyemi and Kayode [7], Olabode [14], Adesanya et al. [4], Kayode and Awoyemi [13]. These authors independently implemented their methods in predictor corrector mode, where the predictor are reducing order predictor hence this affect the accuracy of the method. Other setbacks of these method as extensively discussed by Adesanya et al. [3] and Awoyemi [6].

In order to cater for the shortcoming of the predictor-corrector method, scholars adopted block method which is self-starting and does not need starting value to be implemented. Among these authors are: Jator [12], Jator and Li [11], Omar and Suleiman ([15], Abass [1], Fudziah, Yab and Mohammed [9], Awoyemi et al. [5].

In this paper, we propose continuous block method which has the same properties as the continuous linear multistep method. This continuity properties enable us to evaluate at all points within the interval of integration, hence enable the study of the dynamical system at all the grid points. This paper considers four off step points which makes it an improvement on the work of Jator [12] which considered two off step points.

2. Methodology

We first state the theorem that establishes the uniqueness of solutions of higher order ordinary differential equations.

Theorem 1. *Let*

$$U^{(n)} = f\left(x, u, u', \dots, u^{(n-1)}\right), U^k(x_0) = C_k$$

$k = 0, 1, \dots, (n-1)$, u and f are scalars. Let R be the region defined by the inequalities $x_0 \leq x \leq x_0 + a$, $|s_j - c_j| \leq b$, $j = 0, 1, \dots, n-1$ ($a > 0, b > 0$). Suppose the function $f(x, s_0, s_1, \dots, s_{n-1})$ is defined in R and in addition:

- (a) f is non-negative and non-decreasing in each $x, s_0, s_1, \dots, s_{n-1}$ in R ;
- (b) $f(x, c_0, c_1, \dots, c_{n-1}) > 0$ for $x_0 \leq x \leq x_0 + a$; and
- (c) $C_k \geq 0$, $k = 0, 1, 2, \dots, n-1$.

Then value problem (2) has a unique solution in R (See Wend (1967) for details).

2.1. Specification of the Method

We considered a power series approximate solution of the form

$$y(x) = \sum_{j=0}^{r+s-1} a_j x^j, \quad (2)$$

where r and s are the numbers of interpolation and collocation points respectively. The second derivatives of (2) gives

$$y''(x) = \sum_{j=2}^{r+s-1} j(j-1) a_j x^{j-2}. \quad (3)$$

Substituting (3) into (1) gives

$$f\left(x, y, y'\right) = \sum_{j=2}^{r+s-1} j(j-1) a_j x^{j-2}. \quad (4)$$

Let the solution to (1) be sought on the partition N : $a = x_0 < x_1 < x_2 < \dots < x_N = b$ with a constant step size (h) given as $h = x_{n+1} - x_n$, $n = 0, 1, \dots, N$

Collocating (4) at x_{n+r} , $r = 0\left(\frac{1}{3}\right)2$ and interpolating (2) at x_{n+s} , $s = 0, 1$ gives a system of non linear equation of the form

$$AX = U, \quad (5)$$

where

$$A = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 \end{bmatrix}^T,$$

$$U = \begin{bmatrix} y_n & y_{n+1} & f_n & f_{n+\frac{1}{3}} & f_{n+\frac{2}{3}} & f_{n+1} & f_{n+\frac{4}{3}} & f_{n+\frac{5}{3}} & f_{n+2} \end{bmatrix}^T,$$

and

$$X = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 & x_n^8 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & x_{n+1}^7 & x_{n+1}^8 \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 & 42x_n^5 & 56x_n^6 \\ 0 & 0 & 2 & 6x_{n+\frac{1}{3}} & 12x_{n+\frac{1}{3}}^2 & 20x_{n+\frac{1}{3}}^3 & 30x_{n+\frac{1}{3}}^4 & 42x_{n+\frac{1}{3}}^5 & 56x_{n+\frac{1}{3}}^6 \\ 0 & 0 & 2 & 6x_{n+\frac{2}{3}} & 12x_{n+\frac{2}{3}}^2 & 20x_{n+\frac{2}{3}}^3 & 30x_{n+\frac{2}{3}}^4 & 42x_{n+\frac{2}{3}}^5 & 56x_{n+\frac{2}{3}}^6 \\ 0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 & 20x_{n+1}^3 & 30x_{n+1}^4 & 42x_{n+1}^5 & 56x_{n+1}^6 \\ 0 & 0 & 2 & 6x_{n+\frac{4}{3}} & 12x_{n+\frac{4}{3}}^2 & 20x_{n+\frac{4}{3}}^3 & 30x_{n+\frac{4}{3}}^4 & 42x_{n+\frac{4}{3}}^5 & 56x_{n+\frac{4}{3}}^6 \\ 0 & 0 & 2 & 6x_{n+\frac{5}{3}} & 12x_{n+\frac{5}{3}}^2 & 20x_{n+\frac{5}{3}}^3 & 30x_{n+\frac{5}{3}}^4 & 42x_{n+\frac{5}{3}}^5 & 56x_{n+\frac{5}{3}}^6 \\ 0 & 0 & 2 & 6x_{n+2} & 12x_{n+2}^2 & 20x_{n+2}^3 & 30x_{n+2}^4 & 42x_{n+2}^5 & 56x_{n+2}^6 \end{bmatrix}.$$

