International Journal of Pure and Applied Mathematics
Volume 83 No. 3 2013, 489-499
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v83i3.9

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A $W^{2,p}$-ESTIMATE FOR A CLASS OF ELLIPTIC OPERATORS

Sara Monsurrò[1], Maria Transirico[2 §]

[1,2]Departement of Mathematics
University of Salerno
via Ponte Don Melillo, Fisciano (SA), I-84084, ITALY

Abstract: We prove a $W^{2,p}$-a priori bound, $p > 1$, for a class of uniformly elliptic second order differential operators with discontinuous coefficients in unbounded domains. As an application we obtain the solvability of the related Dirichlet problem.

AMS Subject Classification: 35J25, 35B45, 35R05
Key Words: elliptic equations, discontinuous coefficients, a priori bounds

1. Introduction

Let Ω be an open subset of $\mathbb{R}^n$, $n \geq 2$, with a suitable regularity property. We are interested in the study of the uniformly elliptic second order linear differential operator

$$L = -\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} + a,$$

and of the associated Dirichlet problem

$$\begin{cases} u \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega), \\ Lu = f, \quad f \in L^p(\Omega), \end{cases} \tag{1.1}$$

with $p > 1$, in the framework of discontinuous coefficients.

Received: November 8, 2012

[§]Correspondence author

It is well known that if Ω is bounded, $n \geq 3$ and $p < n$ the sole assumption $a_{ij} = a_{ji} \in L^\infty(\Omega)$ on the leading coefficients do not ensure the uniqueness of the solution of (1.1). While for $n = 2$, it is enough to show existence and uniqueness results just for $p = 2$ or for p sufficiently close to 2 (see [23] and [2]). It has been then a challenging problem to determine what kind of additional conditions on the a_{ij} could be taken into account in order to overcome this difficulty (see for instance [17] for more details on this matter).

Among the various hypotheses, here we consider those of Miranda's type, referring to the classical work [18], where $n \geq 3$, $p = 2$ and the a_{ij} are supposed to have derivatives in $L^n(\Omega)$.

Several authors investigated this setting, both in the cases of bounded and unbounded domains, generalizing the hypotheses of [18]. In particular, for bounded domains we recall for $p = 2$ the results of [1], [12] and [15] and for $p > 1$ the very general case considered in [13] and [14], where the a_{ij} have vanishing mean oscillation (VMO), that is a kind of continuity in the average sense and not in the pointwise sense. Concerning unbounded domains, hypotheses similar to those of [18] have been considered for instance in [25], [26], [5], [6] and [19] for $p = 2$ and in [7] for $p > 1$. Assumptions as those of [13] and [14] have been taken into account in [8], and in [3] and [4] in a weighted case.

Here we deal with unbounded sets and we take the $(a_{ij})_{x_h}$ in suitable Morrey type spaces, that are a generalization to unbounded domains of the classical Morrey spaces, see [27] and [9] for details. Our main result consists in a $W^{2,p}$-bound, $p > 1$, having the only term $\|Lu\|_{L^p(\Omega)}$ in the right hand side,

$$\|u\|_{W^{2,p}(\Omega)} \leq c\|Lu\|_{L^p(\Omega)}, \quad \forall\, u \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega), \tag{1.2}$$

where the dependence of the constant c is explicitly described. It is very rare to obtain this kind of estimate when dealing with unbounded domains. The main point to achieve (1.2) is the existence of the derivatives of the a_{ij} that permits to rewrite the operator L in divergence form and to exploit some estimates for variational operators recently proved in [20], [21] and [22]. Obviously (1.2) immediately takes to the uniqueness of the solution of problem (1.1). An analogous bound, but for the case $p = 2$, can be found in [19], where the $(a_{ij})_{x_h}$ belong to an opportune Morrey type space.

In order to show (1.2) some preliminary tools are needed, they are recalled in Section 2. Section 3 contains our main results and is developed as follows: we start proving a bound as (1.2), but for more regular functions, and then we obtain the claimed $W^{2,p}$-estimate by means of a density argument. We conclude with an application, proving the solvability of the Dirichlet problem associated.

2. Preliminaries

In this section we recall the definitions and some useful properties of the function spaces where the coefficients of our operator belong, we refer the reader also to [25], [27] and [9], for more details.

