International Journal of Pure and Applied Mathematics
Volume 83 No. 4 2013, 613-621
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v83i4.10
ijpam.eu

AMPLY WEAK SEMISIMPLE-SUPPLEMENTED MODULES

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Abstract: Let R be a ring and M be a right R-module. In this paper we will study various properties of amply weak semisimple-supplemented module. It is shown that: (1) every projective weakly semisimple-supplemented module is amply weak semisimple-supplemented; (2) if M is an amply weak semisimple-supplemented module and satisfies DCC on weak semisimple-supplement submodules and on small submodules, then M is Artinian; (3) an amply weak semisimple-supplemented module behaves well with respect to supplements and to homomorphic images.

AMS Subject Classification: 16D10, 16D60, 16D70, 16D99
Key Words: supplement submodule, weak semisimple-supplement submodule, amply weak semisimple-supplemented module

1. Introduction

Throughout this article, all rings are associative with unity and R denotes such a ring. All modules are unital right R-modules unless indicated otherwise. Let M be an R-module. $N \leq M$ will mean N is a submodule of M. $Soc(M)$, $End(M)$ and $Rad(M)$ will denote the Socle of M, the ring of endomorphisms of M and the Jacobson radical of M, respectively. The notions which are not explained here will be found in [7].

Received: January 7, 2013

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Recall that a module M is called *semisimple* if it is a direct sum of simple submodules. A submodule K is called *small* in M (notation $K << M$) if for every submodule L in M, the equality $K + L = M$ implies $L = M$. A module M is called *hollow* if every proper submodule of H is small (see, [7]). Let N and L be submodules of M. N is called a *supplement* of L in M if N is a minimal element in the set of submodules $K \subset M$ with $M = K + L$ (see,[3]). In ([4], Definition 4.4, p.56) M is called *supplemented* if any submodule of M has a supplement in M.

In early years, supplemented modules and the other generalization, *amply supplemented modules* appeared in Helmut Zöschinger's works ([9], [10], [11], [12]). After Zöschinger, many authors (see for example [2], [5], [6] and [8]) studied on variations of supplemented modules. This paper is based on another variation of supplemented modules. We say that a submodule N of M has *ample weak semisimple-supplements* in M if, for every $L \subseteq M$ with $N + L = M$, there exists a weak semisimple-supplement S of N with $S \subseteq L$. We say that M is *amply weak semisimple-supplemented module* if every submodule of M has ample weak semisimple-supplements in M. We proved that every projective weak semisimple-supplemented module is amply weak semisimple-supplemented. It is shown that if M is an amply weak semisimple-supplemented module and satisfies DCC on semisimple-supplement submodules and on small submodules, then M is Artinian. Moreover, it is proven that an amply weak semisimple-supplemented module behaves well with respect to supplements and to homomorphic images.

In this section, we discuss the concept of semisimple-supplement submodules and we give some properties of such type submodules.

Definition 1. Let M be an R-module, N and S be two submodules of M. S is called *semisimple-supplement* of N in M if $N + S = M$, $N \cap S << S$ and $Soc(S) = S$.

Since S is semisimple, every submodule of S is a direct summand. If $S \cap N << S$, then $S \cap N = 0$. Hence, S being a semisimple-supplement of N, we have $M = N \oplus S$, S is semisimple and S is the minimal element in the set of submodules $K \subset M$ with $M = K + N$.

Definition 2. Let M be an R-module. We say that M is semisimple-supplemented if all submodules of M has a semisimple-supplement in M.

Definition 3. Let M be an R-module and $N \subseteq M$. If, for every $L \subseteq M$ with $N + L = M$, there exists a semisimple-supplement S of N with $S \subseteq L$, then we say that N has *ample semisimple-supplements* in M.

Definition 4. Let M be an R-module. If every submodule of M has ample semisimple-supplements in M, then M is called amply semisimple-supplemented module.

It is clear that every amply semisimple-supplemented module is amply supplemented.

Proposition 5. *Let M be an R-module. Then the following statements are equivalent.*

(a) M is semisimple.

(b) M is semisimple-supplemented.

(c) M is amply semisimple-supplemented.

Proof. (a)$\Longrightarrow$ (b). It is clear.

(b)$\Longrightarrow$ (c). Let $M = N + L$. Since M is semisimple-supplemented, there exists a semisimple supplement S of N in M. Then $M = N \oplus S$. Hence $M = (N + L) \cap (N \oplus S) = N \oplus (L \cap S)$. By the minimality of S, $L \cap S = S$, and hence $S \subseteq L$. Thus N has ample semisimple supplement S with $S \subseteq L$.

