LOW SEPARATION AXIOMS IN MINIMAL STRUCTURES

Jesús Ávila¹, Erika Durán², Ximena Moya³
¹,²,³Department of Mathematics and Statistics
University of Tolima
Ibagué, COLOMBIA

Abstract: In this work we generalize some separation notions between $T_0$ and $T_1$ to $m$–spaces. In addition, we study the inclusion relations between the corresponding spaces and prove that the resultant diagram is different from the one in the topological case. Finally, we use the concepts of $m$–kernel, $m$–shell, $m$–derived and $m$–closure to characterize the new separation notions.

AMS Subject Classification: 54A05, 54D10
Key Words: separation axioms, $m$–spaces, $m$–open, $m$–closed, $m$–kernel, $m$–shell, $m$–derived, $m$–closure

1. Introduction

Separation axioms constitute a classical topic in General Topology. Its systematic study began with the works of Urysohn in 1925 ([8]). Van Est and Freudenthal ([9]) studied in more detail some axioms stronger than $T_1$.

The development of the separation axioms between $T_0$ and $T_1$ started with the work of Young in 1943 ([11]). Later in 1961, Aull and Thron ([1]) introduced new axioms and found that they all can be described in terms of the derived set of singletons.

Received: August 4, 2012  © 2013 Academic Publications, Ltd.
Correspondence author
The study of more general concepts than the topological structure has taken several directions over the past fifteen years. Maki in 1996 ([4]), studied minimal structures (or $m-$structures) on a set $X$, that is, collections of subsets of $X$ containing the empty set and $X$, with no other restriction. On the other hand, Császár since 1997 has studied topological notions in collections which are closed under unions ([3]). They constitute the well-known generalized topologies.

Many classical topological notions have been studied in $m-$spaces and generalized topologies (see [5], [6], [2], [7], [10] and the literature quoted therein). However, the study of separation axioms have been limited to $T_0$, $T_1$ and stronger conditions. This paper is, nevertheless, devoted to the study of some low separation axioms between $m-T_0$ and $m-T_1$.

In Section 1 we present the most important separation axioms between $T_0$ and $T_1$ studied in [1] and some basic concepts of $m-$spaces. Following [1], in Section 2 we define some separation notions between $m-T_0$ and $m-T_1$ and study the inclusion relations between the corresponding spaces. Finally, in Section 3, we use the concepts of $m-$kernel, $m-$shell, $m-$derived and $m-$closure to characterize the new separation notions.

2. Preliminaries

In this work we will use the classical notations for the derived and the closure of a set. We start with the well known axioms between $T_0$ and $T_1$ defined in [1]. We shall simply $x$ to denote the singleton $\{x\}$. In addition, we use the notation $A \vdash B$ to indicate that there exists an open set $G$ such that $A \subseteq G$ and $G \cap B = \emptyset$. Moreover, the derived set and the closure set of $A$ will be denoted by $\text{Der}(A)$ and $\text{Cl}(A)$ respectively.

**Definition 1.** The topological space $(X, \tau)$ is called:

1. $T_0$ if for each $p, q \in X$ ($p \neq q$), there exists an open set $G$ such that $p \in G$, $q \notin G$ or $q \in G$, $p \notin G$.

2. $T_1$ if for each $p, q \in X$ ($p \neq q$), there exist open sets $G, H$ such that $p \in G, q \notin G$ and $q \in H, p \notin H$.

3. $T_D$ if for each $x \in X$, $\text{Der}(x)$ is closed.

4. $T_{UD}$ if for each $x \in X$, $\text{Der}(x)$ is the union of disjoint closed sets.

5. $T_{DD}$ if it is $T_D$ and for each $x, y \in X$ ($x \neq y$), $\text{Der}(x) \cap \text{Der}(y) = \emptyset$. 
6. $T_F$ if for any point $x$ and any finite subset $F$ of $X$ such that $x \notin F$, either $x \vdash F$ or $F \vdash x$.

