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NUMERICAL SOLUTIONS OF SINGULAR PERTURBATION PROBLEMS WITH MULTIPLE BOUNDARY LAYERS AND INTERIOR LAYERS

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Abstract: Two singular perturbation problems were considered: One with multiple boundary layers and one with interior layers. Numerical schemes were developed from the improved a'priori bounds. With a constant number of layer adapted grid points, numerical accuracy was maintained at the same level for a family of singular perturbation problems.

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The study of singular perturbation problems is exceptionally useful because they describe the physics of many things of academic and economic interest. They may be used to model the weather, ocean currents, water flow in a pipe, the air's flow around a wing, and motion of stars inside a galaxy. The asymptotic expansion of O'Malley [8] and the a'priori bound theorem of Chang and Howes [1] were among the prominent approaches to solve singular perturbation problems. With the advance of unprecedented computing power, there has been a flow of literatures on numerical solutions from the nineteen eighties. Miller, O'Riordan and Shishkin [7] constructed the Shishkin-type mesh to

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gain the independence of error estimation with respect to the singular perturbation parameter. Schultz and his students [2] and [4], successfully developed the stabilized high order finite difference methods. Lin, Schultz and Zhang [6] developed boundary layer detection theory from improved a'priori bounds for quasilinear singular perturbation problems.

Zhang, Schultz and Lin [12] developed sharp a'priori bounds for semi-linear singular perturbation problems including ones with multiple boundary layers. In this paper, we state one of the theorems in [12] and we study an example of a singular perturbation problem with such layers. Then we extend the boundary layer detection for quasi-linear singular perturbation problems in [6] to those with interior layers.

First, we consider singular perturbation problems in the following semi-linear form,

$$\begin{aligned} &\varepsilon u" = g(x,u) \quad \text{for} \quad x \in (a,b), \\ &u(a) = v_a \quad \text{and} \quad u(b) = v_b, \end{aligned} \tag{1}$$

where the singular perturbation parameter $\varepsilon \ll 1$ is a small positive constant and $g(x,u)$ is sufficiently continuously differentiable in the domain considered. In general, such a singular perturbation problem may display two boundary layers in its solution, and the width of each boundary layer is proportional to $\sqrt{\varepsilon}$.

In the following definition of stability of the solution $R(x)$ for the reduced equation $g(x,u) = 0$of the singular perturbation problem (1), assume that the function has the stated number of continuous partial derivatives with respect to u in $\Omega(R)$. For an integer $q \geq 0$, the function $R = R(x)$ is said to be ***I_q*-stable** in $[a, \ b]$ if there exists a positive constant m such that

$$\partial_u^j g(x, R(x)) \equiv 0 \quad \text{for} \quad x \in [a, \quad b] \quad \text{and} \quad 0 \leq j \leq 2q,$$

and

$$\partial_u^{2q+1} g(x,u) \geq m \quad \text{in} \quad \Omega(R),$$

Chang and Howes [1]. Here is one of the major theorems in [12]. It shows an improved a'priori bounds for the solution of the singular perturbation problem (1) with an an I_q-stable ($q = 0$) solution of its reduced equation $g(x,u) = 0$. A summary of proof is provided to show that lengths of boundary layers are proportional to $\sqrt{\varepsilon}$. Details along with other I_q-stable cases can be found in [12].

Theorem. *Let $R(x)$ of class $C^2([a,b])$ be an an I_q-stable ($q = 0$) solution of the reduced equation $g(x,u) = 0$of the singular perturbation problem (1).*

Then there exist positive constants C and ε_0 such that for $0 < \varepsilon < \varepsilon_0$, it has a solution $u = u(x)$ which satisfies

$$|u(x) - R(x)| < C\varepsilon \quad \text{for} \quad x \in [a + w\sqrt{\varepsilon},\ \ b - w\sqrt{\varepsilon}],$$

where w is a positive constant for a family of small values of ε.

Proof. By Theorem 3.1 of Chang and Howes [1], there exists an $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$ the problem (1) has a solution $u = u(x)$ for $x \in [a, b]$ which satisfies for sufficiently small values of $\varepsilon \geq \varepsilon_1$.

$$|u(x) - R(x)| \leq s(x) + t(x) + c\varepsilon,$$

where $s(x) = |v_a - R(a)|e^{\frac{-\sqrt{m}(x-a)}{\sqrt{\varepsilon}}}$ and $t(x) = |v_b - R(b)|e^{\frac{-\sqrt{m}(b-x)}{\sqrt{\varepsilon}}}$.

