RHEONOMIC LAGRANGE SPACES
WITH $(\alpha, \beta)$-METRIC

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Abstract: The present paper deals with the differential geometry of rheonomic Lagrange spaces with $(\alpha, \beta)$-metric. We obtain the coefficients of a semispray, integral curves of the semispray, canonical nonlinear connection, differential equations of autoparallel curves and canonical metrical $N$-linear connection of rheonomic Lagrange spaces with $(\alpha, \beta)$-metric.

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1. Introduction

The geometry of Lagrange spaces is metric generalization of that of Finsler spaces and was introduced by the great Romanian geometer R. Miron (see [12] and [13]). A Lagrange space is a pair $L^n = (M, L(x, y))$, where $M$ is a smooth manifold and $L(x, y)$, a regular Lagrangian (see [9]). In the last 10-12 years, this
field has attracted worldwide geometers and physicists (see [3]-[4], [14]-[15]) to work on its development and applications in various disciplines of science. The present authors (see [7] and [8]) studied and investigated several properties of almost $\varphi$-Lagrange spaces and their subspaces. B. Nicolaescu (see [1] and [2]) introduced Lagrange spaces with $(\alpha, \beta)$-metric and obtained fundamental entities related to them. Miron [10] discussed Finsler-Lagrange spaces with $(\alpha, \beta)$-metric. In several problems arising from mechanics and physics, time dependent Lagrangians play important role. A rheonomic Lagrange space is a pair $RL^n = (M, L(x,y,t))$, where $L(x,y,t)$ is a time dependent regular Lagrangian. In recent years several mathematicians (see [5] and [11]) contributed significantly towards the development and applications of rheonomic Lagrange spaces.

In Section 2, we discuss basic notion of a rheonomic Lagrange space and introduce rheonomic Lagrange spaces with $(\alpha, \beta)$-metric. In Section 3, we obtain the coefficients of a semispray of a rheonomic Lagrange space with $(\alpha, \beta)$-metric. We also obtain the integral curves of this semispray in the same section. Section 4 deals with the nonlinear connection produced by the semispray and autoparallel curves with respect to the semispray. In Section 5, we discuss the canonical metrical $N$-linear connection of a rheonomic Lagrange spaces with $(\alpha, \beta)$-metric and obtain its coefficients.

2. Preliminaries

Let $(TM, \pi, M)$ be the tangent bundle of an $n$-dimensional smooth manifold $M$. Let $(x^i)$ and $(x^i, y^i)$ be the local coordinates on $M$ and $TM$ respectively. Let us consider the product manifold $E := TM \times \mathbb{R}$ and $(x^i, y^i, t)$ as local coordinates on $E$. A time dependent Lagrangian is a function $L : E \rightarrow \mathbb{R}$ which is smooth on $\tilde{E} = E \setminus \{(x,0,0), x \in M\}$ and continuous on its complement. This Lagrangian is said to be regular if $\text{rank}(g_{ij}(x,y,t)) = n$, where $g_{ij}(x,y,t) := \frac{1}{2} \frac{\partial^2 L(x,y,t)}{\partial y^i \partial y^j}$ are components of a covariant symmetric tensor called the metric tensor of the Lagrangian $L(x,y,t)$. A rheonomic Lagrange space (see [5], [6] and [11]) is a pair $RL^n = (M, L(x,y,t))$, $L(x,y,t)$ being a regular Lagrangian whose metric tensor $g_{ij}$ has constant signature on $\tilde{E}$. The coordinates $(x^i, y^i, t)$ on $E$ change by the following rule:

$$\tilde{x}^i = \tilde{x}^i(x^1, x^2, \ldots, x^n), \quad \tilde{y}^i = \frac{\partial \tilde{x}^i}{\partial x^j} y^j, \quad \tilde{t} = \varphi(t)$$ (1)
with
\[
\text{rank} \left( \frac{\partial \tilde{x}^i}{\partial x^j} \right) = n, \quad \varphi'(t) \neq 0. \tag{2}
\]
In general, \( \varphi(t) = at + b, a \neq 0 \). A semispray \( S \) on \( E \) has the form (see \([5]\) and \([11]\))
\[
S = y^i \frac{\partial}{\partial x^i} - (2G^i + G^i_0) \frac{\partial}{\partial y^i}, \tag{3}
\]
where
\[
2G^i = \frac{1}{2} g^{ik} \left( \frac{\partial^2 L}{\partial y^k \partial x^j} y^j - \frac{\partial L}{\partial x^k} \right), \quad y^i = \frac{dx^i}{d\sigma}, \tag{4}
\]
\[
G^i_0 = \frac{1}{2} g^{ik} \frac{\partial^2 L}{\partial y^k \partial t}. \tag{5}
\]
The pair of functions \( (G^i(x, y, t), G^i_0(x, y, t)) \) is the system of coefficients of the semispray \( S \). The semispray \( S \) is canonical as its coefficients depend on \( L \) only. Thus for a rheonomic Lagrange space \( RL^n \) there is a canonical semispray \( S \) with coefficients \( (G^i(x, y, t), G^i_0(x, y, t)) \) of the form given by (4) and (5).