From now on, we denote by Ω an unbounded open subset of $\mathbb{R}^n$, $n \geq 2$. Let $\Sigma(\Omega)$ be the σ-algebra of all Lebesgue measurable subsets of Ω. Given $E \in \Sigma(\Omega)$, $|E|$ is its Lebesgue measure and $E(x,\rho)$ the intersection $E \cap B(x,\rho)$ ($x \in \mathbb{R}^n, \rho \in \mathbb{R}_+$), where $B(x,\rho)$ is the open ball centered in x and with radius ρ. Moreover $\mathcal{D}^0(\bar{\Omega})$ is the class of restrictions to $\bar{\Omega}$ of the functions $\zeta \in C^0_\circ(\mathbb{R}^n)$ such that $\bar{\Omega} \cap \operatorname{supp} \zeta \subseteq \Omega$.

For $\lambda \in [0, n[$, $q \in [1, +\infty[$, the space of Morrey type $M^{q,\lambda}(\Omega, t)$ ($t \in \mathbb{R}_+$) is the set of all functions g in $L^q_{loc}(\overline{\Omega})$ such that

$$\|g\|_{M^{q,\lambda}(\Omega,t)} = \sup_{\substack{\tau \in]0,t] \\ x \in \Omega}} \tau^{-\lambda/q} \|g\|_{L^q(\Omega(x,\tau))} < +\infty, \tag{2.1}$$

endowed with the norm defined in (2.1). It is easy to verify that, for any $t_1, t_2 \in \mathbb{R}_+$, a function g belongs to $M^{q,\lambda}(\Omega, t_1)$ if and only if it belongs to $M^{q,\lambda}(\Omega, t_2)$ and the norms of g in these two spaces are equivalent. We therefore restrict our attention to the space $M^{q,\lambda}(\Omega) = M^{q,\lambda}(\Omega, 1)$.

Let us introduce three subspaces of $M^{q,\lambda}(\Omega)$ needed in the sequel. By $M^{q,\lambda}_\circ(\Omega)$ and $\tilde{M}^{q,\lambda}(\Omega)$ we denote the closures of $C^\infty_\circ(\Omega)$ and $L^\infty(\Omega)$ in $M^{q,\lambda}(\Omega)$ respectively, while $VM^{q,\lambda}(\Omega)$ is made up of the functions $g \in M^{q,\lambda}(\Omega)$ such that

$$\lim_{t \to 0^+} \|g\|_{M^{q,\lambda}(\Omega,t)} = 0.$$

We explicitly observe that the following inclusions (algebraic and topologic) hold true

$$M^{q,\lambda}_\circ(\Omega) \subset \tilde{M}^{q,\lambda}(\Omega) \subset VM^{q,\lambda}(\Omega).$$

Furthermore

$$M^{q,\lambda}(\Omega) \subseteq M^{q_0,\lambda_0}(\Omega) \quad \text{ if } q_0 \leq q \text{ and } \frac{\lambda_0 - n}{q_0} \leq \frac{\lambda - n}{q}. \tag{2.2}$$

We put $M^q(\Omega) = M^{q,0}(\Omega)$, $VM^q(\Omega) = VM^{q,0}(\Omega)$, $\tilde{M}^q(\Omega) = \tilde{M}^{q,0}(\Omega)$ and $M^q_\circ(\Omega) = M^{q,0}_\circ(\Omega)$.

To define the moduli of continuity of functions belonging to $\tilde{M}^{q,\lambda}(\Omega)$ or $M^{q,\lambda}_\circ(\Omega)$ we set, for $h \in \mathbb{R}_+$ and $g \in M^{q,\lambda}(\Omega)$,

$$F[g](h) = \sup_{\substack{E \in \Sigma(\Omega) \\ \sup_{x \in \Omega} |E(x,1)| \leq \frac{1}{h}}} \|g\, \chi_E\|_{M^{q,\lambda}(\Omega)}.$$