We first consider the above positive function $s(x) = |v_a - R(a)|e^{\frac{-\sqrt{m}(x-a)}{\sqrt{\varepsilon}}}$. Choose a positive constant w, such that $w > -\frac{\ln \varepsilon_1}{\sqrt{m}}$. We can prove that $s(x) = c_1 e^{\frac{-\sqrt{m}(x-a)}{\sqrt{\varepsilon}}} < c_1\varepsilon$ for $x \in [a + w\sqrt{\varepsilon},\quad b]$, where $c_1 = |v_a - R(a)|$. For details, see [12].

Similarly, we can prove that $t(x) = |v_b - R(b)|e^{\frac{-\sqrt{m}(b-x)}{\sqrt{\varepsilon}}} < c_2\varepsilon$ for $x \in [a,\ \ b - w\sqrt{\varepsilon}]$., where $c_2 = |v_b - R(b)|$. Thus, for $C = c_1 + c_2 + c$, we have

$$|u(x) - R(x)| < C\varepsilon \quad \text{for} \quad x \in [a + w\sqrt{\varepsilon},\quad b - w\sqrt{\varepsilon}].$$

Here is our first example with multiple boundary layers.

$$\begin{aligned} &\varepsilon u" = u \quad \text{for} \quad x \in (0, \quad 1), \\ &u(1) = 1 \quad \text{and} \quad u(1) = 2, \end{aligned} \tag{1}$$

O'Malley [8], Chang and Howes [1]. Two boundary layers, one at left and one at right are forced by boundary values. The exact solution u is

$$u(x) = \frac{(e^{\frac{-1}{\sqrt{\varepsilon}}} - 2)e^{\frac{x}{\sqrt{\varepsilon}}} + (2 - e^{\frac{1}{\sqrt{\varepsilon}}})e^{\frac{-x}{\sqrt{\varepsilon}}}}{e^{\frac{-1}{\sqrt{\varepsilon}}} - e^{\frac{1}{\sqrt{\varepsilon}}}}.$$

For small values of the singular perturbation parameter ε, it is approximately

$$u(x) \approx 2e^{\frac{-(1-x)}{\sqrt{\varepsilon}}} + e^{\frac{-x}{\sqrt{\varepsilon}}},$$

since $e^{\frac{-1}{\sqrt{\varepsilon}}} \approx 0$ and $\frac{2-e^{\frac{1}{\sqrt{\varepsilon}}}}{e^{\frac{-1}{\sqrt{\varepsilon}}}-e^{\frac{1}{\sqrt{\varepsilon}}}} \approx 1$.

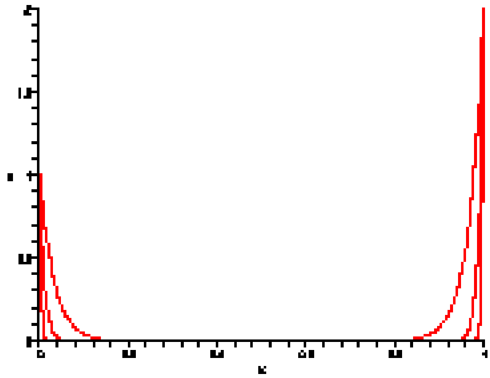


Figure 1.1: Graphs of the solutions of the problem (2), $\varepsilon = .001, .0001$ and .0001

The approximation form of u confirms the observation of multiple boundary layers at both boundaries $x = 0$ and $x = 1$. Let us examine the solution more closely. The solution u is defined for all $\varepsilon > 0$ and moreover, $\lim_{\varepsilon \to 0^+} u(x, \varepsilon) = 0$ for $\delta \leq$ x $\leq$ 1-δ, where δ is a fixed constant $(0, 1)$. The function $u(x)$ attains its limiting value non-uniformly in the neighborhoods of $x = 0$ and $x = 1$ in the sense that

$$\lim_{\varepsilon \to 0^+} \lim_{x \to 0^+} u(x, \varepsilon) = 1 \neq 0 = \lim_{x \to 0^+} \lim_{\varepsilon \to 0^+} u(x, \varepsilon),$$

and

$$\lim_{\varepsilon \to 0^+} \lim_{x \to 1^-} u(x, \varepsilon) = 2 \neq 0 = \lim_{x \to 1^-} \lim_{\varepsilon \to 0^+} u(x, \varepsilon).$$

The graphs of u for small values of ε are shown in Figure 1.1.

