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RHEONOMIC LAGRANGE SPACES WITH (α, β)-METRIC

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Abstract: The present paper deals with the differential geometry of rheonomic Lagrange spaces with (α, β)-metric. We obtain the coefficients of a semispray, integral curves of the semispray, canonical nonlinear connection, differential equations of autoparallel curves and canonical metrical N-linear connection of rheonomic Lagrange spaces with (α, β)-metric.

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1. Introduction

The geometry of Lagrange spaces is metric generalization of that of Finsler spaces and was introduced by the great Romanian geometer R. Miron (see [12] and [13]). A Lagrange space is a pair $L^n = (M, L(x, y))$, where M is a smooth manifold and $L(x, y)$, a regular Lagrangian (see [9]).In the last 10-12 years, this

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field has attracted worldwide geometers and physicists (see [3]-[4], [14]-[15]) to work on its development and applications in various disciplines of science. The present authors (see [7] and [8]) studied and investigated several properties of almost φ-Lagrange spaces and their subspaces. B. Nicolaescu (see [1] and [2]) introduced Lagrange spaces with (α, β)-metric and obtained fundamental entities related to them. Miron [10] discussed Finsler-Lagrange spaces with (α, β)-metric. In several problems arising from mechanics and physics, time dependent Lagrangians play important role. A rheonomic Lagrange space is a pair $RL^n = (M, L(x, y, t))$, where $L(x, y, t)$ is a time dependent regular Lagrangian. In recent years several mathematicians (see [5] and [11]) contributed significantly towards the development and applications of rheonomic Lagrange spaces.

In Section 2, we discuss basic notion of a rheonomic Lagrange space and introduce rheonomic Lagrange spaces with (α, β)-metric. In Section 3, we obtain the coefficients of a semispray of a rheonomic Lagrange space with (α, β)-metric. We also obtain the integral curves of this semispray in the same section. Section 4 deals with the nonlinear connection produced by the semispray and autoparallel curves with respect to the semispray. In Section 5, we discuss the canonical metrical N-linear connection of a rheonomic Lagrange spaces with (α, β)-metric and obtain its coefficients.

2. Preliminaries

Let (TM, π, M) be the tangent bundle of an n-dimensional smooth manifold M. Let (x^i) and (x^i, y^i) be the local coordinates on M and TM respectively. Let us consider the product manifold $E := TM \times \mathbb{R}$ and (x^i, y^i, t) as local coordinates on E. A time dependent Lagrangian is a function $L : E \to \mathbb{R}$ which is smooth on $\tilde{E} = E \setminus \{(x, 0, 0), x \in M\}$ and continuous on its complement. This Lagrangian is said to be regular if $rank\,(g_{ij}(x, y, t)) = n$, where $g_{ij}(x, y, t) := \frac{1}{2}\frac{\partial^2 L(x,y,t)}{\partial y^i \partial y^j}$ are components of a covariant symmetric tensor called the metric tensor of the Lagrangian $L(x, y, t)$. A rheonomic Lagrange space (see [5], [6] and [11]) is a pair $RL^n = (M, L(x, y, t))$, $L(x, y, t)$ being a regular Lagrangian whose metric tensor g_{ij} has constant signature on $\tilde{E}$. The coordinates (x^i, y^i, t) on E change by the following rule:

$$\tilde{x}^i = \tilde{x}^i(x^1, x^2, ..., x^n), \quad \tilde{y}^i = \frac{\partial \tilde{x}^i}{\partial x^j} y^j, \quad \tilde{t} = \varphi(t) \tag{1}$$

with

$$rank\left(\frac{\partial \tilde{x}^i}{\partial x^j}\right) = n, \quad \varphi'(t) \neq 0. \tag{2}$$

In general, $\varphi(t) = at + b, a \neq 0$. A semispray S on E has the form (see [5] and [11])

$$S = y^i \frac{\partial}{\partial x^i} - (2G^i + G_0^i)\frac{\partial}{\partial y^i}, \tag{3}$$

where

$$2G^i = \frac{1}{2} g^{ik}\left(\frac{\partial^2 L}{\partial y^k \partial x^j} y^j - \frac{\partial L}{\partial x^k}\right), \ \ y^i = \frac{dx^i}{d\sigma}, \tag{4}$$

$$G_0^i = \frac{1}{2} g^{ik} \frac{\partial^2 L}{\partial y^k \partial t}. \tag{5}$$

The pair of functions $\left(G^i(x,y,t), G_0^i(x,y,t)\right)$ is the system of coefficients of the semispray S. The semispray S is canonical as its coefficients depend on L only. Thus for a rheonomic Lagrange space RL^n there is a canonical semispray S with coefficients $\left(G^i(x,y,t), G_0^i(x,y,t)\right)$ of the form given by (4) and (5).