7. $T_{FF}$ if for any two finite subsets $F_1$ and $F_2$ of $X$ with $F_1 \cap F_2 = \emptyset$, either $F_1 \vdash F_2$ or $F_2 \vdash F_1$.

8. $T_Y$ if for each $x, y \in X \ (x \neq y)$, $Cl(x) \cap Cl(y)$ is either a singleton or the empty set.

9. $T_{YS}$ if for each $x, y \in X \ (x \neq y)$, $Cl(x) \cap Cl(y)$ is either $\emptyset$ or $x$ or $y$.

From the notions defined above it is obtained the following diagram of strict implications ([1]).

$$
\begin{array}{c}
T_1 \rightarrow T_{DD} \rightarrow T_D \\
\downarrow \\
T_{YS} \downarrow \\
T_{FF} \rightarrow T_Y \rightarrow T_F \rightarrow T_{UD} \rightarrow T_0
\end{array}
$$

The above axioms can be characterized in terms of the derived, the closure, the shell and the kernel of singletons ([1]). The kernel of $A \subseteq X$, denoted by $\hat{A}$, is defined as the intersection of the open sets containing $A$ and the shell of $A$, denoted by $\check{A}$, as $\hat{A} \setminus A$.

**Definition 2.** A minimal structure or an $m-$structure on the nonempty set $X$, is a class $m$ of subsets of $X$ such that $\emptyset, X \in m$.

For a nonempty set $X$ and an $m-$structure $m$ on $X$, the pair $(X, m)$ is called an $m-$space. Each element of $m$ is said to be $m-$open set and the complement of an $m-$open set is an $m-$closed set. Note that the union and intersection of $m-$open sets are not generally $m-$open sets; thus each topological space is an $m-$space and it is easy to see that there exist $m-$spaces that are not topological spaces.

As in the topological case, if $X$ is an $m-$space and $A \subseteq X$ the $m-$closure of $A$ ($m - Cl(A)$) is the intersection of all $m-$closed sets containing $A$. It is clear that if $A$ is $m-$closed then $m - Cl(A) = A$ but the converse is not true in general. We will say that the collection $m$ satisfies the Maki’s condition ([4]) if $m$ is closed under unions. Thus under the Maki’s condition the intersection of $m-$closed sets is an $m-$closed and hence also the $m-$closure of any subset.
The $m-$derived set ($m - Der$) is defined similarly to the topological case and it is proved that $m - Cl(A) = A \cup m - Der(A)$ for any $A \subseteq X$.

We finish this section with some special classes of sets, which we will use to characterize some separation axioms in $m-$spaces.

**Definition 3.** Let $X$ be an $m$-space and $A \subseteq X$. Then:

1. The $m$-kernel of $A$ is $m - \hat{A} = \bigcap \{G \subseteq X : G \in m, A \subseteq G\}$.
2. The $m$-shell of $A$ is $m - \check{A} = m - \hat{A}\setminus A$.
3. $m - \langle A \rangle = m - Cl(A) \cap m - \hat{A}$.

The following properties of the notions defined above will be used in the final section.

**Proposition 4.** Let $X$ be an $m-$space and $x, y \in X$. The following statements hold:

1. If $y \in m - \hat{x}$, then $m - \hat{y} \subseteq m - \hat{x}$.
2. $y \in m - \hat{x}$ iff $x \in m - Cl(y)$.
3. $y \in m - \hat{x}$ iff $x \in m - Der(y)$.
4. For each $p \in X$, $m - \hat{p}$ is either the empty set or a singleton iff for each $x, y \in X$ ($x \neq y$), $m - Der(x) \cap m - Der(y) = \emptyset$.
5. If $y \in m - \langle x \rangle$, then $m - \langle y \rangle = m - \langle x \rangle$.
6. For each $x, y \in X$, either $m - \langle x \rangle \cap m - \langle y \rangle = \emptyset$ or $m - \langle x \rangle = m - \langle y \rangle$.