Note that by setting $\varepsilon = 0$ we obtain the reduced equation $u \equiv 0$ which is the approximation of the singular perturbation problem on its interior domain.

For the length of left boundary layer, we have

$$e^{\frac{-x}{\sqrt{\varepsilon}}} < \varepsilon \quad \Leftrightarrow \quad \frac{-x}{\sqrt{\varepsilon}} < \ln \varepsilon \quad \Leftrightarrow \quad x > (-\ln \varepsilon)\sqrt{\varepsilon},$$

$w = 30 \geq -\ln \varepsilon \quad \text{for} \quad \varepsilon > 10^{-12}$ as shown in the proof of the theorem. Similarly, for the length of the right boundary layer, we have

$$e^{-\frac{1-x}{\sqrt{\varepsilon}}} < \varepsilon \quad \Leftrightarrow \quad -\frac{1-x}{\sqrt{\varepsilon}} < \ln \varepsilon \quad \Leftrightarrow \quad x < 1 - (-\ln \varepsilon)\sqrt{\varepsilon},$$

Number of points	200	400	1000
Maximal error	$3.44*10^{-4}$	$8.61*10^{-5}$	$1.38*10^{-5}$

Table 1.1: The convergence of the new method on the left boundary layer, $\varepsilon = \mathbf{10^{-6}}$

$w = 30 \geq -\ln\varepsilon$ for $\varepsilon > 10^{-12}$. Therefore, the exact solution $u < C\varepsilon$ for $\varepsilon > 10^{-12}, w\sqrt{\varepsilon} \leq x \leq 1 - w\sqrt{\varepsilon}$ and a positive constant C. It follows that the lengths of the boundary layers are proportional to $\sqrt{\varepsilon}$. The proportion constant $w = 30$ is the boundary layer parameter. The singular perturbation problem (1) is approximated by following three problems,

$$\varepsilon u" = u \quad \text{for} \quad x \in (0, \quad t_1), \tag{1.1}$$

$$u(1) = 1 \quad \text{and} \quad u(t_1) = 0,$$

$$u = 0 \quad \text{for} \quad \leq x \in (t_1, \quad t_2), \tag{1.2}$$

and

$$\varepsilon u" = u \quad \text{for} \quad x \in (0, \quad t_1),$$

$$u(t_2) = 0 \quad \text{and} \quad u(1) = 2, \tag{1.3}$$

$$\text{where } t_1 = w\sqrt{\varepsilon} \text{ and } t_2 = 1 - w\sqrt{\varepsilon},$$

We apply the second order central differences on the following boundary layer adapted mesh to solve the singular perturbation problems (1.1) and (1.3), while for (1.2), u is given explicitly or implicitly from the reduced equation $g(x, u) = 0$. Figure 1.2 displays the layer adapted grid points at both boundaries.

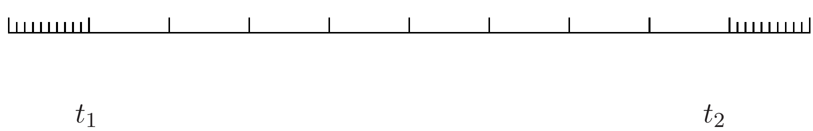


Figure 1.2: The boundary layer adapted mesh at both boundaries

Table 1.1 shows the convergence of the new method with central finite differences on the left boundary layer as the number of mesh points increases. Table 1.2 shows the convergence of the new method on the right boundary layer as the number of mesh points increases.

Number of points	200	400	1000
Maximal error	$6.88*10^{-4}$	$1.72*10^{-4}$	$2.76*10^{-5}$

Table 1.2: The convergence of the new method on the right boundary layer, $\varepsilon = \mathbf{10^{-6}}$

Number of mesh points	Maximal Error		
	$\varepsilon = 10^{-6}$	$\varepsilon = 10^{-8}$	$\varepsilon = 10^{-10}$
N_b =500 for the left boundary layer L.B.L.	$5.52*10^{-5}$	$5.52*10^{-5}$	$5.52*10^{-5}$
N_b =500 for the right boundary layer	$1.10*10^{-4}$	$1.10*10^{-4}$	$1.10*10^{-4}$
N_b =1000 for the left boundary layer L.B.L.	$1.38*10^{-5}$	$1.38*10^{-5}$	$1.38*10^{-5}$
N_b =1000 for the right boundary layer	$2.76*10^{-5}$	$2.76*10^{-5}$	$2.76*10^{-5}$

Table 1.3: Table 1.3. The robustness of the new method

Table 1.3 shows the robustness of the separation method. For a family of diminishing values of the singular perturbation parameter, the error is stabilized for a fixed number of mesh points on both boundary layers.