The integral of action of the Lagrangian \( L(x, y, t) \) along a smooth curve \( c : [0, 1] \to M \times \mathbb{R}, \ I(c) = \int_0^1 L(x, \frac{dx}{d\sigma}, \sigma) d\sigma \) leads, by virtue of variational calculus, to the Euler-Lagrange equations (see \([5]\), \([6]\) and \([11]\)):
\[
E_i(L) \equiv \frac{\partial L}{\partial x^i} - \frac{d}{d\sigma} \left( \frac{\partial L}{\partial y^i} \right) = 0, \quad y^i = \frac{dx^i}{d\sigma}. \tag{6}
\]
For a point \( u = (x, y, t) \in E \), consider the subspaces \( V_u E = \text{span} \left( \frac{\partial}{\partial y^i} \big|_u \right) \) and \( V_{u,0} E = \text{span} \left( \frac{\partial}{\partial t} \big|_u \right) \) of \( T_u E \). Obviously, \( dim V_u E = n \) and \( dim V_{0,u} E = 1 \). Now, the distribution
\[
V \oplus V_0 : u \in E \mapsto V_u E \oplus V_{0,u} E
\]
is vertical. The horizontal distribution \( N : u \mapsto N_u \subset T_u E \), complementary to the above mentioned vertical distribution, is a nonlinear connection on \( E \). Thus, we have the following decomposition of the tangent space \( T_u E \):
\[
T_u E = N_u \oplus V_u \oplus V_{u,0}. \tag{7}
\]
The adapted basis of the decomposition is \( \left( \frac{\delta}{\delta x^i} \equiv \delta_i, \frac{\partial}{\partial y^i} \equiv \hat{\delta}_i, \frac{\partial}{\partial t} \equiv \hat{\delta}_0 \right) \), where
\[
\delta_i = \partial_i - N^j_i \hat{\delta}_j - N_i \hat{\delta}_0, \quad \partial_i \equiv \frac{\partial}{\partial x^i}. \tag{8}
\]
The pair \((N^i_j, N^j_i)\) is the system of coefficients of nonlinear connection \(N\). For a semispray \(S\) of the rheonomic Lagrange space \(RL^n\) with coefficients \((G^i(x, y, t), G^i_0(x, y, t))\) given respectively by \((4)\) and \((5)\), there is a nonlinear connection \(N\) determined only by \(RL^n\). The coefficients of \(N\) are expressed by

\[
N^i_j = \frac{\partial G^i}{\partial y^j} \quad (9)
\]

and

\[
N^j_i = g_{ij} G^i_0 \quad (10)
\]

which, in view of \((4)\) and \((5)\), take the form

\[
N^i_j = \frac{1}{4} \frac{\partial}{\partial y^j} \left[ g^{ih} \left( \frac{\partial^2 L}{\partial y^h \partial x^k} y^k - \frac{\partial L}{\partial x^h} \right) \right] \quad (11)
\]

and

\[
N^j_i = \frac{1}{2} \frac{\partial^2 L}{\partial y^j \partial t}. \quad (12)
\]

The tangent vector field \(\frac{dc}{d\sigma}\) of the curve

\[
c : \sigma \in I \subseteq \mathbb{R} \mapsto (x(\sigma), y(\sigma), t(\sigma)) \in \tilde{E}
\]
on \(\tilde{E}\) is given by (cf. [6]):

\[
\frac{dc}{d\sigma} = \frac{dx^i}{d\sigma} \delta^i_i + \frac{\delta y^\alpha}{d\sigma} \hat{\theta}_\alpha, \quad \alpha \in \{0, 1, \ldots, n\}, \quad (13)
\]

where

\[
\frac{\delta y^\alpha}{d\sigma} = \frac{dy^\alpha}{d\sigma} + N^\alpha_i \frac{dx^i}{d\sigma}, \quad N^0_i = N_i.
\]

The curve \(c\) is said to be horizontal if \(\frac{\delta y^\alpha}{d\sigma} = 0\).

A horizontal curve \(c\) on \(\tilde{E}\) for which \(y^i = \frac{dx^i}{dt}\), is said to be parallel with respect to the nonlinear connection \(N\).