Then we recall that for a function $g \in M^{q,\lambda}(\Omega)$ the following characterizations hold:

$$g \in \tilde{M}^{q,\lambda}(\Omega) \iff \lim_{h\to+\infty} F[g](h) = 0$$

and

$$g \in M_\circ^{q,\lambda}(\Omega) \iff \lim_{h\to+\infty} \big(F[g](h) + ||(1-\zeta_h)g||_{M^{q,\lambda}(\Omega)}\big) = 0,$$

where ζ_h denotes a function of class $C_o^\infty(\mathbb{R}^n)$ such that

$$0 \le \zeta_h \le 1\,, \qquad \zeta_{h|_{\overline{B(0,h)}}} = 1\,, \qquad \operatorname{supp}\zeta_h \subset B(0,2h).$$

Thus if g is a function in $\tilde{M}^{q,\lambda}(\Omega)$ a *modulus of continuity* of g in $\tilde{M}^{q,\lambda}(\Omega)$ is a map $\widetilde{\sigma}^{q,\lambda}[g] : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$F[g](h) \le \widetilde{\sigma}^{q,\lambda}[g](h), \quad \lim_{h\to+\infty} \widetilde{\sigma}^{q,\lambda}[g](h) = 0.$$

While, if g belongs to $M_o^{q,\lambda}(\Omega)$ a *modulus of continuity* of g in $M_o^{q,\lambda}(\Omega)$ is an application $\sigma_o^{q,\lambda}[g] : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$F[g](h) + \|(1-\zeta_h)\, g\|_{M^{q,\lambda}(\Omega)} \le \sigma_o^{q,\lambda}[g](h),$$

$$\lim_{h\to+\infty} \sigma_o^{q,\lambda}[g](h) = 0.$$

Let us also remind the definition of VMO spaces. If Ω has the property

$$|\Omega(x,\rho)| \ge A\,\rho^n \qquad \forall\, x \in \Omega\,, \quad \forall\, \rho \in]0,1]\,,$$

where A is a positive constant independent of x and ρ, it is possible to consider the space $BMO(\Omega,\tau)$ ($\tau \in \mathbb{R}_+$) of functions $g \in L^1_{\text{loc}}(\bar{\Omega})$ such that

$$[g]_{BMO(\Omega,\tau)} = \sup_{\substack{x\in\Omega\\ \rho\in]0,\tau]}} ⨍_{\Omega(x,\rho)} \Big| g - ⨍_{\Omega(x,\rho)} g \Big| < +\infty\,,$$

where

$$⨍_{\Omega(x,\rho)} g = |\Omega(x,\rho)|^{-1} \int_{\Omega(x,\rho)} g.$$

If $g \in BMO(\Omega) = BMO(\Omega,\tau_A)$, where

$$\tau_A = \sup\left\{\tau \in \mathbb{R}_+ \ : \ \sup_{\substack{x\in\Omega\\ \rho\in]0,\tau]}} \frac{\rho^n}{|\Omega(x,\rho)|} \le \frac{1}{A}\right\},$$

we shall say that $g \in VMO(\Omega)$ if $[g]_{BMO(\Omega,\tau)} \to 0$ for $\tau \to 0^+$.
If g belongs to $VMO(\Omega)$, a *modulus of continuity* of g in $VMO(\Omega)$ is function $\eta[g] :]0,1] \to \mathbb{R}_+$ such that

$$[g]_{BMO(\Omega,\tau)} \leq \eta[g](\tau) \qquad \forall\, \tau \in]0,1]\,, \qquad \lim_{\tau\to 0^+} \eta[g](\tau) = 0\,.$$

We will also need the following Lemma, proved in [19]:

Lemma 2.1. *If Ω has the uniform $C^{1,1}$-regularity property and*

$$g, g_x \in \begin{cases} VM^q(\Omega), & q > 2 \text{ for } n = 2, \\ VM^{q,n-q}(\Omega), & q \in]2,n] \text{ for } n > 2, \end{cases}$$

then $g \in VMO(\Omega)$.

We finally adapt to our framework a more general result of [10].

