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DIRECT SUMS AND FREE FACTORS OF PARAFREE LIE ALGEBRAS

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Abstract: We prove that the free factors and the direct sums of parafree Lie algebras are again parafree.

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1. Introduction

In ([2],[3],[4],[5]) Baumslag has introduced the notion of parafree groups and has obtained some results about parafree groups. Many questions about parafree groups have remain unanswered. Baumslag has taken his result ([6],[7]) on one relator groups to formulate several of these questions for one-relator parafree groups.

Because of the close relationship between groups and Lie algebras, one would expect that parafree Lie algebras enjoy properties that are analogous to those of parafree groups. We have taken this opportunity to obtain some results about parafree Lie algebras. Parafree Lie algebras firstly arise in the works of Baur ([8],[9]), Knus and Stammbach [10]. They have answered certain questions and obtained basic results which provide a solid understanding of the structure of parafree Lie algebras. The aim of this work is to construct the direct sum and the free product of parafree Lie algebras. We carry the formal arguments used

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in [3] over to parafree Lie algebras.

2. Notations and Definitions

Let L be a Lie algebra over a field k. The lower central series

$$L = \gamma_1(L) \supseteq \gamma_2(L) \supseteq \cdots \supseteq \gamma_n(L) \supseteq \cdots$$

is defined inductively by

$$\gamma_2(L) = L' = [L, L], \quad \gamma_{n+1}(L) = [\gamma_n(L), L], \quad n \geq 1.$$

If n is the smallest integer satisfying $\gamma_n(L) = 0$ then L is called nilpotent of degree n.

Definition 1. A Lie algebra L is called residually nilpotent if,

$$\bigcap_{n=1}^{\infty} \gamma_n(L) = \{0\}$$

equivalently, given any non-trivial element $u \in L$, there exists an ideal J of L such that $u \notin J$ with L/J nilpotent. We define here the notion of parafree Lie algebras. We associate with the lower central series of L its lower central sequence:$L/\gamma_2(L), L/\gamma_3(L), ...$

We say that two Lie algebras L and H have the same lower central sequence if $L/\gamma_n(L) \cong H/\gamma_n(H)$ for every $n \geq 1$.

Definition 2. The Lie algebra L is called parafree over a set X if,

i L is residually nilpotent, and

ii L has the same lower central sequence as a free Lie algebra generated by the set X.

The cardinality of X is called the rank of L.

3. Direct Sums and Free Factors

We denote the direct sum of any Lie algebras P_1 and P_2 as $P_1 \oplus P_2$.

Lemma 3. *Let P_1 and P_2 be parafree Lie algebras and $P = P_1 \oplus P_2$. Then P is parafree.*

Proof. Let P be the direct sum of P_1 and P_2. We want to prove that P is residually nilpotent and P has the same lower central sequence as a free Lie algebra. In order to prove residually nilpotency of P, we need the equality

$$\gamma_n(P) = \gamma_n(P_1 \oplus P_2) = \gamma_n(P_1) \oplus \gamma_n(P_2).$$

We prove this equality by induction on n.If $n = 1$ then

$$\gamma_1(P) = P = (P_1 \oplus P_2) = \gamma_1(P_1) \oplus \gamma_1(P_2).$$

Now suppose that for every $m < n$

$$\gamma_m(P) = \gamma_m(P_1) \oplus \gamma_m(P_2).$$

Then

$$\gamma_n(P) = [\gamma_{n-1}(P), P] = [\gamma_{n-1}(P_1) \oplus \gamma_{n-1}(P_2), (P_1 \oplus P_2)].$$

Clearly

$$\begin{aligned}\gamma_n(P) =& [\gamma_{n-1}(P_1) + \gamma_{n-1}(P_2), (P_1 + P_2)] \\ =& [\gamma_{n-1}(P_1), P_1] + [\gamma_{n-1}(P_2), P_2] = \gamma_n(P_1) + \gamma_n(P_2).\end{aligned}$$

and

$$\gamma_n(P_1) \bigcap \gamma_n(P_2) = \{0\}.$$

Hence,

$$\gamma_n(P) = \gamma_n(P_1) \oplus \gamma_n(P_2).$$

Now we are going to prove residual nilpotence of P. We compute $\bigcap\limits_{n=1}^{\infty} \gamma_n(P)$:

$$\bigcap_{n=1}^{\infty} \gamma_n(P) = \bigcap_{n=1}^{\infty} (\gamma_n(P_1) \oplus \gamma_n(P_2)).$$

By the definition of direct sum we have

$$\bigcap_{n=1}^{\infty} (\gamma_n(P_1) \oplus \gamma_n(P_2)) = \bigcap_{n=1}^{\infty} (\gamma_n(P_1)) \oplus \bigcap_{n=1}^{\infty} (\gamma_n(P_2)).$$

Since P_1 and P_1 are parafree, then we have

$$\bigcap_{n=1}^{\infty} (\gamma_n(P_1)) = 0 \text{ and } \bigcap_{n=1}^{\infty} (\gamma_n(P_2)) = \{0\}.$$

Thus we obtain

$$\bigcap_{n=1}^{\infty}(\gamma_n(P)) = 0,$$

i.e. P is residually nilpotent.

