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LOW SEPARATION AXIOMS IN MINIMAL STRUCTURES

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Abstract: In this work we generalize some separation notions between T_0 and T_1 to $m-$spaces. In addition, we study the inclusion relations between the corresponding spaces and prove that the resultant diagram is different from the one in the topological case. Finally, we use the concepts of $m-$kernel, $m-$shell, $m-$derived and $m-$closure to characterize the new separation notions.

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1. Introduction

Separation axioms constitute a classical topic in General Topology. Its systematic study began with the works of Urysohn in 1925 ([8]). Van Est and Freudenthal ([9]) studied in more detail some axioms stronger than T_1.

The development of the separation axioms between T_0 and T_1 started with the work of Young in 1943 ([11]). Later in 1961, Aull and Thron ([1]) introduced new axioms and found that they all can be described in terms of the derived set of singletons.

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The study of more general concepts than the topological structure has taken several directions over the past fifteen years. Maki in 1996 ([4]), studied *minimal structures* (or $m-structures$) on a set X, that is, collections of subsets of X containing the empty set and X, with no other restriction. On the other hand, Császár since 1997 has studied topological notions in collections which are closed under unions ([3]). They constitute the well-known *generalized topologies*.

Many classical topological notions have been studied in $m-$spaces and generalized topologies (see [5], [6], [2], [7], [10] and the literature quoted therein). However, the study of separation axioms have been limited to T_0, T_1 and stronger conditions. This paper is, nevertheless, devoted to the study of some low separation axioms between $m-T_0$ and $m-T_1$.

In Section 1 we present the most important separation axioms between T_0 and T_1 studied in [1] and some basic concepts of $m-$spaces. Following [1], in Section 2 we define some separation notions between $m-T_0$ and $m-T_1$ and study the inclusion relations between the corresponding spaces. Finally, in Section 3, we use the concepts of $m-$kernel, $m-$shell, $m-$derived and $m-$closure to characterize the new separation notions.

2. Preliminaries

In this work we will use the classical notations for the derived and the closure of a set. We start with the well known axioms between T_0 and T_1 defined in [1]. We shall simply x to denote the singleton $\{x\}$. In addition, we use the notation $A \vdash B$ to indicate that there exists an open set G such that $A \subseteq G$ and $G \cap B = \emptyset$. Moreover, the derived set and the closure set of A will be denoted by $Der(A)$ and $Cl(A)$ respectively.

Definition 1. The topological space (X, τ) is called:

1. T_0 if for each $p, q \in X$ $(p \neq q)$, there exists an open set G such that $p \in G$, $q \notin G$ or $q \in G$, $p \notin G$.

2. T_1 if for each $p, q \in X$ $(p \neq q)$, there exist open sets G, H such that $p \in G, q \notin G$ and $q \in H, p \notin H$.

3. T_D if for each $x \in X$, $Der(x)$ is closed.

4. T_{UD} if for each $x \in X$, $Der(x)$ is the union of disjoint closed sets.

5. T_{DD} if it is T_D and for each $x, y \in X$ $(x \neq y)$, $Der(x) \cap Der(y) = \emptyset$.

6. T_F if for any point x and any finite subset F of X such that $x \notin F$, either $x \vdash F$ or $F \vdash x$.

7. T_{FF} if for any two finite subsets F_1 and F_2 of X with $F_1 \cap F_2 = \emptyset$, either $F_1 \vdash F_2$ or $F_2 \vdash F_1$.

8. T_Y if for each $x, y \in X$ $(x \neq y)$, $Cl(x) \cap Cl(y)$ is either a singleton or the empty set.

9. T_{YS} if for each $x, y \in X$ $(x \neq y)$, $Cl(x) \cap Cl(y)$ is either $\emptyset$ or x or y.

From the notions defined above it is obtained the following diagram of strict implications ([1]).

$$\begin{array}{ccccccccccc}
T_1 & \longrightarrow & T_{DD} & & \longrightarrow & & T_D & & \\
 & & \downarrow & & & & & & \\
\downarrow & & T_{YS} & & & & \downarrow & & \\
 & & \downarrow & & & & & & \\
T_{FF} & \longrightarrow & T_Y & \longrightarrow & T_F & \longrightarrow & T_{UD} & \longrightarrow & T_0
\end{array}$$

The above axioms can be characterized in terms of the derived, the closure, the shell and the kernel of singletons ([1]). The kernel of $A \subseteq X$, denoted by $\widehat{A}$, is defined as the intersection of the open sets containing A and the shell of A, denoted by $\overset{\frown}{A}$, as $\widehat{A} \setminus A$.