Solving (5) for a_j 's using Gaussian elimination method and substituting back into (2) gives a continuous hybrid linear multistep method of the form

$$y(x) = \sum_{j=0}^1 \alpha_j y_{n+j} + h^2 \left(\sum_{j=0}^1 \beta_j(x) f_{n+j} + \beta_k f_{n+k} \right), \quad k = \frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{5}{3}. \quad (6)$$

The coefficients of y_{n+j} , f_{n+j} and f_{n+k} give

$$\alpha_0 = 1 - t, \quad \alpha_1 = t,$$

$$\beta_0 = \frac{1}{13440} (243t^8 - 2268t^7 + 8820t^6 - 18522t^5 + 22736t^4 - 1644t^3 + 6720t^2 - 1265t),$$

$$\beta_{\frac{1}{3}} = -\frac{1}{2240} (243t^8 - 2160t^7 + 7812t^6 - 14616t^5 + 14616t^4 - 6720t^3 + 825t),$$

$$\beta_{\frac{2}{3}} = \frac{1}{4480} (1215t^8 - 10260t^7 + 34524t^6 - 58086t^5 + 49140t^4 - 16800t^3 - 267t),$$

$$\beta_1 = -\frac{1}{3360} (1215t^8 - 9720t^7 + 30492t^6 - 46872t^5 + 35560t^4 - 11200t^3 + 525t),$$

$$\beta_{\frac{4}{3}} = -\frac{1}{2240} (243t^8 - 1728t^7 - 4788t^6 - 6552t^5 + 4536t^4 - 1344t^3 + 57t),$$

$$\beta_{\frac{5}{3}} = -\frac{1}{2240} (243t^8 - 1728t^7 - 4788t^6 - 6552t^5 + 4536t^4 - 1344t^3 + 57t),$$

$$\beta_2 = \frac{1}{13440} (243t^8 - 1620t^7 + 4284t^6 - 5670t^5 + 3836t^4 - 1120t^3 + 47t).$$

where $t = \frac{x-x_n}{h}$, $y_{n+j} = y(x_n + jh)$ and

$$f_{n+j} = f\left((x_n + jh), y(x_n + jh), y'(x_n + jh)\right).$$

Solving (6) for the independent solution at the grid points gives the continuous block method

$$y(x) = \sum_{j=0}^1 \frac{(jh)^{(m)}}{m!} y_n^{(m)} + h^2 \left(\sum_{j=0}^1 \sigma_j(x) f_{n+j} + \sigma_k f_{n+k} \right), \quad k = \frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{5}{3}. \quad (7)$$

The coefficient of f_{n+j} and f_{n+k} give

$$\sigma_0 = \frac{1}{13440} (243t^8 - 2268t^7 + 8820t^6 - 18522t^5 + 22736t^4 - 1644t^3 + 6720t^2),$$

$$\beta_{\frac{1}{3}} = -\frac{1}{2240} (243t^8 - 2160t^7 + 7812t^6 - 14616t^5 + 14616t^4 - 6720t^3),$$

$$\beta_{\frac{2}{3}} = \frac{1}{4480} (1215t^8 - 10260t^7 + 34524t^6 - 58086t^5 + 49140t^4 - 16800t^3),$$

$$\beta_1 = -\frac{1}{3360} (1215t^8 - 9720t^7 + 30492t^6 - 46872t^5 + 35560t^4 - 11200t^3),$$

$$\beta_{\frac{4}{3}} = -\frac{1}{2240} (243t^8 - 1728t^7 - 4788t^6 - 6552t^5 + 4536t^4 - 1344t^3),$$

$$\beta_{\frac{5}{3}} = -\frac{1}{2240} (243t^8 - 1728t^7 - 4788t^6 - 6552t^5 + 4536t^4 - 1344t^3),$$

$$\beta_2 = \frac{1}{13440} (243t^8 - 1620t^7 + 4284t^6 - 5670t^5 + 3836t^4 - 1120t^3).$$