### 3. Some Low Separation Axioms in $m$-Spaces

Inspired by [1] we define some classes of $m-$spaces, which are related to $m - T_0$ and $m - T_1$ spaces. We recall that the $m-$space $X$ is called $m - T_0$ if for each $x, y \in X$, ($x \neq y$) either $x \vdash y$ or $y \vdash x$; and it is said to be $m - T_1$ if for each $x, y \in X$, ($x \neq y$) one has that $x \vdash y$ and $y \vdash x$.

**Definition 5.** The $m-$space $X$ is called:

1. $m - T_D$ if for each $x \in X$, $m - Der(x)$ is $m$-closed.
2. $m - T_{UD}$ if for each $x \in X$, $m - Der(x)$ is the union of disjoint $m$-closed sets.

3. $m - T_{DD}$ if it is $m - T_D$ and for each $x, y \in X$ ($x \neq y$), $m - Der(x) \cap m - Der(y) = \emptyset$.

4. $m - T_F$ if for any point $x$ and any finite subset $F$ of $X$ such that $x \notin F$, either $x \vdash F$ or $F \vdash x$.

5. $m - T_{FF}$ if for any two finite subsets $F_1$ and $F_2$ of $X$ with $F_1 \cap F_2 = \emptyset$, either $F_1 \vdash F_2$ or $F_2 \vdash F_1$.

6. $m - T_Y$ if for each $x, y \in X$ ($x \neq y$), $m - Cl(x) \cap m - Cl(y)$ is either a singleton or the empty set.

7. $m - T_{YS}$ if for each $x, y \in X$ ($x \neq y$), $m - Cl(x) \cap m - Cl(y)$ is either $\emptyset$ or $x$ or $y$.

8. $m - T(\alpha)$ if for each $y \in X$, $x \in m - Der(y)$ implies $m - Der(x) = \emptyset$.

In the following we will determine the relationship between the classes of $m-$spaces defined above. Note first that the implications $m - T_1 \to m - T_{DD}$, $m - T_D \to m - T_{UD}$ and $m - T_{DD} \to m - T_{YS}$ are evident.

**Proposition 6.** The following affirmations hold:

1. Every $m - T_{UD}$ space is $m - T_0$.

2. Every $m - T_{FF}$ space is $m - T_Y$.

3. Every $m - T_F$ space is $m - T(\alpha)$.

4. Every $m - T(\alpha)$ space is $m - T_0$.

5. Every $m - T_Y$ space is $m - T(\alpha)$.

**Proof.** 1. For $x, y \in X$ ($x \neq y$) if $y \in m - Der(x)$, then there exists an $m-$closed set $H$ such that $x \in H^c$ and $y \notin H^c$. Since the other case is evident we conclude that $X$ is $m - T_0$.

2. If there exist $x, y \in X$ ($x \neq y$) such that the set $m - Cl(x) \cap m - Cl(y)$ is neither $\emptyset$ nor a singleton, then there exist $p, q \in X$ ($p \neq q$) such that $p, q \in m - Cl(x) \cap m - Cl(y)$. If $\{p, q\} \cap \{x, y\} = \emptyset$ then for $F_1 = \{p, x\}$ and $F_2 = \{q, y\}$ we have that $F_1 \not\vdash F_2$ and $F_2 \not\vdash F_1$, a contradiction. If $p = x$ and $q \neq y$ then for $F_1 = \{x\}$, $F_2 = \{q, y\}$ is again obtained a contradiction. Finally, since the
other cases are impossible, we conclude that \(m - Cl(x) \cap m - Cl(y)\) is either \(\emptyset\) or a singleton, that is, \(X\) is \(m - T_Y\).

3. Let \(y \in X\) and \(x \in m - \text{Der}(y)\). If \(p \in m - \text{Der}(x)\) and \(p = y\), then \(F = \{y\} \not\in x\) and \(x \not\in F\) which is a contradiction. The case \(p \in m - \text{Der}(x)\) and \(p \neq y\) also leads to a contradiction. Hence \(m - \text{Der}(x) = \emptyset\) and thus \(X\) is \(m - T_\alpha\).