(c)$\Longrightarrow$ (a). Let $N \leq M$. Since M is amply semisimple-supplemented module, there exists a semisimple supplement S of N in M. Then $S + N = M$ and $S \cap N << S$. Since S is semisimple, every submodule of S is a direct summand. So $S \cap N = 0$ and hence $S \oplus N = M$. Thus M is semisimple. □

Definition 6. Let M be an R-module, N S be two submodules of M. S is called *weak semisimple-supplement* of N in M if $N + S = M$, $N \cap S << M$ and $Soc(S) = S$.

Definition 7. Let M be an R-module. We say that a submodule $S \subset M$ is a *weak semisimple-supplement* if it is a weak semisimple-supplement for some submodule $N \subset M$.

Definition 8. Let M be an R-module. If every nonzero submodule of M has a weak semisimple-supplements in M, then M is called *a weakly semisimple-supplemented module* or briefly a WSS-module.

It is clear that every semisimple-supplemented module is weakly semisimple supplemented.

Proposition 9. *Let M be an R-module, N be a submodule of M where S be a weak semisimple-supplement of N in M. Then the following statements are hold.*

(1) If $K + S = M$ for some $K \subset N$, then S is also a weak semisimple-supplement of K in M.

(2) If M is finitely generated, then S is also finitely generated.

(3) If $K << M$, then S is a weak semisimple-supplement of $N + K$ in M.

(4) For $K \subset N$, $(S + K)/K$ is a weak semisimple-supplement of N/K in M/K.

Proof. (1) By the definition of weak semisimple-supplement, $N+S = M$, $N \cap S << M$ and S is semisimple. If $K + S = M$ for some $K \subset N$, then $K \cap S \subseteq N \cap S << M$. Therefore S is a weak semisimple-supplement of K in M.

(2) From ([7], 41.1(2)).

(3) Let $X \leq S$ and $N + K + X = M$. Since $K << M$, $N + X = M$ and $N \cap X \subseteq N \cap S << M$. By the minimality of S, $X = S$. Then S is a weak semisimple-supplement of $N + K$ in M.

(4) By the definition of weak semisimple-supplement, $M = S+N$, $S \cap N << M$ and S is semisimple. Hence $M = S + N + K$. Therefore $M/K = N/K + [(S+K)/K]$. Now, we show that $(N/K) \cap [(S+K)/K] << M/K$. Let $[(N/K) \cap [(S + K)/K]] + T/K = M/K$ and $K \subset T$. Then $[N \cap (S + K)]+T = M$ and by modular law $K+(N \cap S)+T = M$. Since $N \cap S << M$ and $K \subset T$, $T = T + K = M$. Hence $(N/K) \cap [(S + K)/K] << M/K$. Thus $(S + K)/K$ is a supplement of N/K. Finally, since S is semisimple, $(S + K)/K$ is semisimple submodule of M/K.

□

Lemma 10. *Let M be an R-module and $M_1, M_2, \ldots, M_n$ be submodules of M. Then $M_1 \oplus M_2 \oplus \cdots \oplus M_n$ is WSS-module if and only if every M_i $(1 \leq i \leq n)$ is WSS-module.*

Proof. Let $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$. To prove WSS-module it is sufficient by induction on n to prove this when $n = 2$. Thus suppose $n = 2$.

Assume that M is WSS-module. Let $N_1 \oplus N_2 \leq M = M_1 \oplus M_2$. By assumption N_1 has a weak semisimple-supplement S_1 in M_1 and N_2 has a weak semisimple-supplement S_2 in M_2. Then $N_1 + S_1 = M_1$, $N_1 \cap S_1 << M_1$, $N_2 + S_2 = M_2$ and $N_2 \cap S_2 << M_2$. Hence

$$M = M_1 \oplus M_2 = (N_1 + S_1) \oplus (N_2 + S_2) = (N_1 \oplus N_2) + (S_1 \oplus S_2),$$

and

$$(N_1 \oplus N_2) \cap (S_1 \oplus S_2) \subseteq (S_1 \cap (N_1 \oplus M_2)) + (S_2 \cap (N_1 \oplus M_2)) << M_1 \oplus M_2$$

Since S_1 and S_2 are semisimple, $S_1 \oplus S_2$ is semisimple. Hence $S_1 \oplus S_2$ is weak semisimple-supplement of $N_1 \oplus N_2$. Thus $M = M_1 \oplus M_2$ is WSS-module.