Now we prove that P has the same lower central sequence as a free Lie algebra. Since P_1 and P_2 are parafree Lie algebras then there exist free Lie algebra F_1 and F_2 such that

$$P_i/\gamma_n(P_i) \cong F_i/\gamma_n(F_i), i = 1, 2, n \geq 1.$$

Therefore

$$\begin{aligned} P/\gamma_n \cong & (P_1/\gamma_n(P_1)) \oplus (P_2/\gamma_n(P_2)) \\ \cong & (F_1/\gamma_n(F_1)) \oplus (F_2/\gamma_n(F_2)) \\ \cong & (F_1 \oplus F_2)/\gamma_n(F_1 \oplus F_2). \end{aligned}$$

There is a homomorphism θ from $(F_1 \oplus F_2)/\gamma_n(F_1 \oplus F_2)$ onto isomorphic copy of its in the algebra $(F_1 * F_2)/\gamma_n(F_1 * F_2)$.

That is, $\theta((F_1 \oplus F_2)/\gamma_n(F_1 \oplus F_2))$ is the isomorphic copy of $(F_1 \oplus F_2)/\gamma_n(F_1 \oplus F_2)$ in $(F_1 * F_2)/\gamma_n(F_1 * F_2)$.

Since any subalgebra of a free nilpotent Lie algebra is free nilpotent, then there exists a free Lie algebra G such that

$$\theta((F_1 \oplus F_2)/\gamma_n(F_1 \oplus F_2)) \cong G/\gamma_n(G).$$

Hence

$$P/\gamma_n(P) \cong G/\gamma_n(G),$$

i.e. P is a parafree Lie algebra. □

Theorem 4. *Let A and B be parafree Lie algebras. The free product of A and B is again parafree.*

Proof. For the proof of this lemma see [8]. □

Proposition 5. *Let A and B be parafree Lie algebras and let $P = A * B$ be the free product of A and B.Then*

$$P/\gamma_2\,(P) \cong (A/\gamma_2(A)) \oplus (B/\gamma_2(B)).$$

Proof. Consider the homomorphism φ from $P/\gamma_2(P)$ onto $(A/\gamma_2(A)) \oplus (B/\gamma_2(B))$ defined as

$$\varphi : u + \gamma_2(P) \to a + b + \gamma_2(A) + \gamma_2(B),$$

where $u = a + b \in P, a \in A, b \in B$. It is clear that $ker\varphi = \gamma_2(P)$ and also φ is surjective. Since φ is an isomorphism, hence

$$P/\gamma_2(P) \cong (A/\gamma_2(A)) \oplus (B/\gamma_2(B)).$$ □

Theorem 6. *The free factors of a parafree Lie algebra are parafree.*

Proof. Let P be a parafree Lie algebra which is the free product of subalgebras A and B. By the Proposition 5, we have the equality

$$P/\gamma_2(P) \cong (A/\gamma_2(A)) \oplus (B/\gamma_2(B)).$$

Let X, Y be generating set of A and B respectively such that X is linearly independent modulo $\gamma_2(A)$ and Y is linearly independent modulo $\gamma_2(B)$. Then by Proposition 5, $X \cup Y$ is linearly independent modulo $\gamma_2(P)$ and it freely generates P modulo $\gamma_2(P)$. Since P is parafree, hence $X \cup Y$ freely generates $P/\gamma_2(P)$, X freely generates A modulo $\gamma_2(A)$ and Y freely generates B modulo $\gamma_2(B)$. Moreover $P/\gamma_n(P)$ is a free nilpotent Lie algebra freely generated by $X \cup Y$. Therefore it follows from [1], p. 104, Theorem 9, that X freely generates a free nilpotent Lie algebra modulo $\gamma_n(A)$. So A has the same lower central sequence as a free Lie algebra. But A is residually nilpotent since P is. So A is parafree. A similar argument shows that B is also parafree. Hence the free factors of a parafree Lie algebra are parafree. □

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