An \(N\)-linear connection \(D\Gamma(N) = (L^i_{jk}, C^i_{j\alpha})\), \((\alpha = 0, i)\) for a rheonomic Lagrange space is said to be a metrical \(N\)-linear connection if

\[
g_{ij|k} = 0, \quad g_{ij} |_k = 0, \quad g_{ij} |_0 = 0, \quad (14)
\]

where \(|\cdot|\) and \(|\cdot|\) denote respectively the \(h\)– and \(v\)– covariant derivatives with respect to \(D\Gamma(N)\).
For given nonlinear connection $N$ with coefficients given by (11) and (12), there exists a unique metrical $N$-linear connection with coefficients given by (see [6])

$$L^i_{jk} = \frac{1}{2} g^{ih} (\delta_j g_{hk} + \delta_k g_{jh} - \delta_h g_{jk}),$$  \hspace{1cm} (15)

$$C^i_{jk} = \frac{1}{2} g^{ih} \left( \dot{\delta}_j g_{hk} + \dot{\delta}_k g_{jh} - \dot{\delta}_h g_{jk} \right),$$  \hspace{1cm} (16)

$$C^i_{j0} = \frac{1}{2} g^{ih} \dot{\delta}_0 g_{jh}.$$  \hspace{1cm} (17)

In the present paper, we deal with a Lagrange space whose Lagrangian $L$ is a function of $\alpha(x, y, t)$ and $\beta(x, y, t)$, where $\alpha(x, y, t) = \sqrt{a_{ij}(x, t) y^i y^j}$ and $\beta(x, y, t) = A_i(x, t) y^i$. Let us denote this Lagrangian by $\bar{L}$. Thus

$$L(x, y, t) = \bar{L} (\alpha(x, y, t), \beta(x, y, t)).$$  \hspace{1cm} (18)

The space $(M, L(x, y, t))$ is called a rheonomic Lagrange space with $(\alpha, \beta)$-metric. We shall frequently use the following relations (see [1]) for the product manifold $TM \times \mathbb{R}$:

$$\dot{\alpha} = \alpha^{-1} y_i, \quad \dot{\beta} = A_i(x, t),$$  \hspace{1cm} (19)

$$\dot{\alpha} \dot{\beta} = 0,$$

where $y_i = a_{ij}(x, t) y^j$.

For basic notations and terminology, we refer to the book [6].

3. Semispray, Integral Curves

For any rheonomic Lagrange space, there is a family of semisprays with coefficients $G^i$ given by (4) and with arbitrary coefficients $G^i_0$ (see [5] and [6]). We may consider the coefficients $G^i_0$ of the form given by (5). In this section, we obtain the coefficients of canonical semispray of a rheonomic Lagrange space with $(\alpha, \beta)$-metric, using equations (4) and (5).

B. Nicolaescu (see [2]) obtained the coefficients $G^i$ of a Lagrange space with $(\alpha, \beta)$-metric as

$$2G^i(x, y) = \gamma^i_{jk}(x) y^j y^k - \lambda(x, y) F^i_j(x) y^j,$$  \hspace{1cm} (20)
with

\[
\lambda(x, y) = \frac{L_\beta}{L_\alpha}, \quad F^i_j(x) = a^i{}^h(x)F_{hj}(x), \quad F_{hj} = \frac{1}{2} \left( \frac{\partial A_j}{\partial x^h} - \frac{\partial A_h}{\partial x^j} \right),
\]

\[
\nabla L_\beta = \frac{\partial L}{\partial \beta}, \quad \nabla L_\alpha = \frac{\partial L}{\partial \alpha}.
\]

Equation (20) can be extended to get the coefficients \( G^i(x, y, t) \) for a rheonomic Lagrange space with \((\alpha, \beta)\)-metric:

\[
2G^i(x, y, t) = \gamma^i_{jk}(x, t)y^j y^k - \lambda(x, y, t)F^i_j(x, t)y^j,
\]

with

\[
\lambda(x, y, t) = \frac{L_\beta}{L_\alpha}, \quad F^i_j(x, t) = a^i{}^h(x, t)F_{hj}(x, t),
\]

\[
F_{hj} = \frac{1}{2} \left( \frac{\partial A_j}{\partial x^h} - \frac{\partial A_h}{\partial x^j} \right), \quad \nabla L_\beta = \frac{\partial L}{\partial \beta}, \quad \nabla L_\alpha = \frac{\partial L}{\partial \alpha}.
\]

Here, \( F_{hj}(x, t) \) is the electromagnetic tensor of the space \( L^n(M, \nabla L(\alpha, \beta)) \) and \( \gamma^i_{jk}(x, t) \) are the second kind Christoffel symbols of \( a_{ij}(x, t) \).