Next, we extend the layer detection for quasi-linear singular perturbation problems in [6] to those with interior layers. We consider singular perturbation problems in the quasi-linear form,

$$\begin{aligned} &\varepsilon u" = f(x,u)u' + g(x,u) \quad \text{for} \quad x \in (a,b) \quad \text{and} \;\; f(x,u) \neq 0, \\ &u(a) = v_a \quad \text{and} \quad u(b) = v_b, \end{aligned} \tag{2}$$

where f and g are continuous. By the improved a'priori bounds of Zhang [11], if $f(x,u) \leq -k < 0$ for a positive constant k and $x \in (a,b)$, it can be analytically approximated by the two differential equations.

$$\begin{aligned} &f(x,u)u' + g(x,u) = 0 \quad \text{for} \quad x \in (t,\ b), \\ &u(b) = v_b, \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} &\varepsilon u" = f(x,u)u' + g(x,u) \quad \text{for} \quad x \in (a,\ t), \\ &u(a) = v_a \quad \text{and} \quad u(t) = v_t, \end{aligned} \tag{2.2}$$

where the turning point is $t = a + w\varepsilon$ and w is a constant for a family of values of ε.

If $f(x,u) \geq k > 0$ for a positive constant k and$x \in (a,b)$, the singular perturbation problem (2) can be analytically approximated by the following two differential equations.

$$\begin{aligned} &f(x,u)u' + g(x,u) = 0 \quad \text{for} \quad x \in (a,\ t), \\ &u(a) = v_a, \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} &\varepsilon u" = f(x,u)u' + g(x,u) \quad \text{for} \quad x \in (t,\ b), \\ &u(t) = v_t \quad \text{and} \quad u(b) = v_b, \end{aligned} \tag{2.4}$$

where the turning point is $t = b - w\varepsilon$ and w is a constant for a family of values of ε. Note that the boundary value v_t at the transition point t of the equations (2.2) and (2.4) is not known. Let $R_b(x)$ be the solution of (2.1) and let $R_a(x)$ be the solution of (2.3). We substitute $R_b(t)$ for v_t into equation (2.2) and substitute $R_a(t)$ for v_t into equation (2.4) respectively.

Here is an example of quasi-linear singular perturbation problems with interior layers.

$$\begin{aligned} &\varepsilon u" = uu' - u^3 \quad \text{for} \quad x \in (0,\ 1), \\ &u(1) = \tfrac{2}{3} \quad \text{and} \quad u(1) = -\tfrac{1}{2}, \end{aligned} \tag{3}$$

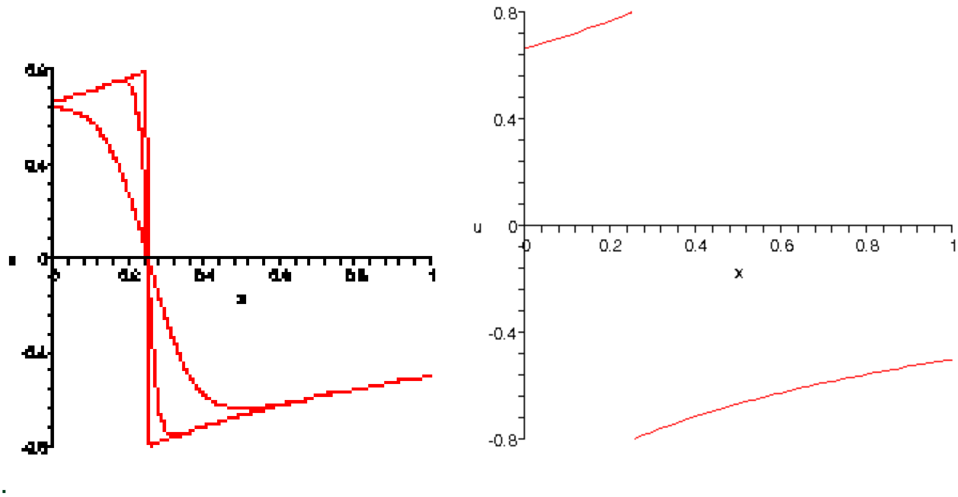
from Smith [9] and Whitman [10]. It is known that a shock occurs as the singular parameter ε decreases. As a result, the interior layer is displayed in its solution.