Lemma 2.2. *Let $p > 1$ and $r,\ t \in [p, +\infty[$. If Ω is an open subset of $\mathbb{R}^n$ having the cone property and $g \in M^r(\Omega)$, with $r > p$ if $p = n$, then*

$$u \longrightarrow g\,u \tag{2.3}$$

is a bounded operator from $W^{1,p}(\Omega)$ to $L^p(\Omega)$. Moreover, there exists a constant $c \in \mathbb{R}_+$, such that

$$\|g\,u\|_{L^p(\Omega)} \leq c\,\|g\|_{M^r(\Omega)}\,\|u\|_{W^{1,p}(\Omega)}\,, \tag{2.4}$$

with $c = c\,(\Omega, n, p, r)$.
Furthermore, if $g \in \tilde{M}^r(\Omega)$, then for any $\varepsilon > 0$ there exists a constant $c_\varepsilon \in \mathbb{R}_+$, such that

$$\|g\,u\|_{L^p(\Omega)} \leq \varepsilon\,\|u\|_{W^{1,p}(\Omega)} + c_\varepsilon \|u\|_{L^p(\Omega)}, \tag{2.5}$$

with $c_\varepsilon = c_\varepsilon(\varepsilon, \Omega, n, p, r, \widetilde{\sigma}^r[\,g])$.

If $g \in M^t(\Omega)$, with $t > p$ if $p = n/2$, then the operator in (2.3) is bounded from $W^{2,p}(\Omega)$ to $L^p(\Omega)$. Moreover, there exists a constant $c' \in \mathbb{R}_+$, such that

$$\|g\,u\|_{L^p(\Omega)} \leq c'\,\|g\|_{M^t(\Omega)}\,\|u\|_{W^{2,p}(\Omega)}\,, \tag{2.6}$$

with $c' = c'(\Omega, n, p, t)$.
Furthermore, if $g \in \tilde{M}^t(\Omega)$, then for any $\varepsilon > 0$ there exists a constant $c'_\varepsilon \in \mathbb{R}_+$, such that

$$\|g\,u\|_{L^p(\Omega)} \leq \varepsilon\,\|u\|_{W^{2,p}(\Omega)} + c'_\varepsilon \|u\|_{L^p(\Omega)}, \tag{2.7}$$

with $c'_\varepsilon = c'_\varepsilon(\varepsilon, \Omega, n, p, t, \widetilde{\sigma}^t[\,g])$.

Proof. The proof can be obtained combining Theorem 3.2 and Corollary 3.3 of [10]. □

3. Main Results

Let $p > 1$ and suppose that

$$\Omega \text{ has the uniform } C^{1,1}\text{-regularity property.} \tag{h_0}$$

We consider the differential operator

$$L = -\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \, \partial x_j} + \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} + a \,, \tag{3.1}$$

with the following conditions on the coefficients:

$$\begin{cases} a_{ij} = a_{ji} \in L^\infty(\Omega), \;\; i,j = 1,\dots,n \,, \\ \exists\, \nu > 0 \;\; : \;\; \displaystyle\sum_{i,j=1}^{n} a_{ij}\, \xi_i\, \xi_j \;\geq\; \nu |\xi|^2 \;\; \text{a.e. in } \Omega \,, \; \forall\, \xi \in \mathbb{R}^n, \\ (a_{ij})_{x_h} \in M_o^{s,\lambda}(\Omega)\,, \;\; i,j,h = 1,\dots,n, \text{ with} \\ s > 2 \text{ and } \lambda = 0 \quad \text{for } n = 2\,, \\ s \in]2,n] \text{ and } \lambda = n - s \quad \text{for } n > 2\,. \end{cases} \tag{h_1}$$

$$\begin{cases} a_i \in M_o^r(\Omega)\,, \;\; i = 1,\dots,n\,, \text{ with} \\ r > 2 \text{ if } p \leq 2 \text{ and } r = p \text{ if } p > 2 \quad \text{for } n = 2, \\ r \geq p \text{ and } r \geq n, \text{ with } r > p \text{ if } p = n \quad \text{for } n > 2, \end{cases} \tag{h_2}$$

$$\begin{cases} a \in \tilde{M}^t(\Omega)\,, \text{ with} \\ t = p \quad \text{for } n = 2\,, \\ t \geq p \text{ and } t \geq \dfrac{n}{2}, \text{ with } t > p \text{ if } p = \dfrac{n}{2} \quad \text{for } n > 2, \\ \operatorname*{ess\,inf}_{\Omega} a = a_0 > 0. \end{cases} \tag{h_3}$$

Under the assumptions (h_0) - (h_3) the operator $L : W^{2,p}(\Omega) \to L^p(\Omega)$ is bounded, in view of Lemma 2.2.