Now, differentiating (18) partially with respect to \( t \), we have

\[
\frac{\partial L}{\partial t} = \nabla L_\alpha \alpha_0 + \nabla L_\beta \beta_0,
\]

where \( \alpha_0 = \dot{\alpha}_0, \quad \beta_0 = \dot{\beta}_0 \).

Differentiating (24) partially with respect to \( y^j \), we get

\[
\frac{\partial^2 L}{\partial y^j \partial t} = \left[ \frac{\partial^2 \alpha}{\partial y^j \partial t} \nabla L_\alpha + \frac{\partial^2 \beta}{\partial y^j \partial t} \nabla L_\beta + \left( \nabla L_{a\alpha} \dot{\alpha} j \alpha + \nabla L_{a\beta} \dot{\beta} j \beta \right) \alpha_0 \right.
\]

\[
+ \left( \nabla L_{\beta\alpha} \dot{\alpha} j \alpha + \nabla L_{\beta\beta} \dot{\beta} j \beta \right) \beta_0,
\]
which in view of (19), yields

\[
\frac{\partial^2 L}{\partial y^j \partial t} = \left[ \frac{\partial}{\partial t} (\alpha^{-1} y_j) \dot{L}_\alpha + \frac{\partial}{\partial t} (A_j(x,t)) \dot{L}_\beta + \left( \dot{L}_{\alpha\alpha} \alpha^{-1} y_j + \dot{L}_{\alpha\beta} A_j(x,t) \right) \alpha_0
\right.
\]
\[
+ \left. \left( \dot{L}_{\beta\alpha} \alpha^{-1} y_j + \dot{L}_{\beta\beta} A_j(x,t) \right) \beta_0 \right]
\]
\[
= [(-\alpha^{-2} \alpha_0 y_j + \alpha^{-1} y_{j,0}) \dot{L}_\alpha + A_{j,0} \dot{L}_\beta + \alpha^{-1} y_j \left( \dot{L}_{\alpha\alpha} \alpha_0 + \dot{L}_{\alpha\beta} \beta_0 \right)
\]
\[
+ A_j \left( \dot{L}_{\alpha\beta} \alpha_0 + \dot{L}_{\beta\beta} \beta_0 \right)],
\]

where \( y_{j,0} = \dot{y}_0 y_j, \ A_{j,0} = \dot{A}_0 A_j. \)

This gives

\[
\frac{1}{2} \frac{\partial^2 L}{\partial y^2 \partial t} = \rho y_{j,0} + \rho_1 A_{j,0} + \rho_2 \alpha_0 y_j + \rho_3 (y_j \beta_0 + \alpha_0 A_j) + \rho_0 \beta_0 A_j, \quad (25)
\]

where

\[
\rho = \frac{1}{2} \alpha^{-1} \dot{L}_\alpha, \quad \rho_1 = \frac{1}{2} \dot{L}_\beta, \quad \rho_2 = \frac{1}{2} \alpha^{-2} (\dot{L}_{\alpha\alpha} - \alpha^{-1} \dot{L}_\alpha),
\]
\[
\rho_3 = \frac{1}{2} \alpha^{-1} \dot{L}_{\alpha\beta}, \quad \rho_0 = \frac{1}{2} \dot{L}_{\beta\beta}. \quad (26)
\]

The metric tensor \( g_{ij} \) of a Lagrange space with \((\alpha, \beta)\)-metric is given by (cf. [1])

\[
g_{ij}(x, y) = \rho a_{ij}(x) + c_i c_j, \quad (27)
\]

where

\[
c_i = q_{-1} y_j + q_0 A_j \quad (28)
\]

and \( q_{-1}, \ q_0 \) satisfy

\[
\rho_0 = (q_0)^2, \quad \rho_{-1} = q_0 q_{-1}, \quad \rho_{-2} = (q_{-1})^2. \quad (29)
\]

The detailed expression for \( g_{ij} \) is as follows (cf. [2]):

\[
g_{ij}(x, y) = \rho a_{ij}(x) + \rho_1 A_i(x) A_j(x) + \rho_3 (y_i A_j + y_j A_i) + \rho_2 y_i y_j. \quad (30)
\]

The inverse tensor \( g^{ij} \) of \( g_{ij} \) is given by (cf. [1])

\[
g^{ij} = \frac{1}{\rho} a^{ij} - \frac{1}{1 + c^j c^j}, \quad (31)
\]
\[ c^i = \rho^{-1} a^{ij} c_j \quad \text{and} \quad c^i c_i = c^2. \]  

Equations (30) and (31) can be extended to obtain the expression for the tensor \( g_{ij} \) and its inverse \( g^{ij} \) for the rheonomic Lagrange space with \((\alpha, \beta)\)-metric:

\[ g_{ij}(x, y, t) = \rho a_{ij}(x, t) A_j(x, t) + \rho_{-1}(y_i A_j + y_j A_i) + \rho_{-2}y_i y_j, \]  

\[ g^{ij} = \frac{1}{\rho} a^{ij}(x, t) - \frac{1}{1 + c^i c^i}, \]  

where \( c^i \) satisfies conditions similar to (32) on \( E = TM \times \mathbb{R} \).