The integral of action of the Lagrangian $L(x,y,t)$ along a smooth curve $c : [0,1] \to M \times \mathbb{R}$, $I(c) = \int_0^1 L\left(x, \frac{dx}{d\sigma}, \sigma\right) d\sigma$ leads, by virtue of variational calculus, to the Euler-Lagrange equations (see [5], [6] and [11]):

$$E_i(L) \equiv \frac{\partial L}{\partial x^i} - \frac{d}{d\sigma}\left(\frac{\partial L}{\partial y^i}\right) = 0, \ \ y^i = \frac{dx^i}{d\sigma}. \tag{6}$$

For a point $u = (x,y,t) \in E$, consider the subspaces $V_u E = span\left(\frac{\partial}{\partial y^i}|_u\right)$ and $V_{u,0}E = span\left(\frac{\partial}{\partial t}|_u\right)$ of $T_u E$. Obviously, $dim V_u E = n$ and $dim V_{0,u} E = 1$. Now, the distribution

$$V \oplus V_0 : u \in E \mapsto V_u E \oplus V_{0,u} E$$

is vertical. The horizontal distribution $N : u \mapsto N_u \subset T_u E$, complementary to the above mentioned vertical distribution, is a nonlinear connection on E. Thus, we have the following decomposition of the tangent space $T_u E$:

$$T_u E = N_u \oplus V_u \oplus V_{u,0}. \tag{7}$$

The adapted basis of the decomposition is $\left(\frac{\delta}{\delta x^i} \equiv \delta_i, \frac{\partial}{\partial y^i} \equiv \dot{\partial}_i, \frac{\partial}{\partial t} \equiv \dot{\partial}_0\right)$, where

$$\delta_i = \partial_i - N_i^j \dot{\partial}_j - N_i \dot{\partial}_0, \ \ \partial_i \equiv \frac{\partial}{\partial x^i}. \tag{8}$$

The pair (N^j_i, N_j) is the system of coefficients of nonlinear connection N. For a semispray S of the rheonomic Lagrange space RL^n with coefficients $\big(G^i(x,y,t), G^i_0(x,y,t)\big)$ given respectively by (4) and (5), there is a nonlinear connection N determined only by RL^n. The coefficients of N are expressed by

$$N^i_j = \frac{\partial G^i}{\partial y^j} \tag{9}$$

and

$$N_j = g_{ij} G^i_0, \tag{10}$$

which, in view of (4) and (5), take the form

$$N^i_j = \frac{1}{4}\frac{\partial}{\partial y^j}\left[g^{ih}\left(\frac{\partial^2 L}{\partial y^h \partial x^k} y^k - \frac{\partial L}{\partial x^h}\right)\right] \tag{11}$$

and

$$N_j = \frac{1}{2}\frac{\partial^2 L}{\partial y^j \partial t}. \tag{12}$$

The tangent vector field $\frac{dc}{d\sigma}$ of the curve

$$c : \sigma \in I \subseteq \mathbb{R} \mapsto (x(\sigma),\, y(\sigma),\, t(\sigma)) \in \tilde{E}$$

on $\tilde{E}$ is given by (cf. [6]):

$$\frac{dc}{d\sigma} = \frac{dx^i}{d\sigma}\delta_i + \frac{\delta y^\alpha}{d\sigma}\dot{\partial}_\alpha,\ \ \alpha \in \{0, 1, ..., n\}, \tag{13}$$

where

$$\frac{\delta y^\alpha}{d\sigma} = \frac{dy^\alpha}{d\sigma} + N^\alpha_i \frac{dx^i}{d\sigma},\ N^0_i = N_i.$$

The curve c is said to be horizontal if $\frac{\delta y^\alpha}{d\sigma} = 0$.

A horizontal curve c on $\tilde{E}$ for which $y^i = \frac{dx^i}{dt}$, is said to be parallel with respect to the nonlinear connection N.

An N-linear connection $D\Gamma(N) = \left(L^i_{jk}, C^i_{j\alpha}\right)$, $(\alpha = 0, i)$ for a rheonomic Lagrange space is said to be a metrical N-linear connection if

$$g_{ij|k} = 0,\ \ g_{ij}|_k = 0,\ \ g_{ij}|_0 = 0, \tag{14}$$

where '$_|$' and '$|$' denote respectively the $h-$ and $v-$ covariant derivatives with respect to $D\Gamma(N)$.