Let us show a preliminary lemma.

Lemma 3.1. *Let L be defined in* (3.1). *If hypotheses* (h_0)-(h_3) *are satisfied, then there exists a constant $c \in \mathbb{R}_+$ such that*

$$\|u\|_{W^{2,p}(\Omega)} \leq c \|Lu\|_{L^p(\Omega)}, \quad \forall\, u \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,2}(\Omega) \cap \mathcal{D}^0(\bar{\Omega})\,, \tag{3.2}$$

with $c = c(\Omega, n, \nu, p, r, t, ||a_{ij}||_{L^\infty(\Omega)}, \sigma_o^{s,\lambda}[(a_{ij})_{x_h}], \sigma_o^r[a_i], \widetilde{\sigma}^t[a], a_0)$.

Proof. We put

$$L_0 = -\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \, \partial x_j}$$

and fix $u \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,2}(\Omega) \cap \mathcal{D}^0(\bar{\Omega})$.

By Lemma 2.1 one has that the coefficients a_{ij} belong also to $VMO(\Omega)$, hence Lemma 3.1 of [11] (for $n = 2$) and Theorem 5.1 of [10] (for $n > 2$) apply and therefore there exists a constant $c_1 \in \mathbb{R}_+$ such that

$$\|u\|_{W^{2,p}(\Omega)} \leq c_1(\|L_0 u\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}),$$

with $c_1 = c_1(\Omega, n, \nu, p, ||a_{ij}||_{L^\infty(\Omega)}, \sigma_o^{s,\lambda}[(a_{ij})_{x_h}])$.

Thus one easily has

$$\begin{aligned} \|u\|_{W^{2,p}(\Omega)} \leq c_1(\|Lu\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)} + \\ \textstyle\sum_{i=1}^n \|a_i u_{x_i}\|_{L^p(\Omega)} + \|au\|_{L^p(\Omega)}). \end{aligned} \tag{3.3}$$

Furthermore, Lemma 2.2 gives

$$\begin{cases} \|a_i u_{x_i}\|_{L^p(\Omega)} \leq \varepsilon \|u\|_{W^{2,p}(\Omega)} + c_\varepsilon \|u_{x_i}\|_{L^p(\Omega)}, \\ \\ \|au\|_{L^p(\Omega)} \leq \varepsilon \|u\|_{W^{2,p}(\Omega)} + c'_\varepsilon \|u\|_{L^p(\Omega)}, \end{cases} \tag{3.4}$$

with $c_\varepsilon = c_\varepsilon(\varepsilon, \Omega, n, p, r, \sigma_o^r[a_i])$ and $c'_\varepsilon = c'_\varepsilon(\varepsilon, \Omega, n, p, t, \widetilde{\sigma}^t[a])$.
Moreover, by classical interpolation results one has that there exists a constant $K \in \mathbb{R}_+$ such that

$$\|u_x\|_{L^p(\Omega)} \leq K\varepsilon \|u\|_{W^{2,p}(\Omega)} + \frac{K}{\varepsilon} \|u\|_{L^p(\Omega)}, \tag{3.5}$$

with $K = K(\Omega, p)$. Putting together (3.3), (3.4) and (3.5) we obtain that there exists $c_2 \in \mathbb{R}_+$ such that

$$\|u\|_{W^{2,p}(\Omega)} \leq c_2(\|Lu\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}), \tag{3.6}$$

with $c_2 = c_2(\Omega, n, \nu, p, r, t, ||a_{ij}||_{L^\infty(\Omega)}, \sigma_o^{s,\lambda}[(a_{ij})_{x_j}], \sigma_o^r[a_i], \widetilde{\sigma}^t[a])$.