In view of (25) and (5), we have

\[ G_i^0 = g^{ij}[\rho y_j, 0 + \rho_1 A_{j,0} + \rho_{-2} \alpha \alpha_0 y_j + \rho_{-1} (y_j \beta_{,0} + \alpha \alpha_0 A_j) + \rho_0 \beta_{,0} A_j], \]  

where \( g^{ij} \) is given by (34).

Thus, we have:

**Theorem 1.** There is a semispray \( S \) of a rheonomic Lagrange space with \((\alpha, \beta)\)-metric which depends upon the Lagrange space only and whose coefficients \((G^i(x, y, t), G_0^i(x, y, t))\) are given by (22) and (35).

The integral curves of the semisprays \( S \) are given by the Euler-Lagrange equations \( E_i(L) = 0 \), which are equivalent to (see [5] and [11])

\[
\frac{d^2 x^i}{d\sigma^2} + G_i^i \left( x, \frac{dx}{d\sigma}, \sigma \right) + G_0^i \left( x, \frac{dx}{d\sigma}, \sigma \right) = 0. \tag{36}
\]

In view of (22) and (35), equations (36) take the form

\[
\frac{d^2 x^i}{d\sigma^2} + \gamma_{jk}^i(x, \sigma) y^j y^k - \lambda(x, y, \sigma) F_j^i(x, \sigma) y^j \\
+ g^{ij}[\rho y_j, 0 + \rho_1 A_{j,0} + \rho_{-2} \alpha \alpha_0 y_j + \rho_{-1} (y_j \beta_{,0} + \alpha \alpha_0 A_j) + \rho_0 \beta_{,0} A_j] = 0, \quad y^j = \frac{dx^j}{d\sigma}, \tag{37}
\]

i.e.

\[
\frac{d^2 x^i}{d\sigma^2} + \gamma_{jk}^i(x, \sigma) y^j y^k \\
= \lambda(x, y, \sigma) F_j^i(x, \sigma) y^j - g^{ij}[\rho y_j, 0 + \rho_1 A_{j,0} + \rho_{-2} \alpha \alpha_0 y_j \\
+ \rho_{-1} (y_j \beta_{,0} + \alpha \alpha_0 A_j) + \rho_0 \beta_{,0} A_j], \quad y^j = \frac{dx^j}{d\sigma}. \tag{38}
\]

Thus, we have
Theorem 2. The integral curves of the semispray $S$ of a rheonomic Lagrange space with $(\alpha, \beta)$-metric are given by the second order differential equations (SODE) (38).

4. Canonical Nonlinear Connection, Autoparallel Curves

In this section, we obtain the coefficients of a canonical nonlinear connection $N \left( N^i_j, N_j \right)$ for the semispray $S$ (discussed in the preceding section) of a rheonomic Lagrange space with $(\alpha, \beta)$-metric $L^n \left( M, \tilde{\varepsilon}L(\alpha, \beta) \right)$. We also obtain differential equations of the autoparallel curves with respect to this nonlinear connection.

The coefficients of canonical nonlinear connection $N$ for the semispray $S$ are given by (11) and (12). Nicolaescu (see [2]) obtained the following form of the coefficients $N^i_j$ of the canonical nonlinear connection for a Lagrange space with $(\alpha, \beta)$-metric:

$$N^i_j (x, y) = \gamma^i_{jk}(x)y^k - \frac{1}{2}\lambda^k_j F^i_k(x),$$

(39)

where $\lambda^k_j = \lambda \delta^k_j + \frac{\partial \lambda}{\partial y^j}y^k$ with $\lambda$ given by (21).

The coefficients $N^i_j (x, y, t)$ for a rheonomic Lagrange space with $(\alpha, \beta)$-metric can be written as

$$N^i_j (x, y, t) = \gamma^i_{jk}(x, t)y^k - \frac{1}{2}\lambda^k_j(x, y, t)F^i_k(x, t),$$

(40)

where $\lambda^k_j(x, y, t) = \lambda(x, y, t)\delta^k_j + \frac{\partial \lambda}{\partial y^j}y^k$ with $\lambda$ given by (23).

In view of (12) and (25), we get

$$N_j = \rho y_{j,0} + \rho_1 A_{j,0} + \rho_{-2} \alpha \alpha_0 y_j + \rho_{-1} (y_j \beta_{,0} + \alpha \alpha_0 A_j) + \rho_0 \beta_{,0} A_j,$$

(41)

where $\rho, \rho_1, \rho_{-2}, \rho_{-1}$ and $\rho_0$ are given by (26).