For given nonlinear connection N with coefficients given by (11) and (12), there exists a unique metrical N-linear connection with coefficients given by (see [6])

$$L^i_{jk} = \frac{1}{2} g^{ih} \left(\delta_j g_{hk} + \delta_k g_{jh} - \delta_h g_{jk} \right), \tag{15}$$

$$C^i_{jk} = \frac{1}{2} g^{ih} \left(\dot{\partial}_j g_{hk} + \dot{\partial}_k g_{jh} - \dot{\partial}_h g_{jk} \right), \tag{16}$$

$$C^i_{j0} = \frac{1}{2} g^{ih} \dot{\partial}_0 g_{jh}. \tag{17}$$

In the present paper, we deal with a Lagrange space whose Lagrangian L is a function of $\alpha(x,y,t)$ and $\beta(x,y,t)$, where $\alpha(x,y,t) = \sqrt{a_{ij}(x,t)y^i y^j}$ and $\beta(x,y,t) = A_i(x,t)y^i$. Let us denote this Lagrangian by $\check{L}$. Thus

$$L(x,y,t) = \check{L} \left(\alpha(x,y,t), \beta(x,y,t) \right). \tag{18}$$

The space $(M, L(x,y,t))$ is called a rheonomic Lagrange space with (α,β)-metric. We shall frequently use the following relations (see [1]) for the product manifold $TM \times \mathbb{R}$:

$$\begin{aligned} \dot{\partial}_i \alpha = & \alpha^{-1} y_i, \;\; \dot{\partial}_i \dot{\partial}_j \alpha = \alpha^{-1} a_{ij}(x,t) - \alpha^{-3} y_i y_j, \;\; \dot{\partial}_i \beta = A_i(x,t), \\ & \dot{\partial}_i \dot{\partial}_j \beta = 0, \end{aligned} \tag{19}$$

where $y_i = a_{ij}(x,t)y^j$.

For basic notations and terminology, we refer to the book [6].

3. Semispray, Integral Curves

For any rheonomic Lagrange space, there is a family of semisprays with coefficients G^i given by (4) and with arbitrary coefficients G^i_0 (see [5] and [6]). We may consider the coefficients G^i_0 of the form given by (5). In this section, we obtain the coefficients of canonical semispray of a rheonomic Lagrange space with (α,β)-metric, using equations (4) and (5).

B. Nicolaescu (see [2]) obtained the coefficients G^i of a Lagrange space with (α,β)-metric as

$$2G^i(x,y) = \gamma^i_{jk}(x) y^j y^k - \lambda(x,y) F^i_j(x) y^j, \tag{20}$$

with

$$\begin{aligned}&\lambda(x,y)=\frac{\overset{\vee}{L}_\beta}{\overset{\vee}{L}_\alpha},\;\; F^i_j(x)=a^{ih}(x)F_{hj}(x),\;\; F_{hj}=\frac{1}{2}\left(\frac{\partial A_j}{\partial x^h}-\frac{\partial A_h}{\partial x^j}\right),\\&\overset{\vee}{L}_\beta=\frac{\partial\overset{\vee}{L}}{\partial\beta},\;\;\overset{\vee}{L}_\alpha=\frac{\partial\overset{\vee}{L}}{\partial\alpha}.\end{aligned}\tag{21}$$

Equation (20) can be extended to get the coefficients $G^i(x,y,t)$ for a rheonomic Lagrange space with (α,β)-metric:

$$2G^i(x,y,t)=\gamma^i_{jk}(x,t)y^jy^k-\lambda(x,y,t)F^i_j(x,t)y^j,\tag{22}$$

with

$$\begin{aligned}&\lambda(x,y,t)=\frac{\overset{\vee}{L}_\beta}{\overset{\vee}{L}_\alpha},\;\; F^i_j(x,t)=a^{ih}(x,t)F_{hj}(x,t),\\&F_{hj}=\frac{1}{2}\left(\frac{\partial A_j}{\partial x^h}-\frac{\partial A_h}{\partial x^j}\right),\;\;\overset{\vee}{L}_\beta=\frac{\partial\overset{\vee}{L}}{\partial\beta},\;\;\overset{\vee}{L}_\alpha=\frac{\partial\overset{\vee}{L}}{\partial\alpha}.\end{aligned}\tag{23}$$

Here, $F_{hj}(x,t)$ is the electromagnetic tensor of the space $L^n\ (M,\overset{\vee}{L}(\alpha,\beta))$ and $\gamma^i_{jk}(x,t)$ are the second kind Christoffel symbols of $a_{ij}(x,t)$.

Now, differentiating (18) partially with respect to t, we have

$$\frac{\partial L}{\partial t}=\overset{\vee}{L}_\alpha\alpha_{.0}+\overset{\vee}{L}_\beta\beta_{.0},\tag{24}$$

where $\alpha_{.0}=\dot{\partial}_0\alpha,\;\;\beta_{.0}=\dot{\partial}_0\beta$.