Definition 2. A minimal structure or an $m-$structure on the nonempty set X, is a class m of subsets of X such that $\emptyset, X \in m$.

For a nonempty set X and an $m-$structure m on X, the pair (X, m) is called $m-$space. Each element of m is said to be $m-$open set and the complement of an $m-$open set is an $m-$closed set. Note that the union and intersection of $m-$open sets are not generally $m-$open sets; thus each topological space is an $m-$space and it is easy to see that there exist $m-$spaces that are not topological spaces.

As in the topological case, if X is an $m-$space and $A \subseteq X$ the $m-$closure of A $(m - Cl(A))$ is the intersection of all $m-$closed sets containing A. It is clear that if A is $m-$closed then $m - Cl(A) = A$ but the converse is not true in general. We will say that the collection m satisfies the Maki's condition ([4]) if m is closed under unions. Thus under the Maki's condition the intersection of $m-$closed sets is an $m-$closed and hence also the $m-$closure of any subset.

The $m-$derived set $(m - Der)$ is defined similarly to the topological case and it is proved that $m - Cl(A) = A \cup m - Der(A)$ for any $A \subseteq X$.

We finish this section with some special classes of sets, which we will use to characterize some separation axioms in $m-$spaces.

Definition 3. Let X be an m-space and $A \subseteq X$. Then:

1. The m-kernel of A is $m - \widehat{A} = \bigcap\{G \subseteq X : G \in m, A \subseteq G\}$.
2. The m-shell of A is $m - \overset{\frown}{A} = m - \widehat{A} \backslash A$.
3. $m - \langle A \rangle = m - Cl(A) \cap m - \widehat{A}$.

The following properties of the notions defined above will be used in the final section.

Proposition 4. *Let X be an $m-$space and $x, y \in X$. The following statements hold:*

1. *If $y \in m - \widehat{x}$, then $m - \widehat{y} \subseteq m - \widehat{x}$.*
2. *$y \in m - \widehat{x}$ iff $x \in m - Cl(y)$.*
3. *$y \in m - \overset{\frown}{x}$ iff $x \in m - Der(y)$.*
4. *For each $p \in X$, $m - \overset{\frown}{p}$ is either the empty set or a singleton iff for each $x, y \in X$ $(x \neq y)$, $m - Der(x) \cap m - Der(y) = \emptyset$.*
5. *If $y \in m - \langle x \rangle$, then $m - \langle y \rangle = m - \langle x \rangle$.*
6. *For each $x, y \in X$, either $m - \langle x \rangle \cap m - \langle y \rangle = \emptyset$ or $m - \langle x \rangle = m - \langle y \rangle$.*

3. Some Low Separation Axioms in m-Spaces

Inspired by [1] we define some classes of $m-$spaces, which are related to $m - T_0$ and $m - T_1$ spaces. We recall that the $m-$space X is called $m - T_0$ if for each $x, y \in X$, $(x \neq y)$ either $x \vdash y$ or $y \vdash x$; and it is said to be $m - T_1$ if for each $x, y \in X$, $(x \neq y)$ one has that $x \vdash y$ and $y \vdash x$.

Definition 5. The $m-$space X is called:

1. $m - T_D$ if for each $x \in X$, $m - Der(x)$ is m-closed.

2. $m - T_{UD}$ if for each $x \in X$, $m - Der(x)$ is the union of disjoint m-closed sets.

3. $m - T_{DD}$ if it is $m - T_D$ and for each $x, y \in X$ $(x \neq y)$, $m - Der(x) \cap m - Der(y) = \emptyset$.

4. $m - T_F$ if for any point x and any finite subset F of X such that $x \notin F$, either $x \vdash F$ or $F \vdash x$.

5. $m - T_{FF}$ if for any two finite subsets F_1 and F_2 of X with $F_1 \cap F_2 = \emptyset$, either $F_1 \vdash F_2$ or $F_2 \vdash F_1$.

6. $m - T_Y$ if for each $x, y \in X$ $(x \neq y)$, $m - Cl(x) \cap m - Cl(y)$ is either a singleton or the empty set.