Thus, we have:

Theorem 3. The coefficients of canonical nonlinear connection $N$ produced by the semispray $S$ of a rheonomic Lagrange space with $(\alpha, \beta)$-metric are given by (40) and (41).

The autoparallel curves with respect to the nonlinear connection $N$ produced by a semispray of a rheonomic Lagrange space are solution curves of the
following differential equations (cf. [6]):

\[
\frac{d^2 x^i}{dt^2} + N^i_j \left( x, \frac{dx}{dt}, t \right) \frac{dx^j}{dt} = 0, \quad N^i_j \left( x, \frac{dx}{dt}, t \right) \frac{dx^i}{d\sigma} + 1 = 0. \tag{42}
\]

In view of (40) and (41), equations (42) take the form

\[
\frac{d^2 x^i}{dt^2} + \gamma^i_{jk}(x, t) y^j y^k = \frac{1}{2} \lambda^i_j(x, y, t) F^i_k(x, t) y^j, \tag{43}
\]

\[
[\rho y_{j,0} + \rho_1 A_{j,0} + \rho_{-2} \alpha \alpha_0 y_j + \rho_{-1} (y_j \beta_0 + \alpha \alpha_0 A_j) + \rho_0 \beta_0 A_j] \frac{dx^i}{d\sigma} + 1 = 0.
\]

Thus, we have:

**Theorem 4.** The autoparallel curves with respect to the nonlinear connection \(N\) produced by the semispray \(S\) of a rheonomic Lagrange space with \((\alpha, \beta)\)-metric are solution curves of the system of differential equations (43).

5. Canonical Metrical \(N\)-linear Connection

In this section we deal with the canonical metrical \(N\)-linear connection \(C\Gamma(N) = (L^i_{jk}, C^i_{jk}, C^i_{j0})\) of a rheonomic Lagrange space with \((\alpha, \beta)\)-metric and obtain its coefficients which are given by (15)-(17).

If we partially differentiate the quantities appearing in (26) with respect to \(x^j, y^j\) and \(t\), we respectively find the following sets of quantities:

\[
\partial_j \rho = \frac{1}{2} \rho_{-2} \xi_j + \rho_{-1} \zeta_j, \quad \partial_j \rho_0 = \frac{1}{2} \mu_{-1} \xi_j + \mu_0 \zeta_j, \tag{44}
\]

\[
\partial_j \rho_{-1} = \frac{1}{2} \mu_{-2} \xi_j + \mu_{-1} \zeta_j, \quad \partial_j \rho_{-2} = \frac{1}{2} \mu_{-3} \xi_j + \mu_{-2} \zeta_j,
\]

\[
\tilde{\partial}_j \rho = \rho_{-2} y_j + \rho_{-1} A_j, \quad \tilde{\partial}_j \rho_0 = \rho_{-1} y_j + \rho_0 A_j, \tag{45}
\]

\[
\tilde{\partial}_j \rho_{-1} = \mu_{-2} y_j + \mu_{-1} A_j, \quad \tilde{\partial}_j \rho_{-2} = \mu_{-3} y_j + \mu_{-2} A_j
\]

and

\[
\dot{\partial}_0 \rho = \rho_{-2} \alpha \alpha_0 + \rho_{-1} \beta_0, \quad \dot{\partial}_0 \rho_0 = \mu_{-1} \alpha \alpha_0 + \mu_0 \beta_0, \tag{46}
\]

\[
\dot{\partial}_0 \rho_{-1} = \mu_{-2} \alpha \alpha_0 + \mu_{-1} \beta_0, \quad \dot{\partial}_0 \rho_{-2} = \mu_{-3} \alpha \alpha_0 + \mu_{-2} \beta_0,
\]

where

\[
\xi_j = \partial_j a_{rs} y^r y^s, \quad \zeta_j = \partial_j A_{r} y^r,
\]
Applying (49) in (16), we obtain

\[ \mu_0 = \frac{1}{2} \alpha^{-1} L_{\beta \beta}, \quad \mu_{-1} = \frac{1}{2} \alpha^{-3} L_{\alpha \beta}, \quad \mu_{-2} = \frac{1}{2} \alpha^{-2} \left( \frac{\nabla}{\alpha} - \frac{1}{\alpha} L_{\alpha \beta} \right), \]

\[ \mu_{-3} = \frac{1}{2} \alpha^{-3} \left( L_{\alpha \alpha} - 3 \alpha^{-1} L_{\alpha \alpha} + 3 \alpha^{-2} L_{\alpha} \right). \]

(47)