To conclude it remains to estimate $\|u\|_{L^p(\Omega)}$ in terms of $\|Lu\|_{L^p(\Omega)}$. To this aim we consider the problem

$$\begin{cases} u \in \overset{\circ}{W}{}^{1,2}(\Omega)\,, \\ -\sum_{i,j=1}^{n}(a_{ij}u_{x_i})_{x_j} + \sum_{i=1}^{n}\big(\sum_{j=1}^{n}(a_{ij})_{x_j} + a_i\big)u_{x_i} + au = Lu, \\ Lu \in W^{-1,2}(\Omega)\,. \end{cases} \tag{3.7}$$

Taking into account hypotheses $(h_1) - (h_3)$ and in view of (2.2), we obtain by Corollary 4.1 of [24] that this problem is uniquely solvable. Moreover, since $Lu \in L^2(\Omega) \cap L^p(\Omega)$, from Theorem 4.1 of [22] we get

$$\|u\|_{L^p(\Omega)} \le c_3\|Lu\|_{L^p(\Omega)}, \tag{3.8}$$

with $c_3 = c_3(n, \nu, p, r, t, \sigma_o^{s,\lambda}[(a_{ij})_{x_j}], \sigma_o^{r}[a_i], a_0)$. Combining (3.6) and (3.8) we get (3.2). □

Let us prove now the claimed $W^{2,p}$-estimate, $p > 1$.

Theorem 3.2. *Let L be defined in* (3.1). *If hypotheses* (h_0)-(h_3) *are satisfied, then there exists a constant $c \in \mathbb{R}_+$ such that*

$$\|u\|_{W^{2,p}(\Omega)} \le c\|Lu\|_{L^p(\Omega)}, \quad \forall\, u \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega)\,, \tag{3.9}$$

with $c = c(\Omega, n, \nu, p, r, t, ||a_{ij}||_{L^\infty(\Omega)}, \sigma_o^{s,\lambda}[(a_{ij})_{x_h}], \sigma_o^{r}[a_i], \widetilde{\sigma}^t[a], a_0)$.

Proof. Let $u \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega)$. By Lemma 4.4 in [7] there exists a sequence $(u_h)_{h\in\mathbb{N}}$ such that

$$u_h \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,2}(\Omega) \cap \mathcal{D}^0(\bar{\Omega}), \quad u_h \to u \in W^{2,p}(\Omega). \tag{3.10}$$

Therefore from Lemma 3.1 we deduce that

$$\begin{gathered} \|u_h\|_{W^{2,p}(\Omega)} \le c_0\|Lu_h\|_{L^p(\Omega)} \le \\ c_0\big(\|L(u_h - u)\|_{L^p(\Omega)} + \|Lu\|_{L^p(\Omega)}\big), \end{gathered} \tag{3.11}$$

with $c_0 = c_0(\Omega, n, \nu, p, r, t, ||a_{ij}||_{L^\infty(\Omega)}, \sigma_o^{s,\lambda}[(a_{ij})_{x_h}], \sigma_o^{r}[a_i], \widetilde{\sigma}^t[a], a_0)$.

The boundedness of the operator $L : W^{2,p}(\Omega) \to L^p(\Omega)$ gives then

$$\|u_h\|_{W^{2,p}(\Omega)} \le c\big(\|u_h - u\|_{W^{2,p}(\Omega)} + \|Lu\|_{L^p(\Omega)}\big), \tag{3.12}$$

with c depending on the same constants as c_0. Letting $h \to +\infty$ we conclude the proof, as a consequence of (3.10). □

Corollary 3.3. *If conditions* (h_0)-(h_3) *are satisfied, then the problem*

$$\begin{cases} u \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega), \\ \\ Lu = f, \quad f \in L^p(\Omega), \end{cases} \tag{3.13}$$

is uniquely solvable.

Proof. Let us consider the problem

$$\begin{cases} u \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega)\,, \\ \\ -\Delta u + u = f\,, \quad f \in L^p(\Omega). \end{cases} \tag{3.14}$$

By Theorem 5.2 in [11] (for $n = 2$), Theorem 4.3 of [8] (for $n > 2$) one has that (3.14) is uniquely solvable.

Thus, if we put for each $\tau \in [0, 1]$

$$L_\tau = \tau(L) + (1 - \tau)(-\Delta + 1),$$

in view of Theorem 3.2

$$||u||_{W^{2,p}(\Omega)} \le c||L_\tau u||_{L^p(\Omega)},$$

$$\forall\, u \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega)\,,\ \forall \tau \in [0, 1]\,.$$

The thesis follows then by the method of continuity along a parameter (see, for instance, Theorem 5.2 of [16]). □

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