7. $m - T_{YS}$ if for each $x, y \in X$ $(x \neq y)$, $m - Cl(x) \cap m - Cl(y)$ is either $\emptyset$ or x or y.

8. $m - T(\alpha)$ if for each $y \in X$, $x \in m - Der(y)$ implies $m - Der(x) = \emptyset$.

In the following we will determine the relationship between the classes of $m-$spaces defined above. Note first that the implications $m - T_1 \to m - T_{DD}$, $m - T_D \to m - T_{UD}$ and $m - T_{DD} \to m - T_{YS}$ are evident.

Proposition 6. *The following affirmations hold:*

1. *Every $m - T_{UD}$ space is $m - T_0$.*

2. *Every $m - T_{FF}$ space is $m - T_Y$.*

3. *Every $m - T_F$ space is $m - T(\alpha)$.*

4. *Every $m - T(\alpha)$ space is $m - T_0$.*

5. *Every $m - T_Y$ space is $m - T(\alpha)$.*

Proof. 1. For $x, y \in X$ $(x \neq y)$ if $y \in m - Der(x)$, then there exists an $m-$closed set H such that $x \in H^c$ and $y \notin H^c$. Since the other case is evident we conclude that X is $m - T_0$.

2. If there exist $x, y \in X$ $(x \neq y)$ such that the set $m - Cl(x) \cap m - Cl(y)$ is neither $\emptyset$ nor a singleton, then there exist $p, q \in X$ $(p \neq q)$ such that $p, q \in m - Cl(x) \cap m - Cl(y)$. If $\{p, q\} \cap \{x, y\} = \emptyset$ then for $F_1 = \{p, x\}$ and $F_2 = \{q, y\}$ we have that $F_1 \not\vdash F_2$ and $F_2 \not\vdash F_1$, a contradiction. If $p = x$ and $q \neq y$ then for $F_1 = \{x\}$, $F_2 = \{q, y\}$ is again obtained a contradiction. Finally, since the

other cases are impossible, we conclude that $m - Cl(x) \cap m - Cl(y)$ is either $\emptyset$ or a singleton, that is, X is $m - T_Y$.

3. Let $y \in X$ and $x \in m - Der(y)$. If $p \in m - Der(x)$ and $p = y$, then $F = \{y\} \not\vdash x$ and $x \not\vdash F$ which is a contradiction. The case $p \in m - Der(x)$ and $p \neq y$ also leads to a contradiction. Hence $m - Der(x) = \emptyset$ and thus X is $m - T(\alpha)$.

4. Let $x, y \in X$ with $x \neq y$. If $x \in m - Der(y)$, then by assumption $y \notin m - Der(x)$ and the result follows. The other case is analogous, so that X is $m - T_0$.

5. Let $y \in X$ and $x \in m - Der(y)$. If $p \in m - Der(x)$ and $p = y$, then $\{x, y\} \subseteq m - Cl(x) \cap m - Cl(y)$ which is a contradiction. Analogously the case $p \neq y$ leads to a contradiction. Hence $m - Der(x) = \emptyset$ and thus X is $m - T(\alpha)$. □

The topological version of the $m - T(\alpha)$ notion was introduced in [1] (Theorem 3.2). However, it is equivalent to the notion T_F, which is not true in $m-$spaces (Example 7 (2)).

In the topological case the implications $T_1 \to T_{FF}$, $T_Y \to T_F$, $T_Y \to T_{UD}$, $T_F \to T_{UD}$ hold. However, these implications are not true in general in $m-$spaces, as we show in the following example.

Example 7. 1. Consider $\mathbb{R}$ and $m = \{\emptyset, \mathbb{R}\} \cup \{\mathbb{R} - \{x\} : x \in \mathbb{R}\}$. It is clear that this space is $m - T_1$. Let $F_1 = \{0, 1\}$, $F_2 = \{2, 3\}$. The $m-$open sets containing F_1 have the form $\mathbb{R} - \{z\}$ with $z \neq 0, 1$. If $z \neq 2, 3$ then $\mathbb{R} - \{z\} \cap F_2 = F_2$ and so $F_1 \not\vdash F_2$. If either $z = 2$ or 3 then $\mathbb{R} - \{z\} \cap F_2 = F_2 - \{z\}$ and $F_1 \not\vdash F_2$. Analogously $F_2 \not\vdash F_1$ and so this space is not $m - T_{FF}$. In conclusion $m - T_1 \not\to m - T_{FF}$.