Evaluating (7) at $t = \frac{1}{3}, \frac{2}{3}$ gives a discrete block formula of the form

$$\mathbf{A}^{(0)} \mathbf{Y}_m^{(i)} = \sum_{i=0}^1 h^i \mathbf{e}_i y_n^{(i)} + h^2 \mathbf{d}_i f(y_n) + h^2 \mathbf{b}_i f(\mathbf{Y}_m), \quad i = 0, 1, \quad (8)$$

where

$$\mathbf{Y}_m = \begin{bmatrix} y_{n+\frac{1}{3}} & y_{n+\frac{2}{3}} & y_{n+1} & y_{n+\frac{4}{3}} & y_{n+\frac{5}{3}} & y_{n+2} \end{bmatrix}^T,$$

$$\mathbf{f}(\mathbf{Y}_m) = \begin{bmatrix} f_{n+\frac{1}{3}} & f_{n+\frac{2}{3}} & f_{n+1} & f_{n+\frac{4}{3}} & y_{n+\frac{5}{3}} & y_{n+2} \end{bmatrix}^T,$$

$$\mathbf{y}_n^{(i)} = \begin{bmatrix} y_{n-1}^{(i)} & y_{n-2}^{(i)} & y_{n-3}^{(i)} & y_{n-4}^{(i)} & y_{n-5}^{(i)} & y_n^{(i)} \end{bmatrix}^T,$$

$$\mathbf{f}(\mathbf{y}_n) = \begin{bmatrix} f_{n-1} & f_{n-2} & f_{n-3} & f_{n-4} & f_{n-5} & f_n \end{bmatrix}^T,$$

and $A^{(0)} = 6 \times 6$ Identity matrix.

If $i = 0$, then:

$$\mathbf{e}_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{e}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \frac{4}{3} \\ 0 & 0 & 0 & 0 & 0 & \frac{5}{3} \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix},$$

$$\mathbf{d}_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{28549}{1088640} \\ 0 & 0 & 0 & 0 & 0 & \frac{1027}{17010} \\ 0 & 0 & 0 & 0 & 0 & \frac{253}{2688} \\ 0 & 0 & 0 & 0 & 0 & \frac{1088}{8505} \\ 0 & 0 & 0 & 0 & 0 & \frac{35225}{217728} \\ 0 & 0 & 0 & 0 & 0 & \frac{41}{210} \end{bmatrix},$$

$$\mathbf{b}_0 = \begin{bmatrix} \frac{275}{5184} & \frac{-5717}{120960} & \frac{10621}{272160} & \frac{-7703}{362880} & \frac{403}{60480} & \frac{-199}{217728} \\ \frac{194}{945} & \frac{-8}{81} & \frac{788}{8505} & \frac{-97}{1890} & \frac{46}{2835} & \frac{-19}{8505} \\ \frac{165}{-267} & \frac{81}{-267} & \frac{5}{9} & \frac{-363}{-363} & \frac{57}{-57} & \frac{-47}{-47} \\ \frac{448}{1504} & \frac{4480}{-8} & \frac{32}{2624} & \frac{4480}{-8} & \frac{2240}{32} & \frac{13440}{-8} \\ \frac{2835}{8375} & \frac{945}{3125} & \frac{8505}{25625} & \frac{81}{-625} & \frac{945}{275} & \frac{1701}{-1375} \\ \frac{12096}{6} & \frac{72576}{3} & \frac{54432}{68} & \frac{24192}{3} & \frac{5184}{6} & \frac{217728}{0} \\ \frac{7}{7} & \frac{35}{35} & \frac{105}{105} & \frac{70}{70} & \frac{35}{35} & 0 \end{bmatrix}.$$