In view of (33), we have

\[ 2C_{j h k} = a_{h k} \dot{\partial}_j \rho + A_h A_k \dot{\partial}_j \rho_0 + (\dot{\partial}_j \rho_{-1}) \mathcal{G}_{(h k)} \{ y_h A_k \} \]

\[ + \rho_{-1} \mathcal{G}_{(h k)} \{ a_{h j} A_k \} + (\dot{\partial}_j \rho_{-2}) y_h y_k \]

\[ + \rho_{-2} \mathcal{G}_{(h k)} \{ a_{h j} y_k \}, \]

(48)

where \( \mathcal{G}_{(h k)} \) denotes the interchange of indices \( h \) and \( k \) and addition. Using (45) in (48), we get

\[ 2C_{j h k} = a_{h k} (\rho_{-2} y_j + \rho_{-1} A_j) + A_h A_k (\mu_{-1} y_j + \mu_0 A_j) \]

\[ + (\mu_{-2} y_j + \mu_{-1} A_j) \mathcal{G}_{(h k)} \{ y_h A_k \} + \rho_{-1} \mathcal{G}_{(h k)} \{ a_{h j} A_k \} \]

\[ + (\mu_{-3} y_j + \mu_{-2} A_j) y_h y_k + \rho_{-2} \mathcal{G}_{(h k)} \{ a_{h j} y_k \}. \]

(49)

Applying (49) in (16), we obtain

\[ C_{i j k} = \frac{1}{2} g^{i h} \left[ a_{h k} (\rho_{-2} y_j + \rho_{-1} A_j) + A_h A_k (\mu_{-1} y_j + \mu_0 A_j) \right] \]

\[ + (\mu_{-2} y_j + \mu_{-1} A_j) \mathcal{G}_{(h k)} \{ y_h A_k \} + \rho_{-1} \mathcal{G}_{(h k)} \{ a_{h j} A_k \} \]

\[ + (\mu_{-3} y_j + \mu_{-2} A_j) y_h y_k + \rho_{-2} \mathcal{G}_{(h k)} \{ a_{h j} y_k \} \]

(50)

Differentiating (33) partially with respect to \( t \), we have

\[ 2C_{j h 0} = \dot{\partial}_0 g_{j h} = a_{j h} \dot{\partial}_0 \rho + \alpha a_{j h,0} + A_j A_h \dot{\partial}_0 \rho_0 + \rho_0 \mathcal{G}_{(j h)} \{ A_{j,0} A_h \} \]

\[ + (\dot{\partial}_0 \rho_{-1}) \mathcal{G}_{(j h)} \{ y_j A_h \} + \rho_{-1} \mathcal{G}_{(j h)} \{ y_{j,0} A_h + y_j A_{h,0} \} \]

\[ + (\dot{\partial}_0 \rho_{-2}) y_j y_h + \rho_{-2} \mathcal{G}_{(j h)} \{ y_{j,0} y_h \}, \]

(51)

which, in view of (46) becomes

\[ 2C_{j h 0} = a_{j h} (\rho_{-2} \alpha \alpha + \rho_{-1} \beta_0) + \rho a_{j h,0} + A_j A_h (\mu_{-1} \alpha \alpha + \mu_0 \beta_0) \]

\[ + \rho_0 \mathcal{G}_{(j h)} \{ A_{j,0} A_h \} + (\mu_{-2} \alpha \alpha + \mu_{-1} \beta_0) \mathcal{G}_{(j h)} \{ y_j A_h \} \]

\[ + \rho_{-1} \mathcal{G}_{(j h)} \{ y_{j,0} A_h + y_j A_{h,0} \} + (\mu_{-3} \alpha \alpha + \mu_{-2} \beta_0) y_j y_h \]

\[ + \rho_{-2} \mathcal{G}_{(j h)} \{ y_{j,0} y_h \}. \]
Using (51) in (17), we get

\[ C^i_{j0} = \frac{1}{2} g^{ih} [ a_{jh} (\rho - \alpha, \beta, 0) + \rho a_{jh} + A_j A_h (\mu - \alpha, \beta, 0) + \rho_0 \mathfrak{S}_{(jh)} \{ A_j A_h \} + (\mu - \alpha, \beta, 0) \mathfrak{S}_{(jh)} \{ y_j A_h \} + \rho_0 \mathfrak{S}_{(jh)} \{ y_j A_h \} + (\mu - \alpha, \beta, 0) y_j y_h + \rho_0 \mathfrak{S}_{(jh)} \{ y_j y_h \}] \tag{52} \]