2. Let $X = \{1, 2, 3\}$ and $m = \{\emptyset, X, \{1\}, \{1, 2\}, \{2, 3\}, \{3\}\}$. Since for each $x \in X$, $m - Der(x) = \emptyset$ then this space is $m - T_Y$. For $2 \in X$ and $F = \{1, 3\}$ we have that $\{2\} \not\vdash F$ and $F \not\vdash \{2\}$, so this space is not $m - T_F$. This example shows that $m - T_Y \not\to m - T_F$ and $m - T(\alpha) \not\to m - T_F$.

3. Consider $\mathbb{R}$ and $m = \{\emptyset, \mathbb{R}\} \cup \{A \subseteq \mathbb{R} : 0 \notin A,\ A\ is\ finite\}$. If $x \neq 0$ then $m - Der(x) = \{0\}$ and if $x = 0$ then $m - Der(0) = \emptyset$, so this space is $m - T_Y$. Since the nonempty $m-$closed sets are infinite, the set $m - Der(x) = \{0\}$ can not be expressed as a union of disjoint $m-$closed sets. Then this space is not $m - T_{UD}$ and thus $m - T_Y \not\to m - T_{UD}$.

4. Let X be an infinite set, $p \in X$ a fixed element and $m = \{\emptyset, X\} \cup \{A \subseteq X : p \in A,\ |A| \geq 3\ and\ A\ is\ finite\}$. Let $x \in X$ and F a finite subset of X with $x \notin F$. If $x = p$ then $p \vdash F$. If $x \neq p$ and $p \in F$ then $F \vdash x$ and if $p \notin F$ then $x \vdash F$. Hence this space is $m - T_F$. Since the set $m - Der(p) = X - \{p\}$

can not be expressed as a union of disjoint $m-$closed sets we conclude that X is not $m-T_{UD}$. Thus, $m-T_F \not\rightarrow m-T_{UD}$ and $m-T(\alpha) \not\rightarrow m-T_{UD}$.

In conclusion we obtain the following diagram of implications.

$$\begin{array}{ccccc}
m-T_1 & \longrightarrow & m-T_{DD} & \longrightarrow & m-T_D \\
 & & \downarrow & & \\
 & & m-T_{YS} & & \downarrow \\
 & & \downarrow & & \\
m-T_{FF} & \longrightarrow & m-T_Y & & m-T_{UD} \\
\downarrow & & \downarrow & & \downarrow \\
m-T_F & \longrightarrow & m-T(\alpha) & \longrightarrow & m-T_0
\end{array}$$

Since each topological space is in particular an $m-$space the implications $m-T_1 \to m-T_{DD} \to m-T_D \to m-T_{UD} \to m-T_0$, $m-T_{DD} \to m-T_{YS} \to m-T_Y$ and $m-T_{FF} \to m-T_Y$ are strict. Moreover, since each $m-T_Y$ space is $m-T(\alpha)$ (Proposition 6 (5)) the Example 7 (2) shows that $m-T(\alpha) \not\rightarrow m-T_F$. The remaining implications are also strict as we show below.

Example 8. 1. For $X=\{1,2,3,4\}$ and $m=\{\emptyset, X, \{1\}, \{2\}, \{1,2,3\}, \{1,2,4\}\}$ we have that $m-Der(1)=m-Der(2)=\{3,4\}$ and $m-Der(3)=m-Der(4)=\emptyset$. So, this space is $m-T(\alpha)$. Moreover, $m-Cl(1) \cap m-Cl(2)=\{3,4\}$, that is, X is not $m-T_Y$. Then $m-T(\alpha) \not\rightarrow m-T_Y$.

2. Consider $\mathbb{R}$ with $m=\{\emptyset,\mathbb{R}\} \cup \{[x,\infty) : x \in \mathbb{R}\}$. It is clear that this space is $m-T_0$ and it is not $m-T(\alpha)$. Thus, $m-T_0 \not\rightarrow m-T(\alpha)$.