Differentiating (24) partially with respect to y^j, we get

$$\begin{aligned}\frac{\partial^2 L}{\partial y^j\partial t}=&[\frac{\partial^2\alpha}{\partial y^j\partial t}\overset{\vee}{L}_\alpha+\frac{\partial^2\beta}{\partial y^j\partial t}\overset{\vee}{L}_\beta+\left(\overset{\vee}{L}_{\alpha\alpha}\dot{\partial}_j\alpha+\overset{\vee}{L}_{\alpha\beta}\dot{\partial}_j\beta\right)\alpha_{.0}\\&+\left(\overset{\vee}{L}_{\beta\alpha}\dot{\partial}_j\alpha+\overset{\vee}{L}_{\beta\beta}\dot{\partial}_j\beta\right)\beta_{.0}],\end{aligned}$$

which in view of (19), yields

$$\frac{\partial^2 L}{\partial y^j \partial t} = [\frac{\partial}{\partial t}(\alpha^{-1}y_j)\overset{\vee}{L}_\alpha + \frac{\partial}{\partial t}\left(A_j(x,t)\right)\overset{\vee}{L}_\beta + \left(\overset{\vee}{L}_{\alpha\alpha}\alpha^{-1}y_j + \overset{\vee}{L}_{\alpha\beta}A_j(x,t)\right)\alpha_{.0}$$
$$+ \left(\overset{\vee}{L}_{\beta\alpha}\alpha^{-1}y_j + \overset{\vee}{L}_{\beta\beta}A_j(x,t)\right)\beta_{.0}]$$
$$= [(-\alpha^{-2}\alpha_{.0}y_j + \alpha^{-1}y_{j.0})\overset{\vee}{L}_\alpha + A_{j.0}\overset{\vee}{L}_\beta + \alpha^{-1}y_j\left(\overset{\vee}{L}_{\alpha\alpha}\alpha_{.0} + \overset{\vee}{L}_{\alpha\beta}\beta_{.0}\right)$$
$$+ A_j\left(\overset{\vee}{L}_{\alpha\beta}\alpha_{.0} + \overset{\vee}{L}_{\beta\beta}\beta_{.0}\right)],$$

where $y_{j.0} = \dot{\partial}_0 y_j$, $A_{j.0} = \dot{\partial}_0 A_j$.

This gives

$$\frac{1}{2}\frac{\partial^2 L}{\partial y^j \partial t} = \rho y_{j.0} + \rho_1 A_{j.0} + \rho_{-2}\alpha\alpha_{.0}y_j + \rho_{-1}(y_j\beta_{.\,0} + \alpha\alpha_{.0}A_j) + \rho_0\beta_{.0}A_j, \tag{25}$$

where

$$\rho = \frac{1}{2}\alpha^{-1}\overset{\vee}{L}_\alpha, \ \ \rho_1 = \frac{1}{2}\overset{\vee}{L}_\beta, \ \ \rho_{-2} = \frac{1}{2}\alpha^{-2}(\overset{\vee}{L}_{\alpha\alpha} - \alpha^{-1}\overset{\vee}{L}_\alpha),$$
$$\rho_{-1} = \frac{1}{2}\alpha^{-1}\overset{\vee}{L}_{\alpha\beta}, \ \ \rho_0 = \frac{1}{2}\overset{\vee}{L}_{\beta\beta}. \tag{26}$$

The metric tensor g_{ij} of a Lagrange space with (α, β)-metric is given by (cf. [1])

$$g_{ij}(x,y) = \rho a_{ij}(x) + c_i c_j, \tag{27}$$

where

$$c_i = q_{-1}y_j + q_0 A_j \tag{28}$$

and q_{-1}, q_0 satisfy

$$\rho_0 = (q_0)^2, \ \rho_{-1} = q_0 q_{-1}, \ \rho_{-2} = (q_{-1})^2. \tag{29}$$

The detailed expression for g_{ij} is as follows (cf. [2]):

$$g_{ij}(x,y) = \rho a_{ij}(x) + \rho_0 A_i(x)A_j(x) + \rho_{-1}(y_i A_j + y_j A_i) + \rho_{-2}y_i y_j. \tag{30}$$

The inverse tensor g^{ij} of g_{ij} is given by (cf. [1])

$$g^{ij} = \frac{1}{\rho}a^{ij} - \frac{1}{1+c^2}c^i c^j, \tag{31}$$

$$c^i = \rho^{-1} a^{ij} c_j \ \ \texttt{and} \ \ c^i c_i = c^2. \tag{32}$$

Equations (30) and (31) can be extended to obtain the expression for the tensor g_{ij} and its inverse g^{ij} for the rheonomic Lagrange space with (α, β)-metric:

$$g_{ij}(x,y,t) = \rho a_{ij}(x,t) + \rho_0 A_i(x,t) A_j(x,t) + \rho_{-1}(y_i A_j + y_j A_i) + \rho_{-2} y_i y_j, \tag{33}$$

$$g^{ij} = \frac{1}{\rho} a^{ij}(x,t) - \frac{1}{1+c^2} c^i c^j, \tag{34}$$

where c^i satisfies conditions similar to (32) on $E = TM \times \mathbb{R}$.

In view of (25) and (5), we have

$$\begin{aligned} G_0^i = & g^{ij}[\rho y_{j.0} + \rho_1 A_{j.0} + \rho_{-2} \alpha \alpha_{.0} y_j + \rho_{-1}\left(y_j \beta_{.0} + \alpha \alpha_{.0} A_j\right) \\ & + \rho_0 \beta_{.0} A_j], \end{aligned} \tag{35}$$

where g^{ij} is given by (34).