If $i = 1$, then:

$$\mathbf{e}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{d}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{19087}{181440} \\ 0 & 0 & 0 & 0 & 0 & \frac{1139}{11340} \\ 0 & 0 & 0 & 0 & 0 & \frac{137}{1344} \\ 0 & 0 & 0 & 0 & 0 & \frac{286}{2835} \\ 0 & 0 & 0 & 0 & 0 & \frac{3715}{36288} \\ 0 & 0 & 0 & 0 & 0 & \frac{41}{420} \end{bmatrix},$$

$$\mathbf{b}_1 = \begin{bmatrix} \frac{2713}{7560} & \frac{-15487}{60480} & \frac{586}{2835} & \frac{-6737}{60480} & \frac{263}{7560} & \frac{-863}{181440} \\ \frac{94}{11} & \frac{332}{11} & \frac{332}{11} & \frac{-269}{11} & \frac{22}{11} & \frac{-37}{11} \\ \frac{189}{27} & \frac{3780}{-387} & \frac{2835}{34} & \frac{3780}{-243} & \frac{945}{9} & \frac{11340}{-29} \\ \frac{56}{464} & \frac{2240}{-128} & \frac{105}{1504} & \frac{2240}{-58} & \frac{280}{16} & \frac{6720}{-8} \\ \frac{945}{725} & \frac{945}{-2125} & \frac{2835}{250} & \frac{945}{-3875} & \frac{945}{235} & \frac{2835}{-275} \\ \frac{1512}{18} & \frac{12096}{9} & \frac{567}{68} & \frac{12096}{9} & \frac{1572}{18} & \frac{36288}{41} \\ \frac{35}{35} & \frac{140}{140} & \frac{105}{105} & \frac{140}{140} & \frac{35}{35} & \frac{420}{420} \end{bmatrix}.$$

3. Analysis of Basic Properties of the Method

3.1. Order of the Block

Let the linear operator $\{y(x) : h\}$ associated with the discrete block method (8) be defined as

$$\{y(x) : h\} = \mathbf{A}^{(0)} \mathbf{Y}_m^{(i)} - \sum_{i=0}^1 h^i e_i y_n^{(i)} - h^2 (\mathbf{d}\mathbf{f}(y_n) + \mathbf{b}\mathbf{F}(Y_m)). \quad (9)$$

Expanding (9) in Taylor series and comparing the coefficients of h gives

$$\begin{aligned} \{y(x) : h\} &= C_0 y(x) + C_1 y'(x) + \dots + C_p h^p y^p(x) + C_{p+1} h^{p+1} y^{p+1}(x) \\ &\quad + C_{p+2} h^{p+2} y^{p+2}(x) + \dots \end{aligned}$$

Definition 2. The linear operator and associated block formula are said to be of order p if $C_0 = C_1 = \dots = C_p = C_{p+1} = 0, C_{p+2} \neq 0$. C_{p+2} is called the error constant and implies that the truncation error is given by $t_{n+k} = C_{p+2} h^{p+2} y^{p+2}(x) + O(h^{p+3})$.

For our method, expanding (8) in Taylor series gives

$$\begin{aligned} &\sum_{j=0} \frac{(\frac{1}{3}h)^j}{j!} y_n^{(j)} - y_n - \frac{1}{3} h y_n' - \frac{28549}{10088640} h^2 y_n'' \\ &- \sum_{j=0} \frac{h^{j+2}}{j!} y_n^{j+2} \left[\frac{275}{5184} \left(\frac{1}{3}\right)^j - \frac{5717}{120960} \left(\frac{2}{3}\right)^j + \frac{10621}{272160} (1)^j \right. \\ &\quad \left. - \frac{7703}{362880} \left(\frac{4}{3}\right)^j + \frac{403}{60480} \left(\frac{5}{3}\right)^j - \frac{199}{217728} (2)^j \right] \\ &\sum_{j=0} \frac{(\frac{2}{3}h)^j}{j!} y_n^{(j)} - y_n - \frac{2}{3} h y_n' - \frac{1027}{1707} h^2 y_n'' - \end{aligned}$$