3. Let X be a set such that $|X| \geq 4$ and $m=\{\emptyset, X\} \cup \{\{x\} : x \in X\}$. For different elements $a,b,c,d \in X$ one has that $F_1=\{a,b\} \not\vdash F_2=\{c,d\}$ and $F_2 \not\vdash F_1$. So, this space is not $m-T_{FF}$ but it is $m-T_F$. Hence $m-T_F \not\rightarrow m-T_{FF}$.

Finally, it is easy to see that each topological space T_i is an $m-T_i$ space and this implication is strict. Therefore, the class of topological spaces T_i is strictly contained in the class of $m-T_i$ spaces.

4. Characterizations

In what follows, we use the concepts of $m-$derived, $m-$closure, $m-$shell and $m-$kernel to characterize some low separation axioms in $m-$spaces. We start with several characterizations of the $m-T_0$ spaces. This result extends the

Theorem 2.3 of [1], the proof follows of the definitions and the application of Proposition 4.

Theorem 9. *Let X be an m-space. The following conditions are equivalent:*

1. *X is m-T_0.*
2. *If $y \in m - Cl(x)$ with $y \neq x$, then $x \notin m - Cl(y)$.*
3. *If $y \in m - Der(x)$, then $m - Cl(y) \subseteq m - Der(x)$.*
4. *If $y \in m - \overset{\frown}{x}$, then $m - \widehat{y} \subseteq m - \overset{\frown}{x}$.*
5. *For each $x \in X$, $m - Der(x) \cap m - \overset{\frown}{x} = \emptyset$.*
6. *For each $x \in X$, $m - \langle x \rangle = x$.*

In a similar way it can be obtained the following characterizations of the $m - T_1$ spaces. This result extends Theorem 2.4 of [1].

Theorem 10. *Let X be an m-space. The following conditions are equivalent:*

1. *X is m-T_1.*
2. *For each $x \in X$, $m - Cl(x) = x$.*
3. *For each $x \in X$, $m - Der(x) = \emptyset$.*
4. *For each $x \in X$, $m - \widehat{x} = x$.*
5. *For each $x \in X$, $m - \overset{\frown}{x} = \emptyset$.*
6. *For each $x, y \in X$ with $x \neq y$, $m - Cl(x) \cap m - Cl(y) = \emptyset$.*
7. *For each $x, y \in X$ with $x \neq y$, $m - \widehat{x} \cap m - \widehat{y} = \emptyset$.*
8. *$m - N_D = X$, where $m - N_D = \{x \in X : m - Der(x) = \emptyset\}$.*
9. *$m - N_S = X$, where $m - N_S = \{x \in X : m - \overset{\frown}{x} = \emptyset\}$.*

It is well known that the topological spaces T_D are characterized as those where each singleton is the intersection of a closed set with an open set ([1], Theorem 3.1). However, in the $m-$spaces this property is not true in general but if the Maki's condition is satisfied then the result is verified.

Theorem 11. *If X is m-T_D, then each singleton is the intersection of an m-open set with the m-closure of some subset of X. The converse is true if m satisfies the Maki's condition.*

Proof. For each $x \in X$ we have that $(m - Der(x))^c \cap (m - Cl(x)) = x$ and so the result follows.

Conversely, if for each $x \in X$, $x = G_x \cap m - Cl(A_x)$ where G_x is an $m-$open and $A_x \subseteq X$, then $x = G_x \cap m - Cl(x)$ and so $m - Der(x) = m - Cl(x) \cap G_x^c$. Thus by the Maki's condition we obtain that $m - Der(x)$ is $m-$closed. □

In $m-$spaces the $m - T(\alpha)$ notion can be used to characterize the $m - T_Y$ axiom. This situation is analogous to the topological case where the T_F notion is used ([1], Theorem 3.6).

Theorem 12. *Let X be an $m-$space. The following conditions are equivalent:*

1. *X is $m - T_Y$.*
2. *X is $m - T(\alpha)$ and for each $x, y \in X$ with $x \neq y$ the set $m - Der(x) \cap m - Der(y)$ is either empty or a singleton.*
3. *X is $m - T(\alpha)$ and for each $x, y \in X$ with $x \neq y$ the set $m - \overset{\frown}{x} \cap m - \overset{\frown}{y}$ is either empty or a singleton.*
4. *For each $x, y \in X$ with $x \neq y$ the set $m - \widehat{x} \cap m - \widehat{y}$ is either empty or a singleton.*

Proof. The first implication is evident. For the remaining it is enough to reason by contradiction and to apply 3 and 4 of Proposition 4. □

Similar to the last theorem the $m - T(\alpha)$ notion can be used to characterize the $m - T_{YS}$ axiom.