4. Let \(x, y \in X\) with \(x \neq y\). If \(x \in m - \text{Der}(y)\), then by assumption \(y \notin m - \text{Der}(x)\) and the result follows. The other case is analogous, so that \(X\) is \(m - T_0\).

5. Let \(y \in X\) and \(x \in m - \text{Der}(y)\). If \(p \in m - \text{Der}(x)\) and \(p = y\), then \(\{x, y\} \subseteq m - Cl(x) \cap m - Cl(y)\) which is a contradiction. Analogously the case \(p \neq y\) leads to a contradiction. Hence \(m - \text{Der}(x) = \emptyset\) and thus \(X\) is \(m - T_\alpha\). \(\blacksquare\)

The topological version of the \(m - T(\alpha)\) notion was introduced in [1] (Theorem 3.2). However, it is equivalent to the notion \(m - T_F\), which is not true in \(m - \text{spaces}\) (Example 7 (2)).

In the topological case the implications \(T_1 \rightarrow T_{FF}, T_Y \rightarrow T_F, T_Y \rightarrow T_{UD}, T_F \rightarrow T_{UD}\) hold. However, these implications are not true in general in \(m - \text{spaces}\), as we show in the following example.

Example 7. 1. Consider \(\mathbb{R}\) and \(m = \{\emptyset, \mathbb{R}\} \cup \{\mathbb{R} - \{x\} : x \in \mathbb{R}\}\). It is clear that this space is \(m - T_1\). Let \(F_1 = \{0, 1\}, F_2 = \{2, 3\}\). The \(m - \text{open}\) sets containing \(F_1\) have the form \(\mathbb{R} - \{z\}\) with \(z \neq 0, 1\). If \(z \neq 2, 3\) then \(\mathbb{R} - \{z\} \cap F_2 = F_2\) and so \(F_1 \not\in F_2\). If either \(z = 2\) or \(3\) then \(\mathbb{R} - \{z\} \cap F_2 = F_2 - \{z\}\) and \(F_1 \not\in F_2\). Analogously \(F_2 \not\in F_1\) and so this space is not \(m - T_{FF}\). In conclusion \(m - T_1 \not\rightarrow m - T_{FF}\).

2. Let \(X = \{1, 2, 3\}\) and \(m = \{\emptyset, X, \{1\}, \{1, 2\}, \{2, 3\}, \{3\}\}\). Since for each \(x \in X\), \(m - \text{Der}(x) = \emptyset\) then this space is \(m - T_Y\). For \(2 \in X\) and \(F = \{1, 3\}\) we have that \(\{2\} \not\in F\) and \(F \not\in \{2\}\); so this space is not \(m - T_F\). This example shows that \(m - T_Y \not\rightarrow m - T_F\) and \(m - T(\alpha) \not\rightarrow m - T_F\).

3. Consider \(\mathbb{R}\) and \(m = \{\emptyset, \mathbb{R}\} \cup \{A \subseteq \mathbb{R} : 0 \notin A, A\ is\ finite\}\). If \(x \neq 0\) then \(m - \text{Der}(x) = \{0\}\) and if \(x = 0\) then \(m - \text{Der}(0) = \emptyset\), so this space is \(m - T_Y\). Since the nonempty \(m - \text{closed}\) sets are infinite, the set \(m - \text{Der}(x) = \{0\}\) can not be expressed as a union of disjoint \(m - \text{closed}\) sets. Then this space is not \(m - T_{UD}\) and thus \(m - T_Y \not\rightarrow m - T_{UD}\).