$$\begin{aligned}
& \sum_{j=0} \frac{h^{j+2}}{j!} y_n^{j+2} \left[\frac{194}{945} \left(\frac{1}{3}\right)^j - \frac{8}{81} \left(\frac{2}{3}\right)^j + \frac{788}{8505} (1)^j \right. \\
& \quad \left. - \frac{97}{1890} \left(\frac{4}{3}\right)^j + \frac{46}{2835} \left(\frac{5}{3}\right)^j - \frac{19}{8505} (2)^j \right] \\
& \quad \sum_{j=0} \frac{(h)^j}{j!} y_n^{(j)} - y_n - h y_n' - \frac{25}{2688} h^2 y_n'' - \\
& \sum_{j=0} \frac{h^{j+2}}{j!} y_n^{j+2} \left[\frac{165}{448} \left(\frac{1}{3}\right)^j - \frac{267}{4480} \left(\frac{2}{3}\right)^j + \frac{5}{32} (1)^j \right. \\
& \quad \left. - \frac{363}{4480} \left(\frac{4}{3}\right)^j + \frac{57}{2240} \left(\frac{5}{3}\right)^j - \frac{47}{13440} (2)^j \right] \\
& \quad \sum_{j=0} \frac{\left(\frac{4}{3}h\right)^j}{j!} y_n^{(j)} - y_n - \frac{4}{3} h y_n' - \frac{1088}{8505} h^2 y_n'' - \\
& \sum_{j=0} \frac{h^{j+2}}{j!} y_n^{j+2} \left[\frac{1504}{2835} \left(\frac{1}{3}\right)^j - \frac{8}{945} \left(\frac{2}{3}\right)^j + \frac{2624}{8505} (1)^j \right. \\
& \quad \left. - \frac{8}{81} \left(\frac{4}{3}\right)^j + \frac{32}{945} \left(\frac{5}{3}\right)^j - \frac{8}{1701} (2)^j \right] \\
& \quad \sum_{j=0} \frac{\left(\frac{5}{3}h\right)^j}{j!} y_n^{(j)} - y_n - \frac{5}{3} h y_n' - \frac{35225}{217728} h^2 y_n'' - \\
& \sum_{j=0} \frac{h^{j+2}}{j!} y_n^{j+2} \left[\frac{8375}{12096} \left(\frac{1}{3}\right)^j + \frac{3125}{72576} \left(\frac{2}{3}\right)^j + \frac{25625}{54432} (1)^j \right. \\
& \quad \left. - \frac{625}{24192} \left(\frac{4}{3}\right)^j + \frac{275}{5184} \left(\frac{5}{3}\right)^j - \frac{1375}{217728} (2)^j \right] \\
& \quad \sum_{j=0} \frac{(2h)^j}{j!} y_n^{(j)} - y_n + 2h y_n' - \frac{41}{210} h^2 y_n'' - \\
& \sum_{j=0} \frac{h^{j+2}}{j!} y_n^{j+2} \left[\frac{6}{7} \left(\frac{1}{3}\right)^j + \frac{3}{35} \left(\frac{2}{3}\right)^j + \frac{68}{105} (1)^j - \frac{10}{70} \left(\frac{4}{3}\right)^j + \frac{6}{5} \left(\frac{5}{3}\right)^j \right] = 0,
\end{aligned}$$

comparing the coefficient efficient of h gives $C_0 = C_1 = \dots = C_8 = 0$ and

$$C_9 = \left[\frac{6031}{17856417600} \quad \frac{233}{279006525} \quad \frac{1}{765450} \quad \frac{496}{2796006525} \quad \frac{1621}{714256704} \quad \frac{1}{382725} \right]^T$$

hence, our method is of order 7.

3.2. Zero Stability of the Method

Definition 3. A block method is said to be zero stable if as $h \rightarrow 0$, the roots $r_j = 1$ ($1 \leq j \leq k$) of the first characteristic polynomial $\rho(r) = 0$ that is $\rho(r) = \det [\sum A^{(0)} R^{k-1}] = 0$ satisfying $|R| \leq 1$, must have multiplicity equal to unity (see Fatunla [8] for details)

For our method

$$\rho(R) = \left[R \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right] = 0,$$

$$\rho(R) = R^5(R - 1) = 0, \quad R = 0, 0, 0, 0, 0, 1.$$

Hence the method is zero stable.

4. Stability Interval

Definition 4. The block (8) is said to be absolutely stable within a given interval if for a given h , all roots z_s of the characteristic polynomial $\pi(z, h) = \rho(z) + h^2 \sigma(z) = 0$, satisfies $|z_s| < 1$, $s = 1, 2, \dots, n$. where $h = \lambda^2 h^2$ and $\lambda = \frac{\partial f}{\partial y}$. We adopted the boundary locus method to determine the stability interval of our block.