Thus, we have:

Theorem 1. *There is a semispray S of a rheonomic Lagrange space with (α, β)-metric which depends upon the Lagrange space only and whose coefficients $\left(G^i(x,y,t), G_0^i(x,y,t)\right)$ are given by (22) and (35).*

The integral curves of the semispray S are given by the Euler-Lagrange equations $E_i(L) = 0$, which are equivalent to (see [5] and [11])

$$\frac{d^2 x^i}{d\sigma^2} + G^i\left(x, \frac{dx}{d\sigma}, \sigma\right) + G_0^i\left(x, \frac{dx}{d\sigma}, \sigma\right) = 0. \tag{36}$$

In view of (22) and (35), equations (36) take the form

$$\begin{aligned} & \frac{d^2 x^i}{d\sigma^2} + \gamma^i_{jk}(x,\sigma) y^j y^k - \lambda(x,y,\sigma) F^i_j(x,\sigma) y^j \\ & \quad + g^{ij}[\rho y_{j.0} + \rho_1 A_{j.0} + \rho_{-2} \alpha \alpha_{.0} y_j + \rho_{-1}\left(y_j \beta_{.0} + \alpha \alpha_{.0} A_j\right) \\ & \quad + \rho_0 \beta_{.0} A_j] = 0, \qquad y^j = \frac{dx^j}{d\sigma}, \end{aligned} \tag{37}$$

i.e.

$$\begin{aligned} & \frac{d^2 x^i}{d\sigma^2} + \gamma^i_{jk}(x,\sigma) y^j y^k \\ & \quad = \lambda(x,y,\sigma) F^i_j(x,\sigma) y^j - g^{ij}[\rho y_{j.0} + \rho_1 A_{j.0} + \rho_{-2} \alpha \alpha_{.0} y_j \\ & \qquad + \rho_{-1}\left(y_j \beta_{.0} + \alpha \alpha_{.0} A_j\right) + \rho_0 \beta_{.0} A_j], \ y^j = \frac{dx^j}{d\sigma}. \end{aligned} \tag{38}$$

Thus, we have

Theorem 2. *The integral curves of the semispray S of a rheonomic Lagrange space with (α, β)-metric are given by the second order differential equations (SODE) (38).*

4. Canonical Nonlinear Connection, Autoparallel Curves

In this section, we obtain the coefficients of a canonical nonlinear connection $N\left(N^i_j, N_j\right)$ for the semispray S (discussed in the preceding section) of a rheonomic Lagrange space with (α, β)-metric $L^n\left(M, \overset{\vee}{L}(\alpha, \beta)\right)$. We also obtain differential equations of the autoparallel curves with respect to this nonlinear connection.

The coefficients of canonical nonlinear connection N for the semispray S are given by (11) and (12). Nicolaescu (see [2]) obtained the following form of the coefficients N^i_j of the canonical nonlinear connection for a Lagrange space with (α, β)-metric:

$$N^i_j(x, y) = \gamma^i_{jk}(x) y^k - \frac{1}{2} \lambda^k_j F^i_k(x), \tag{39}$$

where $\lambda^k_j = \lambda \delta^k_j + \frac{\partial \lambda}{\partial y^j} y^k$ with λ given by (21).

The coefficients $N^i_j(x, y, t)$ for a rheonomic Lagrange space with (α, β)-metric can be written as

$$N^i_j(x, y, t) = \gamma^i_{jk}(x, t) y^k - \frac{1}{2} \lambda^k_j(x, y, t) F^i_k(x, t), \tag{40}$$

where $\lambda^k_j(x, y, t) = \lambda(x, y, t)\delta^k_j + \frac{\partial \lambda}{\partial y^j} y^k$ with λ given by (23).

In view of (12) and (25), we get

$$N_j = \rho y_{j.0} + \rho_1 A_{j.0} + \rho_{-2}\alpha\alpha_{.0} y_j + \rho_{-1}\left(y_j \beta_{.0} + \alpha\alpha_{.0} A_j\right) + \rho_0 \beta_{.0} A_j, \tag{41}$$

where $\rho, \rho_1, \rho_{-2}, \rho_{-1}$ and ρ_0 are given by (26).