4. Let \(X\) be an infinite set, \(p \in X\) a fixed element and \(m = \{\emptyset, X\} \cup \{A \subseteq X : p \in A, |A| \geq 3\ and\ A\ is\ finite\}\). Let \(x \in X\) and \(F\) a finite subset of \(X\) with \(x \notin F\). If \(x = p\) then \(p \not\in F\). If \(x \neq p\) and \(p \in F\) then \(F \not\in x\) and if \(p \notin F\) then \(x \not\in F\). Hence this space is \(m - T_F\). Since the set \(m - \text{Der}(p) = X - \{p\}\)
can not be expressed as a union of disjoint \( m \)-closed sets we conclude that \( X \) is not \( m - T_{UD} \). Thus, \( m - T_F \nleftrightarrow m - T_{UD} \) and \( m - T(\alpha) \nleftrightarrow m - T_{UD} \).

In conclusion we obtain the following diagram of implications.

\[
\begin{array}{cccc}
m - T_1 & \rightarrow & m - T_{DD} & \rightarrow & m - T_D \\
& & \downarrow & & \downarrow \\
& & m - T_{YS} & & \\
& & \downarrow & & \\
m - T_{FF} & \rightarrow & m - T_Y & & m - T_{UD} \\
& & \downarrow & & \downarrow \\
& & m - T_F & & m - T_0 \\
\end{array}
\]

Since each topological space is in particular an \( m \)-space the implications \( m - T_1 \rightarrow m - T_{DD} \rightarrow m - T_D \rightarrow m - T_{UD} \rightarrow m - T_0 \), \( m - T_{DD} \rightarrow m - T_{YS} \rightarrow m - T_Y \) and \( m - T_{FF} \rightarrow m - T_Y \) are strict. Moreover, since each \( m - T_Y \) space is \( m - T(\alpha) \) (Proposition 6 (5)) the Example 7 (2) shows that \( m - T(\alpha) \nleftrightarrow m - T_F \). The remaining implications are also strict as we show below.

**Example 8.** 1. For \( X = \{1, 2, 3, 4\} \) and \( m = \{\emptyset, X, \{1\}, \{2\}, \{1, 2, 3\}, \{1, 2, 4\}\} \) we have that \( m - \text{Der}(1) = m - \text{Der}(2) = \{3, 4\} \) and \( m - \text{Der}(3) = m - \text{Der}(4) = \emptyset \). So, this space is \( m - T(\alpha) \). Moreover, \( m - \text{Cl}(1) \cap m - \text{Cl}(2) = \{3, 4\} \), that is, \( X \) is not \( m - T_Y \). Then \( m - T(\alpha) \nleftrightarrow m - T_Y \).

2. Consider \( \mathbb{R} \) with \( m = \{\emptyset, \mathbb{R}\} \cup \{[x, \infty) : x \in \mathbb{R}\} \). It is clear that this space is \( m - T_0 \) and it is not \( m - T(\alpha) \). Thus, \( m - T_0 \nleftrightarrow m - T(\alpha) \).

3. Let \( X \) be a set such that \(|X| \geq 4\) and \( m = \{\emptyset, X\} \cup \{\{x\} : x \in X\} \). For different elements \( a, b, c, d \in X \) one has that \( F_1 = \{a, b\} \not\subseteq F_2 = \{c, d\} \) and \( F_2 \not\subseteq F_1 \). So, this space is not \( m - T_{FF} \) but it is \( m - T_F \). Hence \( m - T_F \nleftrightarrow m - T_{FF} \).

Finally, it is easy to see that each topological space \( T_i \) is an \( m - T_i \) space and this implication is strict. Therefore, the class of topological spaces \( T_i \) is strictly contained in the class of \( m - T_i \) spaces.

**4. Characterizations**

In what follows, we use the concepts of \( m \)-derived, \( m \)-closure, \( m \)-shell and \( m \)-kernel to characterize some low separation axioms in \( m \)-spaces. We start with several characterizations of the \( m - T_0 \) spaces. This result extends the
Theorem 2.3 of [1], the proof follows of the definitions and the application of Proposition 4.

**Theorem 9.** Let $X$ be an $m$-space. The following conditions are equivalent:

1. $X$ is $m$-$T_0$.