Substituting $y'' = -\lambda^2 y$ into the block (8) and substituting $e^{i\theta} = \cos \theta + i \sin \theta$, gives

$$h^- = \frac{7437184000 \cos \theta - 7437184000}{781 \cos \theta - 12159}$$

Hence the stability interval is given as $\theta \in [0, 60^\circ]$

5. Numerical Example

Problem I. We consider the non-linear initial value problems

$$y'' = \frac{(y')^2}{2y} - 2y, y\left(\frac{\pi}{6}\right) = \frac{1}{4}, y'\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}, h = 0.05.$$

$$\text{ExactSolution: } y(x) = (\sin x)^2$$

This problem was solved by Awoyemi[6] and Awoyemi and Kayode [7], where they both proposed method implemented in predictor corrector method for stepsize $h = 0.003125$. Awoyemi et al. [5] also solved this problem where a method implemented in block method was proposed for $h = 0.003125$. Jator [10] and Adesanya et al. [2] also solved this problem where methods implemented in block method for stepsize $h = 0.04913$ and 0.05 respectively were proposed. It was shown that the later method gave better approximation despite high value of h used. We solved this problem using our method and compared our result with Adesanya et al. [2] and Jator [10] as shown in Table I.

Problem II. We consider a non linear initial value problem

$$y'' - x(y')^2 = 0, y(0), y'(0) = \frac{1}{2}, h = 0.05.$$

$$\text{ExactSolution: } y(x) = 1 + \frac{1}{2} \ln \left(\frac{2+x}{2-x} \right)$$

This problem was solved by Jator [10] and Adesanya et al. [2], where a block method was proposed with stepsize 0.05 . It was shown that the result of these authors are better than the existing methods. We solved this problem and compared our result with the results of these authors as shown in Table II.

Note: $\text{Error} = |\text{Exact result} - \text{computed result}|$

EAE= Error in Adesanya et al [2]

EJE= Error in Jator [10]

Table 1 showing results for problem 1

X	Exact solution	Computed Solution	Error in new method	EJE	EAE
1.10	0.77982668476381558	0.79982668475924446	4.5711(-12)	2.8047(-10)	1.8811(-10)
1.20	0.87334365786466683	0.87334365785785987	4.7869(-12)	2.7869(-10)	2.4539(-10)
1.30	0.93197659748049533	0.93197659747586414	6.6311(-12)	2.1490(-10)	3.0306(-10)
1.40	0.97338799333525854	0.97338799333108894	4.6695(-12)	5.4975(-10)	3.5819(-10)
1.50	0.99592690375892334	0.99592690375564233	3.2810(-12)	1.1545(-10)	4.0838(-10)
1.60	0.99869477351706315	0.99869477351106584	5.9973(-12)	4.4825(-10)	4.5128(-10)
1.70	0.98158125637743932	0.98158125637252467	4.9146(-12)	7.7969(-10)	4.8473(-10)
1.80	0.94526861426358267	0.94526861425983988	3.7421(-12)	1.1840(-09)	5.0696(-10)
1.90	0.89120451762540223	0.89120451762278163	2.6205(-12)	1.6318(-09)	5.1697(-10)
2.00	0.82154433138674121	0.82154433137749383	9.2473(-12)	2.0567(-09)	5.1381(-10)

Table 1

Table 11 showing results for problem 11

X	Exact solution	Computed Solution	Error in new method	EJE	EAE
0.1	1.0500417292784914	1.0500417292784903	1.1102(-15)	3.1624(-12)	7.5028(-13)
0.2	1.1003353477310756	1.1003353477310696	5.9952(-15)	1.5091(-11)	9.7410(-12)
0.3	1.151140435936467	1.1511404359356441	2.5535(-14)	4.5286(-11)	3.7638(-11)
0.4	1.2027325540540823	1.2027325540540112	7.1054(-14)	1.0808(-10)	9.7765(-11)
0.5	1.2554128118829955	1.2554128118828796	1.1568(-13)	1.7818(-10)	2.0825(-10)
0.6	1.3095196042031119	1.3095196041202992	1.1990(-13)	4.4424(-10)	3.9604(-10)
0.7	1.3654437542713964	1.3654437542720822	6.8567(-13)	7.4446(-10)	7.0460(-10)
0.8	1.4236489301936022	1.4236489301970776	3.4754(-12)	1.5009(-09)	1.2095(-09)
0.9	1.484700278594052	1.4847002786062715	1.2219(-11)	3.7579(-09)	2.0511(-09)
1.0	1.5493061443340554	1.5493061443713383	3.7282(-11)	4.7410(-09)	3.5066(-09)

Table 2

6. Conclusion

We have proposed a direct method for the solution of second order initial value problems of ordinary differential equations. It has been shown that our new method is zero stable, consistent and convergent. The numerical examples have shown clearly that our method performs better in term of accuracy than the existing methods.

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