Conversely, assume that M_1 and M_2 are WSS-module. Let $L \leq M_1$. By assumption $L \oplus M_2$ has a weak semisimple-supplement S in M. Then $(L \oplus M_2) + S = M$ and $(L \oplus M_2) \cap S << M$. Hence

$$M_1 = M_1 \cap ((L \oplus M_2) + S) = L + (M_1 \cap S),$$

and

$$L \cap S = L \cap (M_1 \cap S) \subseteq (L \oplus M_2) \cap S << M.$$

Hence $L \cap (M_1 \cap S) << M_1$. Note that $M_1 \cap S$ is semisimple since it is a submodule of semisimple submodule S. Thus $M_1 \cap S$ is a weak semisimple-supplement of L in M_1. □

2. Amply Weak Semisimple-Supplemented Modules

In this section, we discuss the concept of amply weak semisimple-supplemented modules and we give some properties of such type modules.

Definition 11. Let M be an R-module and $N \subseteq M$. If, for every $L \subseteq M$ with $N + L = M$, there exists a weak semisimple-supplement S of N with $S \subseteq L$, then we say that N has *ample weak semisimple-supplements* in M.

Definition 12. Let M be an R-module. If every submodule of M has ample weak semisimple-supplements in M, then M is called *an amply weak semisimple-supplemented module* or briefly an AWSS-module.

Proposition 13. *Every AWSS-module is WSS-module.*

Proof. Let M be an AWSS-module and N be a submodule of M. Then $N + M = M$. Since M is AWSS-module, M contains a weak semisimple-supplement of N. Hence M is WSS-module. □

Proposition 14. *Let M be an R-module. If every submodule of M is a WSS-module, then M is AWSS-module.*

Proof. Let $L, N \leq M$ and $M = N + L$. By assumption, there is a weak semisimple-supplement submodule S of $L \cap N$ in L. Then $(L \cap N) + S = L$ and $(L \cap N) \cap S = N \cap S << L$. Thus $N \cap S << M$ and $S + N \geq S + (L \cap N) = L$ and hence $S + N \geq N + L = M$. Therefore $M = S + N$, as required. □

Proposition 15. *Every factor module of an AWSS-module is AWSS-module.*

Proof. Let M be an AWSS-module and M/K be any factor module of M. Let $N/K \subseteq M/K$. For $L/K \subseteq M/K$, let $N/K + L/K = M/K$. Then $N + L = M$. Since M is AWSS-module, there exists a weak semisimple-supplement S of N with $S \subseteq L$. By Proposition 9(4), $(S + K)/K$ is a weak semisimple-supplement of N/K in M/K. Since $(S + K)/K \subseteq L/K$, N/K has ample weak semisimple-supplements in M/K. Thus M/K is AWSS-module. □

Corollary 16. *Every homomorphic image of an AWSS-module is AWSS-module.*

Proof. Let M be an AWSS-module. Since every homomorphic image of M is isomorphic to a factor module of M, every homomorphic image of M is AWSS-module by Proposition 15. □

Proposition 17. *Every supplement submodule of an AWSS-module is AWSS-module.*

Proof. Let M be an AWSS-module and S be any supplement submodule of M. Then there exists a submodule N of M such that S is a supplement of N. Let $L \subseteq S$ and $L + S' = S$ for $S' \subseteq S$. Then $N + L + S' = M$. Since M is AWSS-module, $N+L$ has a weak semisimple-supplement S'' in M with $S'' \subseteq S'$. In this case $(N + L) + S'' = M$. Since $L + S'' \subseteq S$ and S is a supplement of N in M, $L + S'' = S$. On the other hand, since $L \cap S'' \subseteq (N + L) \cap S'' << M$, $L \cap S^{''} << M$. Hence L has ample weak semisimple-supplements in S. Thus S is AWSS-module. □

Corollary 18. *Every direct summand of an AWSS-module is AWSS-module.*

Proof. Let M be an AWSS-module. Since every direct summand of M is supplement in M, then by Proposition 17, every direct summand of M is AWSS-module. □

A module M is said to be π-*projective* if, for every two submodules N, L of M with $L + N = M$, there exists $f \in End(M)$ with $Imf \leq L$ and $Im(1 - f) \leq N$, see [7].