2. If $y \in m - Cl(x)$ with $y \neq x$, then $x \notin m - Cl(y)$.

3. If $y \in m - Der(x)$, then $m - Cl(y) \subseteq m - Der(x)$.

4. If $y \in m - \hat{x}$, then $m - \hat{y} \subseteq m - \hat{x}$.

5. For each $x \in X$, $m - Der(x) \cap m - \hat{x} = \emptyset$.

6. For each $x \in X$, $m - \langle x \rangle = x$.

In a similar way it can be obtained the following characterizations of the $m$-$T_1$ spaces. This result extends Theorem 2.4 of [1].

**Theorem 10.** Let $X$ be an $m$-space. The following conditions are equivalent:

1. $X$ is $m$-$T_1$.

2. For each $x \in X$, $m - Cl(x) = x$.

3. For each $x \in X$, $m - Der(x) = \emptyset$.

4. For each $x \in X$, $m - \hat{x} = x$.

5. For each $x \in X$, $m - \hat{x} = \emptyset$.

6. For each $x, y \in X$ with $x \neq y$, $m - Cl(x) \cap m - Cl(y) = \emptyset$.

7. For each $x, y \in X$ with $x \neq y$, $m - \hat{x} \cap m - \hat{y} = \emptyset$.

8. $m - N_D = X$, where $m - N_D = \{x \in X : m - Der(x) = \emptyset\}$.

9. $m - N_S = X$, where $m - N_S = \{x \in X : m - \hat{x} = \emptyset\}$.

It is well known that the topological spaces $T_D$ are characterized as those where each singleton is the intersection of a closed set with an open set ([1], Theorem 3.1). However, in the $m$-spaces this property is not true in general but if the Maki’s condition is satisfied then the result is verified.
Theorem 11. If $X$ is $m$-$T_D$, then each singleton is the intersection of an $m$-open set with the $m$-closure of some subset of $X$. The converse is true if $m$ satisfies the Maki’s condition.

Proof. For each $x \in X$ we have that $(m - \text{Der}(x))^c \cap (m - \text{Cl}(x)) = x$ and so the result follows.

Conversely, if for each $x \in X$, $x = G_x \cap m - \text{Cl}(A_x)$ where $G_x$ is an $m$-open and $A_x \subseteq X$, then $x = G_x \cap m - \text{Cl}(x)$ and so $m - \text{Der}(x) = m - \text{Cl}(x) \cap G_x^c$. Thus by the Maki’s condition we obtain that $m - \text{Der}(x)$ is $m$-closed.

In $m$-spaces the $m - T(\alpha)$ notion can be used to characterize the $m - T_Y$ axiom. This situation is analogous to the topological case where the $T_F$ notion is used ([1], Theorem 3.6).

Theorem 12. Let $X$ be an $m$-space. The following conditions are equivalent:

1. $X$ is $m - T_Y$.

2. $X$ is $m - T(\alpha)$ and for each $x, y \in X$ with $x \neq y$ the set $m - \text{Der}(x) \cap m - \text{Der}(y)$ is either empty or a singleton.

3. $X$ is $m - T(\alpha)$ and for each $x, y \in X$ with $x \neq y$ the set $m - \hat{x} \cap m - \hat{y}$ is either empty or a singleton.

4. For each $x, y \in X$ with $x \neq y$ the set $m - \hat{x} \cap m - \hat{y}$ is either empty or a singleton.

Proof. The first implication is evident. For the remaining it is enough to reason by contradiction and to apply 3 and 4 of Proposition 4.

Similar to the last theorem the $m - T(\alpha)$ notion can be used to characterize the $m - T_{YS}$ axiom.

Theorem 13. $X$ is $m - T_{YS}$ iff $X$ is $m - T(\alpha)$ and for each $x, y \in X$ with $x \neq y$, $m - \text{Der}(x) \cap m - \text{Der}(y) = \emptyset$.