Thus, we have:

Theorem 3. *The coefficients of canonical nonlinear connection N produced by the semispray S of a rheonomic Lagrange space with (α, β)-metric are given by (40) and (41).*

The autoparallel curves with respect to the nonlinear connection N produced by a semispray of a rheonomic Lagrange space are solution curves of the following differential equations (cf. [6]):

$$\frac{d^2x^i}{dt^2} + N^i_j\left(x, \frac{dx}{dt}, t\right)\frac{dx^j}{dt} = 0,\ N_j\left(x, \frac{dx}{dt}, t\right)\frac{dx^i}{d\sigma} + 1 = 0. \tag{42}$$

In view of (40) and (41), equations (42) take the form

$$\begin{aligned}&\frac{d^2x^i}{dt^2} + \gamma^i_{jk}(x,t)y^jy^k = \frac{1}{2}\lambda^k_j(x,y,t)F^i_k(x,t)y^j,\\ &[\rho y_{j.0} + \rho_1 A_{j.0} + \rho_{-2}\alpha\alpha_{.0}y_j + \rho_{-1}(y_j\beta_{.0} + \alpha\alpha_{.0}A_j) + \rho_0\beta_{.0}A_j]\frac{dx^i}{d\sigma}\\ &+ 1 = 0.\end{aligned} \tag{43}$$

Thus, we have:

Theorem 4. *The autoparallel curves with respect to the nonlinear connection N produced by the semispray S of a rheonomic Lagrange space with (α,β)-metric are solution curves of the system of differential equations (43).*

5. Canonical Metrical *N*-linear Connection

In this section we deal with the canonical metrical N-linear connection $C\Gamma(N) = (L^i_{jk}, C^i_{jk}, C^i_{j0})$ of a rheonomic Lagrange space with (α,β)-metric and obtain its coefficients which are given by (15)-(17).

If we partially differentiate the quantities appearing in (26) with respect to x^j, y^j and t, we respectively find the following sets of quantities:

$$\begin{aligned}&\partial_j\rho = \frac{1}{2}\rho_{-2}\xi_j + \rho_{-1}\zeta_j,\ \ \partial_j\rho_0 = \frac{1}{2}\mu_{-1}\xi_j + \mu_0\zeta_j,\\ &\partial_j\rho_{-1} = \frac{1}{2}\mu_{-2}\xi_j + \mu_{-1}\zeta_j,\ \ \partial_j\rho_{-2} = \frac{1}{2}\mu_{-3}\xi_j + \mu_{-2}\zeta_j,\end{aligned} \tag{44}$$

$$\begin{aligned}&\dot\partial_j\rho = \rho_{-2}y_j + \rho_{-1}A_j,\ \ \dot\partial_j\rho_0 = \mu_{-1}y_j + \mu_0A_j,\\ &\dot\partial_j\rho_{-1} = \mu_{-2}y_j + \mu_{-1}A_j,\ \ \dot\partial_j\rho_{-2} = \mu_{-3}y_j + \mu_{-2}A_j\end{aligned} \tag{45}$$

and

$$\begin{aligned}&\dot\partial_0\rho = \rho_{-2}\alpha\alpha_{.0} + \rho_{-1}\beta_{.0},\ \ \dot\partial_0\rho_0 = \mu_{-1}\alpha\alpha_{.0} + \mu_0\beta_{.0},\\ &\dot\partial_0\rho_{-1} = \mu_{-2}\alpha\alpha_{.0} + \mu_{-1}\beta_{.0},\ \ \dot\partial_0\rho_{-2} = \mu_{-3}\alpha\alpha_{.0} + \mu_{-2}\beta_{.0},\end{aligned} \tag{46}$$

where

$$\xi_j = \partial_j a_{rs}y^ry^s,\ \ \zeta_j = \partial_j A_r y^r,$$

$$\begin{aligned}\mu_0 &= \frac{1}{2}\overset{\vee}{L}_{\beta\beta\beta}, \ \ \mu_{-1} = \frac{1}{2}\alpha^{-1}\overset{\vee}{L}_{\alpha\beta\beta}, \ \ \mu_{-2} = \frac{1}{2}\alpha^{-2}\left(\overset{\vee}{L}_{\alpha\alpha\beta} - \alpha^{-1}\overset{\vee}{L}_{\alpha\beta}\right),\\ \mu_{-3} &= \frac{1}{2}\alpha^{-3}\left(\overset{\vee}{L}_{\alpha\alpha\alpha} - 3\alpha^{-1}\overset{\vee}{L}_{\alpha\alpha} + 3\alpha^{-2}\overset{\vee}{L}_{\alpha}\right).\end{aligned} \tag{47}$$

In view of (33), we have

$$\begin{aligned}2C_{jhk} := \dot{\partial}_j g_{hk} =& a_{hk}\dot{\partial}_j\rho + A_hA_k\dot{\partial}_j\rho_0 + (\dot{\partial}_j\rho_{-1})\mathfrak{S}_{(hk)}\left\{y_hA_k\right\}\\ &+ \rho_{-1}\mathfrak{S}_{(hk)}\left\{a_{hj}A_k\right\} + (\dot{\partial}_j\rho_{-2})y_hy_k\\ &+ \rho_{-2}\mathfrak{S}_{(hk)}\left\{a_{hj}y_k\right\},\end{aligned} \tag{48}$$