Theorem 19. *Let M be a WSS-module and π-projective module. Then M is AWSS-module.*

Proof. Let $N \leq M$ and $L + N = M$ for $L \leq M$. Since M is WSS-module, there exists a weak semisimple-supplement S of N in M. Then $N + S = M$, $N \cap S << M$ and S is semisimple. Since M is π-projective, there exists an endomorphism f such that $f(M) \leq L$ and $(1 - f)(M) \leq N$. Note that $f(N) \subseteq N$ and $(1 - f)(L) \subseteq L$. Then

$$M = f(M) + (1 - f)(M) \leq f(N \oplus S) + N = N + f(S).$$

Let $n \in N \cap f(S)$. Then there exists $s \in S$ with $n = f(s)$. In this case $s - n = s - f(s) = (1 - f)(s) \in N$ and then $s \in N$. Hence $s \in N \cap S$ and $N \cap f(S) \subseteq f(N \cap S)$. Since $N \cap S << M$, then by Lemma ([7], 19.3(4)) $f(N \cap S) << f(M)$. Then $N \cap f(S) \leq f(N \cap S) << M$. Since S is semisimple, $f(S)$ is semisimple. Hence $f(S)$ is a weak semisimple-supplement of N in M. Since $f(S) \leq L$, N has ample weak semisimple-supplements in M. Thus M is AWSS-module. □

Corollary 20. *Every projective and WSS-module is AWSS-module.*

Proof. Since every projective module is π-projective, every projective and WSS-module is AWSS-module by theorem 19. □

Corollary 21. *Let $M_1, M_2, \cdots, M_n$ be projective modules. Then $\overset{n}{\underset{i=1}{\oplus}} M_i$ is AWSS-module if and only if for every $1 \leq i \leq n$, M_i is AWSS-module.*

Proof. ($\Longrightarrow$) It is clear from Corollary 18.

($\Longleftarrow$) Since, for every $1 \leq i \leq n$, M_i is AWSS-module, M_i is WSS-module. Then $\overset{n}{\underset{i=1}{\oplus}} M_i$ is also WSS-module by Lemma 10. Since, for every $1 \leq i \leq n$, M_i is projective, $\overset{n}{\underset{i=1}{\oplus}} M_i$ is also projective. Then $\overset{n}{\underset{i=1}{\oplus}} M_i$ is AWSS-module by Corollary 20. □

Corollary 22. *Let R be a ring. Then the following statements are equivalent.*

(a) R is weakly semisimple-supplemented.

(b) R is amply weak semisimple-supplemented.

(c) Every finitely generated R-module is AWSS-module.

Proof. $(a) \Longleftrightarrow (b)$. Clear from Corollary 20.

$(a) \Longleftrightarrow (c)$. Clear from Corollary 16 and Corollary 21. □

Theorem 23. *([1],Theorem 5) Let R be any ring and M be a module. Then $Rad(M)$ is Artinian if and only if M satisfies DCC on small submodules.*

Proposition 24. *Let M be an R-module. If M is an AWSS-module and satisfies DCC on weak semisimple-supplement submodules and on small submodules then M is Artinian.*

Proof. Let M be an AWSS-module which satisfies DCC on weak semisimple-supplement submodules and on small submodules. Then $Rad(M)$ is Artinian by Theorem 23. It suffices to show that $M/Rad(M)$ is Artinian. Let N be any submodule of M containing $Rad(M)$. Then there exists a weak semisimple-supplement S of N in M. Hence $M = N + S$, $N \cap S << M$. Since $N \cap S \leq Rad(M)$, $M/Rad(M) = (N/Rad(M)) \oplus ((S + Rad(M))/Rad(M))$ and so every submodule of $M/Rad(M)$ is a direct summand. Therefore $M/Rad(M)$ is semisimple.

Now suppose that $Rad(M) \leq N_1 \leq N_2 \leq N_3 \leq \cdots$ is an ascending chain of submodules of M. Because M is AWSS-module, there exists a descending chain of submodules $S_1 \geq S_2 \geq S_3 \geq \cdots$ such that S_i is a weak semisimple-supplement of N_i in M for each $i \geq 1$. By hypothesis, there exists a positive integer t such that $S_t = S_{t+1} = S_{t+2} = \cdots$. Because $M/Rad(M) = N_i/Rad(M) \oplus (S_i + Rad(M))/Rad(M)$ for all $i \geq t$, it follows that $N_t = N_{t+1} = \cdots$. Thus $M/Rad(M)$ is Noetherian, and hence finitely generated. So $M/Rad(M)$ is Artinian, as desired. □

Corollary 25. *Let M be a finitely generated AWSS-module. If M satisfies DCC on small submodules, then M is Artinian.*

Proof. Since $M/Rad(M)$ is semisimple and M is finitely generated, $M/Rad(M)$ is Artinian. Now that M satisfies DCC on small submodules, $Rad(M)$ is Artinian by Theorem 23. Thus M is Artinian. □

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