Theorem 13. *X is $m - T_{YS}$ iff X is $m - T(\alpha)$ and for each $x, y \in X$ with $x \neq y$, $m - Der(x) \cap m - Der(y) = \emptyset$.*

Proof. It is clear that X is $m - T(\alpha)$. If for each $x, y \in X$ $(x \neq y)$ there exists $p \in m - Der(x) \cap m - Der(y)$, then $p \in m - Cl(x) \cap m - Cl(y)$, which is false. So the result follows.

Conversely, for each $x, y \in X$ $(x \neq y)$ we have that $m - Cl(x) \cap m - Cl(y) = (x \cap m - Der(y)) \cup (y \cap m - Der(x))$. Since X es $m - T(\alpha)$, that union of sets is necessarily different from $\{x, y\}$. Thus X is $m - T_{YS}$. □

Theorem 14. *Let X be an $m-$space. If for each $x, y \in X$ $(x \neq y)$, $m - Der(x) \vdash m - Der(y)$ or $m - Der(y) \vdash m - Der(x)$, then X is $m - T_{YS}$. The converse is true if for each $x \in X$, $m - Cl(x)$ is $m-$closed.*

Proof. If there exist $x, y \in X$ $(x \neq y)$ and p different from x and y such that $p \in m - Cl(x) \cap m - Cl(y)$, then $p \in m - Der(x)$ and $p \in m - Der(y)$. Thus we obtain a contradiction $m - Der(x) \nvdash m - Der(y)$ and $m - Der(y) \nvdash m - Der(x)$.

Conversely, if $m - Cl(x) \cap m - Cl(y) = \emptyset$ for $x \neq y$ then $m - Der(y) \vdash m - Der(x)$ with the $m-$open set $(m - Cl(x))^c$. If $m - Cl(x) \cap m - Cl(y) = x$, then $m - Der(x) \vdash m - Der(y)$ with the $m-$open set $(m - Cl(y))^c$. Since the other case is analogous we obtain the result. □

Note that the previous result suggests the existence of a new separation notion between $m - T_{DD}$ and $m - T_{YS}$, since in each $m - T_{DD}$ space for each $x \neq y$ one has that $m - Der(x) \vdash m - Der(y)$ and $m - Der(y) \vdash m - Der(x)$.

We finish this work with the characterization of the $m - T(\alpha)$ notion, which extends Theorem 3.4 of [1] to $m-$spaces.

Theorem 15. *Let X be an $m-$space. The following conditions are equivalent:*

1. *X is $m - T(\alpha)$.*
2. *For each $x, y \in X$ $(x \neq y)$, $m - Der(x) \cap m - \widehat{y} = \emptyset$.*
3. *$N_S \cup N_D = X$.*
4. *For each $x \in X$, $y \in m - \widehat{x}$ implies $m - \widehat{y} = \emptyset$.*

Proof. $1 \Rightarrow 2$. If $p \in m - Der(x) \cap m - \widehat{y}$ for $x \neq y$, then we would have $p \in m - Der(x)$ and $y \in m - Der(p)$ (Proposition 4 (4)), which contradicts 1. Thus 2 follows.

$2 \Rightarrow 3$. If there exists $x \in X$ such that $m - Der(x) \neq \emptyset$ and $m - \widehat{x} \neq \emptyset$, then there exist $p, q \in X$ $(p \neq q)$ such that $x \in m - \widehat{p} \cap m - Der(q)$ (Proposition 4 (4)), which contradicts 2. Hence $N_S \cup N_D = X$.

$3 \Rightarrow 4$. If $y \in m - \widehat{x}$ and $m - \widehat{y} \neq \emptyset$, then $x \in m - Der(y)$ (Proposition 4 (4)) and by 3 $m - Der(y) = \emptyset$, which is a contradiction. Then 4 follows.

$4 \Rightarrow 1$. If $x \in m - Der(y)$ and $m - Der(x) \neq \emptyset$, then we would have $y \in m - \widehat{x}$ and $x \in m - \widehat{p}$ for some $p \in X$ (Proposition 4 (4)), which contradicts 4. It follows that X is $m - T(\alpha)$. □

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