As the singular perturbation vanishes, for $u \neq 0$, we have $u' = u^2$ whose general solution is $u(x) = \frac{1}{c-x}$ where c is constant. At the left boundary x =0, c is $\frac{3}{2}$ since $u(1) = \frac{2}{3}$. At the right boundary$x = 1$, c is -1 since $u(1) = -\frac{1}{2}$. Let R_a and R_b be the solutions of the reduced differential equation to the left boundary and the right boundary respectively. Then we have,

$$R_a(x) = \frac{1}{\frac{3}{2} - x} = \frac{2}{3 - 2x}, \text{ and } R_b(x) = \frac{1}{-1 - x} = -\frac{1}{1 + x}.$$

Further analysis shows that the interior boundary layer or the shock occurs at the unique point x_s such that $R_a(x_s) + R_b(x_s) = 0$. Thus, we have

$$R_a(x_s) + R_b(x_s) = 0 \quad \Leftrightarrow \quad \frac{2}{3 - 2x_s} + (-\frac{1}{1 + x_s}) = 0 \quad \Leftrightarrow \quad x_s = \frac{1}{4}.$$



(a) (b)

Figure 2.3: Graphs of solutions of the problem (3) and its reduced equations; (a) from left to right ε =.05,.01 and .001; (b)the reduced equations

Now, the singularly perturbed differential equation (3) is equivalent to

$$\begin{array}{l} \varepsilon u'' = uu' - u^3 \quad \text{for} \quad x \in (0,\ x_s), \\ u(1) = \frac{2}{3} \quad \text{and} \quad u(x_s) = 0, \end{array} \tag{3.1}$$

and

$$\begin{array}{l} \varepsilon u'' = uu' - u^3 \quad \text{for} \quad x \in (x_s,\ 1), \\ u(x_s) = 0 \quad \text{and} \quad u(1) = -\frac{1}{2}. \end{array} \tag{3.2}$$

The equation (3.1) has a boundary layer at the right boundary $x_s = \frac{1}{4}$ while the equation (3.2) has one at left $x_s = \frac{1}{4}$. The interior boundary layer of the singular perturbation problem (3) is formed by the coexistence of the above two boundary layers, Figure 2.1.

In general, a reduced equation can be solved numerically with high order Runge Kutta methods. For (3.1) and (3.2), their reduced equations are solved analytically above. On their boundary layer domains, each is solved with the second order central finite difference approximation. The stability of a boundary value perturbation for such problems is proved in Zhang [11]. On the non layer domain, we get almost perfect results from the reduced equations. On the layer domain, the maximal error is controlled at 10^{-4} level for $\varepsilon = 10^{-5}$ by using a few hundred mesh points, Table 2.1.

Number of points on the layer	200	400	600
Maximal error	$1.35*10^{-3}$	$2.91*10^{-4}$	$2.51*10^{-4}$

Table 2.4: The convergence of the new method, $\varepsilon = \mathbf{10^{-5}}$

For a family of extremely small $\varepsilon \leq 10^{-10}$ values, we controlled the error at 10^{-3} level with 200 mesh points and at 10^{-4} level with 400 points, Table 2.2.

Number of Points on the boundary layer	Maximal Error					
	$\varepsilon = 10^{-5}$	$\varepsilon = 10^{-6}$	$\varepsilon = 10^{-7}$	$\varepsilon = 10^{-8}$	$\varepsilon = 10^{-9}$	$\varepsilon = 10^{-10}$
200	$1.35*10^{-3}$	$1.69*10^{-3}$	$1.72*10^{-3}$	$1.72*10^{-3}$	$1.72*10^{-3}$	$1.72*10^{-3}$
400	$2.91*10^{-4}$	$3.77*10^{-4}$	$4.07*10^{-4}$	$4.10*10^{-4}$	$4.10*10^{-4}$	$4.10*10^{-4}$

Table 2.5: The maximal error of the new method

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