where $\mathfrak{S}_{(hk)}$ denotes the interchange of indices h and k and addition. Using (45) in (48), we get

$$\begin{aligned}2C_{jhk} =& a_{hk}(\rho_{-2}y_j + \rho_{-1}A_j) + A_hA_k(\mu_{-1}y_j + \mu_0A_j)\\ &+ (\mu_{-2}y_j + \mu_{-1}A_j)\mathfrak{S}_{(hk)}\left\{y_hA_k\right\} + \rho_{-1}\mathfrak{S}_{(hk)}\left\{a_{hj}A_k\right\}\\ &+ (\mu_{-3}y_j + \mu_{-2}A_j)y_hy_k + \rho_{-2}\mathfrak{S}_{(hk)}\left\{a_{hj}y_k\right\}.\end{aligned} \tag{49}$$

Applying (49) in (16), we obtain

$$\begin{aligned}C^i_{jk} =& \frac{1}{2}g^{ih}[a_{hk}(\rho_{-2}y_j + \rho_{-1}A_j) + A_hA_k(\mu_{-1}y_j + \mu_0A_j)\\ &+ (\mu_{-2}y_j + \mu_{-1}A_j)\mathfrak{S}_{(hk)}\left\{y_hA_k\right\} + \rho_{-1}\mathfrak{S}_{(hk)}\left\{a_{hj}A_k\right\}\\ &+ (\mu_{-3}y_j + \mu_{-2}A_j)y_hy_k + \rho_{-2}\mathfrak{S}_{(hk)}\left\{a_{hj}y_k\right\}].\end{aligned} \tag{50}$$

Differentiating (33) partially with respect to t, we have

$$\begin{aligned}2C_{jh0} := \dot{\partial}_0 g_{jh} =& a_{jh}\dot{\partial}_0\rho + \rho a_{jh.0} + A_jA_h\dot{\partial}_0\rho_0 + \rho_0\mathfrak{S}_{(jh)}\left\{A_{j.0}A_h\right\}\\ &+ (\dot{\partial}_0\rho_{-1})\mathfrak{S}_{(jh)}\left\{y_jA_h\right\} + \rho_{-1}\mathfrak{S}_{(jh)}\left\{y_{j.0}A_h + y_jA_{h.0}\right\}\\ &+ (\dot{\partial}_0\rho_{-2})y_jy_h + \rho_{-2}\mathfrak{S}_{(jh)}\left\{y_{j.0}y_h\right\},\end{aligned}$$

which, in view of (46) becomes

$$\begin{aligned}2C_{jh0} =& a_{jh}(\rho_{-2}\alpha\alpha_{.0} + \rho_{-1}\beta_{.0}) + \rho a_{jh.0} + A_jA_h(\mu_{-1}\alpha\alpha_{.0} + \mu_0\beta_{.0})\\ &+ \rho_0\mathfrak{S}_{(jh)}\left\{A_{j.0}A_h\right\} + (\mu_{-2}\alpha\alpha_{.0} + \mu_{-1}\beta_{.0})\mathfrak{S}_{(jh)}\left\{y_jA_h\right\}\\ &+ \rho_{-1}\mathfrak{S}_{(jh)}\left\{y_{j.0}A_h + y_jA_{h.0}\right\} + (\mu_{-3}\alpha\alpha_{.0} + \mu_{-2}\beta_{.0})y_jy_h\\ &+ \rho_{-2}\mathfrak{S}_{(jh)}\left\{y_{j.0}y_h\right\}.\end{aligned} \tag{51}$$

Using (51) in (17), we get

$$\begin{aligned} C^i_{j0} =&\frac{1}{2}g^{ih}[a_{jh}(\rho_{-2}\alpha\alpha_{.0}+\rho_{-1}\beta_{.0})+\rho a_{jh.0}+A_jA_h(\mu_{-1}\alpha\alpha_{.0}+\mu_0\beta_{.0}) \\ &+\rho_0\mathfrak{S}_{(jh)}\left\{A_{j.0}A_h\right\}+(\mu_{-2}\alpha\alpha_{.0}+\mu_{-1}\beta_{.0})\mathfrak{S}_{(jh)}\left\{y_jA_h\right\} \\ &+\rho_{-1}\mathfrak{S}_{(jh)}\left\{y_{j.0}A_h+y_jA_{h.0}\right\}+(\mu_{-3}\alpha\alpha_{.0}+\mu_{-2}\beta_{.0})y_jy_h \\ &+\rho_{-2}\mathfrak{S}_{(jh)}\left\{y_{j.0}y_h\right\}]. \end{aligned} \tag{52}$$