Proof. It is clear that $X$ is $m - T(\alpha)$. If for each $x, y \in X$ ($x \neq y$) there exists $p \in m - \text{Der}(x) \cap m - \text{Der}(y)$, then $p \in m - \text{Cl}(x) \cap m - \text{Cl}(y)$, which is false. So the result follows.

Conversely, for each $x, y \in X$ ($x \neq y$) we have that $m - \text{Cl}(x) \cap m - \text{Cl}(y) = (x \cap m - \text{Der}(y)) \cup (y \cap m - \text{Der}(x))$. Since $X$ es $m - T(\alpha)$, that union of sets is necessarily different from $\{x, y\}$. Thus $X$ is $m - T_{YS}$.
Theorem 14. Let $X$ be an $m$–space. If for each $x, y \in X$ $(x \neq y)$, $m – Der(x) \vdash m – Der(y)$ or $m – Der(y) \vdash m – Der(x)$, then $X$ is $m – TYS$. The converse is true if for each $x \in X$, $m – Cl(x)$ is $m$–closed.

Proof. If there exist $x, y \in X$ $(x \neq y)$ and $p$ different from $x$ and $y$ such that $p \in m – Cl(x) \cap m – Cl(y)$, then $p \in m – Der(x)$ and $p \in m – Der(y)$. Thus we obtain a contradiction $m – Der(x) \not\vdash m – Der(y)$ and $m – Der(y) \not\vdash m – Der(x)$.

Conversely, if $m – Cl(x) \cap m – Cl(y) = \emptyset$ for $x \neq y$ then $m – Der(y) \vdash m – Der(x)$ with the $m$–open set $(m – Cl(x))^c$. If $m – Cl(x) \cap m – Cl(y) = x$, then $m – Der(x) \vdash m – Der(y)$ with the $m$–open set $(m – Cl(y))^c$. Since the other case is analogous we obtain the result.

Note that the previous result suggests the existence of a new separation notion between $m – TDD$ and $m – TYS$, since in each $m – TDD$ space for each $x \neq y$ one has that $m – Der(x) \vdash m – Der(y)$ and $m – Der(y) \vdash m – Der(x)$.

We finish this work with the characterization of the $m – T(\alpha)$ notion, which extends Theorem 3.4 of [1] to $m$–spaces.

Theorem 15. Let $X$ be an $m$–space. The following conditions are equivalent:

1. $X$ is $m – T(\alpha)$.
2. For each $x, y \in X$ $(x \neq y)$, $m – Der(x) \cap m – \overset{\sim}{y} = \emptyset$.
3. $N_S \cup N_D = X$.
4. For each $x \in X$, $y \in m – \overset{\sim}{x}$ implies $m – \overset{\sim}{y} = \emptyset$.

Proof. 1 ⇒ 2. If $p \in m – Der(x) \cap m – \overset{\sim}{y}$ for $x \neq y$, then we would have $p \in m – Der(x)$ and $y \in m – Der(p)$ (Proposition 4 (4)), which contradicts 1. Thus 2 follows.

2 ⇒ 3. If there exists $x \in X$ such that $m – Der(x) \neq \emptyset$ and $m – \overset{\sim}{x} \neq \emptyset$, then there exist $p, q \in X$ $(p \neq q)$ such that $x \in m – \overset{\sim}{p} \cap m – Der(q)$ (Proposition 4 (4)), which contradicts 2. Hence $N_S \cup N_D = X$.

3 ⇒ 4. If $y \in m – \overset{\sim}{x}$ and $m – \overset{\sim}{y} \neq \emptyset$, then $x \in m – Der(y)$ (Proposition 4 (4)) and by 3 $m – Der(y) = \emptyset$, which is a contradiction. Then 4 follows.

4 ⇒ 1. If $x \in m – Der(y)$ and $m – Der(x) \neq \emptyset$, then we would have $y \in m – \overset{\sim}{x}$ and $x \in m – \overset{\sim}{p}$ for some $p \in X$ (Proposition 4 (4)), which contradicts 4. It follows that $X$ is $m – T(\alpha)$. □
References