Differentiating (33) partially with respect to \( x_j \), we have

\[ \partial_j g_{hk} = X_{hk} \zeta_j + Y_{hk} \zeta_j + \rho \partial_j a_{hk} + \rho_0 \mathfrak{S}_{(hk)} \{ A_k \partial_j A_h \} + \rho_1 \mathfrak{S}_{(hk)} \{ y_h \partial_j A_k + A_k \partial_j y_h \} + \rho_2 \mathfrak{S}_{(hk)} \{ y_h \partial_j y_k \} \tag{53} \]

where

\[ X_{hk} = \frac{1}{2} \left( \rho - \alpha, \beta, 0 \right) A_h A_k + \mu - \alpha, \beta, 0 \mathfrak{S}_{(hk)} \{ y_h A_k \} + \mu - \alpha, \beta, 0 y_h y_k, \]

\[ Y_{hk} = \rho - \alpha, \beta, 0 A_h A_k + \mu - \alpha, \beta, 0 \mathfrak{S}_{(hk)} \{ y_h A_k \} + \mu - \alpha, \beta, 0 y_h y_k. \]

Now from (8), we have \( \delta_j g_{hk} = (\partial_j - N_j^r \dot{\partial}_r - N_j \dot{\partial}_0) g_{hk} \), which, in view of (49), (51) and (53) yields

\[ \delta_j g_{hk} = X_{hk} \zeta_j + Y_{hk} \zeta_j + \rho \partial_j a_{hk} + \rho_0 \mathfrak{S}_{(hk)} \{ A_k \partial_j A_h \} + \rho_1 \mathfrak{S}_{(hk)} \{ y_h \partial_j A_k + A_k \partial_j y_h \} + \rho_2 \mathfrak{S}_{(hk)} \{ y_h \partial_j y_k \} - 2 N_j^r C_{rhk} - 2 N_j C_{hk0}. \tag{54} \]

Similarly, we have

\[ \delta_k g_{jh} = X_{jh} \zeta_k + Y_{jh} \zeta_k + \rho \partial_k a_{jh} + \rho_0 \mathfrak{S}_{(jh)} \{ A_j \partial_k A_h \} + \rho_1 \mathfrak{S}_{(jh)} \{ y_j \partial_k A_h + A_h \partial_k y_j \} + \rho_2 \mathfrak{S}_{(jh)} \{ y_j \partial_k y_h \} - 2 N_j^r C_{rjh} - 2 N_j C_{j0} \tag{55} \]

and

\[ \delta_h g_{jk} = X_{jk} \zeta_h + Y_{jk} \zeta_h + \rho \partial_h a_{jk} + \rho_0 \mathfrak{S}_{(jk)} \{ A_j \partial_h A_k \} + \rho_1 \mathfrak{S}_{(jk)} \{ y_j \partial_h A_k + A_k \partial_h y_j \} + \rho_2 \mathfrak{S}_{(jk)} \{ y_j \partial_h y_k \} - 2 N_j^r C_{rjk} - 2 N_j C_{j0}. \tag{56} \]
Using (54)-(56) in (15), we have

\[ L_{ij}^k = \rho_{ijk} - \mathcal{S}_{(jk)} \left\{ N_j^m C^i_{mk} + N_j C^i_{k0} \right\} + N_{im} C_{mjk} - N^i C_{jki} \]

\[ + \frac{1}{2} g^{ih} \left[ X_{hk} \xi_j + X_{jh} \xi_k - X_{jk} \xi_h + Y_{hk} \zeta_j + Y_{jh} \zeta_k - Y_{jk} \zeta_h \right] \]

\[ + \rho_0 (A_h \mathcal{S}_{(jk)} \{ \partial_j A_k \} + 2 \mathcal{S}_{(jk)} \{ A_k F_{jh} \}) + \rho_{-1} (y_h \mathcal{S}_{(jk)} \{ \partial_j y_k \}) \]

\[ + A_h \mathcal{S}_{(jk)} \{ \partial_j y_k \} + 2 \mathcal{S}_{(jk)} \{ y_k F_{jh} + A_k K_{jh} \} \]

\[ + \rho_{-2} (y_h \mathcal{S}_{(jk)} \{ \partial_j y_k \} + 2 \mathcal{S}_{(jk)} \{ y_j K_{kh} \}) \]  

(57)

where

\[ N_{im} = g^{ih} N_j^m, \quad N^i = g^{ih} N_h, \quad K_{kh} = \frac{1}{2} (\partial_k y_h - \partial_h y_k) \]

Thus, we have

**Theorem 5.** For a rheonomic Lagrange space with \((\alpha, \beta)\)-metric, endowed with a nonlinear connection whose coefficients are given by (40) and (41), there is a unique canonical metrical \(N\)-linear connection \( \mathcal{C}(N) = (L^i_{jk}, C^i_{jk}, C^i_{j0}) \) with the coefficients given by (50), (52) and (57).

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**References**