Differentiating (33) partially with respect to x^j, we have

$$\begin{aligned} \partial_j g_{hk} =&X_{hk}\xi_j+Y_{hk}\zeta_j+\rho\partial_j a_{hk}+\rho_0\mathfrak{S}_{(hk)}\left\{A_k\partial_jA_h\right\} \\ &+\rho_{-1}\mathfrak{S}_{(hk)}\left\{y_h\partial_jA_k+A_k\partial_jy_h\right\}+\rho_{-2}\mathfrak{S}_{(hk)}\left\{y_h\partial_jy_k\right\}, \end{aligned} \tag{53}$$

where

$$X_{hk}=\frac{1}{2}\left(\rho_{-2}a_{hk}+\mu_{-1}A_hA_k+\mu_{-2}\mathfrak{S}_{(hk)}\left\{y_hA_k\right\}+\mu_{-3}y_hy_k\right),$$

$$Y_{hk}=\rho_{-1}a_{hk}+\mu_0A_hA_k+\mu_{-1}\mathfrak{S}_{(hk)}\left\{y_hA_k\right\}+\mu_{-2}y_hy_k.$$

Now from (8), we have $\delta_j g_{hk}=\left(\partial_j-N^r_j\dot\partial_r-N_j\dot\partial_0\right)g_{hk}$, which, in view of (49), (51) and (53) yields

$$\begin{aligned} \delta_j g_{hk} =&X_{hk}\xi_j+Y_{hk}\zeta_j+\rho\partial_j a_{hk}+\rho_0\mathfrak{S}_{(hk)}\left\{A_k\partial_jA_h\right\} \\ &+\rho_{-1}\mathfrak{S}_{(hk)}\left\{y_h\partial_jA_k+A_k\partial_jy_h\right\}+\rho_{-2}\mathfrak{S}_{(hk)}\left\{y_h\partial_jy_k\right\} \\ &-2N^r_jC_{rhk}-2N_jC_{hk0}. \end{aligned} \tag{54}$$

Similarly, we have

$$\begin{aligned} \delta_k g_{jh} =&X_{jh}\xi_k+Y_{jh}\zeta_k+\rho\partial_k a_{jh}+\rho_0\mathfrak{S}_{(jh)}\left\{A_j\partial_kA_h\right\} \\ &+\rho_{-1}\mathfrak{S}_{(jh)}\left\{y_j\partial_kA_h+A_h\partial_ky_j\right\}+\rho_{-2}\mathfrak{S}_{(jh)}\left\{y_j\partial_ky_h\right\} \\ &-2N^r_kC_{rjh}-2N_kC_{jh0} \end{aligned} \tag{55}$$

and

$$\begin{aligned} \delta_h g_{jk} =&X_{jk}\xi_h+Y_{jk}\zeta_h+\rho\partial_h a_{jk}+\rho_0\mathfrak{S}_{(jk)}\left\{A_j\partial_hA_k\right\} \\ &+\rho_{-1}\mathfrak{S}_{(jk)}\left\{y_j\partial_hA_k+A_k\partial_hy_j\right\}+\rho_{-2}\mathfrak{S}_{(jk)}\left\{y_j\partial_hy_k\right\} \\ &-2N^r_hC_{rjk}-2N_hC_{jk0}. \end{aligned} \tag{56}$$

Using (54)-(56) in (15), we have

$$\begin{aligned} L^i_{jk} =&\rho\gamma^i_{jk} - \mathfrak{S}_{(jk)}\left\{N^m_j C^i_{mk} + N_j C^i_{k0}\right\} + N^{im}C_{mjk} - N^i C_{jk0} \\ &+ \frac{1}{2}g^{ih}[X_{hk}\xi_j + X_{jh}\xi_k - X_{jk}\xi_h + Y_{hk}\zeta_j + Y_{jh}\zeta_k - Y_{jk}\zeta_h \\ &+ \rho_0(A_h\mathfrak{S}_{(jk)}\left\{\partial_j A_k\right\} + 2\mathfrak{S}_{(jk)}\left\{A_k F_{jh}\right\}) + \rho_{-1}(y_h\mathfrak{S}_{(jk)}\left\{\partial_j A_k\right\} \\ &+ A_h\mathfrak{S}_{(jk)}\left\{\partial_j y_k\right\} + 2\mathfrak{S}_{(jk)}\left\{y_k F_{jh} + A_k K_{jh}\right\}) \\ &+ \rho_{-2}(y_h\mathfrak{S}_{(jk)}\left\{\partial_j y_k\right\} + 2\mathfrak{S}_{(jk)}\left\{y_j K_{kh}\right\})], \end{aligned} \tag{57}$$

where

$$N^{im} = g^{ih}N^m_h,\ \ N^i = g^{ih}N_h,\ \ K_{kh} = \frac{1}{2}\left(\partial_k y_h - \partial_h y_k\right).$$

Thus, we have

Theorem 5. *For a rheonomic Lagrange space with (α,β)-metric, endowed with a nonlinear connection whose coefficients are given by (40) and (41), there is a unique canonical metrical N-linear connection $C\Gamma(N) = (L^i_{jk}, C^i_{jk}, C^i_{j0})$ with the coefficients given by (50), (52) and (57).*

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