STUDY ON N-IDEAL OF A BF-ALGEBRAS

P. Muralikrishna\(^1\), M. Chandramouleeswaran\(^2\)

\(^1\)School of Advanced Sciences
VIT University
Vellore, 632014, Tamilnadu, INDIA

\(^2\)Department of Mathematics
SBK College
Aruppukottai, 626101, Tamilnadu, INDIA

Abstract: This paper introduces the notion of N-structured ideal of a BF-algebra by using the idea of an N-function.

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1. Introduction


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2. Preliminaries

For any two elements $a, b$ in $[-1, 0]$, we use the notation $a \lor b$ to represent $\max(a, b)$.

**Definition 2.1.** (N-Fuction and N-Structure) Consider a non-empty Set S. Denote the collection of functions from S to $[-1, 0]$ by $F(S, [-1, 0])$. We say that a member of $F(S, [-1, 0])$ is a negative valued function from S to $[-1, 0]$, briefly N-function and by an N-structure (NS) on S, we mean that an ordered pair $(S, \eta)$ of S and N-function $\eta$ on S.

**Definition 2.2.** [1] A non-empty set $X$ with a constant 0 and a single binary operation $\ast$, is said to be a BF-Algebra if it satisfies the following:

(i) $x \ast x = 0$ ; (ii) $x \ast 0 = x$ and (iii) $0 \ast (x \ast y) = y \ast x \forall x, y \in X$.

**Example 2.3.** Let $X = \{0, 1, 2, 3, 4\}$ be a set with the following table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
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<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>4</td>
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<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
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</table>

Then $(X, \ast, 0)$ is a BF-Algebra.

**Definition 2.4.** A partial ordering ”$\leq$” on $X$ can be defined as $x \leq y$ if and only if $x \ast y = 0$.

**Definition 2.5.** [1] A non-empty subset $S$ of a BF-algebra $X$ is said to be a subalgebra if $x \ast y \in S \forall x, y \in S$.

**Definition 2.6.** [1] A non-empty subset $I$ of a BF-algebra $X$ is said to be an ideal of $X$ if (i) $0 \in I$ ; (ii) $x \ast y \in I$ and $y \in I$ \Rightarrow $x \in I \forall x, y \in X$.

**Definition 2.7.** [1] An ideal $I$ is called closed if $0 \ast x \in I \forall x \in X$.

**Definition 2.8.** [4] An N-structure $(X, \eta)$ on a BF-algebra $X$ is called an N-subalgebra of $X$ if $\eta(x \ast y) \leq \eta(x) \lor \eta(y) \forall x, y \in X$.

**Example 2.9.** The N-structure, $(X, \eta)$ on a BF-algebra $X$ in the Examplle 2.3 defined by $\eta(x) = \left\{ \begin{array}{ll} -0.8 & ; x \neq 2 \\ -0.2 & ; x = 2 \end{array} \right.$ is an N-subalgebra of $X$. 
3. N-Structured Ideal of a BF-Algebra

In this section we introduce the notion of N-structured ideal of a BF-algebra and discuss some of its properties. X represents a BF-algebra unless otherwise specified.

**Definition 3.1.** An N-structure \((X, \eta)\) on a BF-algebra X is said to be a N-structured Ideal (N-Ideal) of X if

(i) \(\eta(0) \leq \eta(x)\) and (ii) \(\eta(x) \leq \eta(x * y) \lor \eta(y) \forall x, y \in X\).

**Example 3.2.** The N-structure, \((X, \eta)\) on a BF-algebra X in the Example 2.3 defined by

\[
\eta(x) = \begin{cases} 
-0.3 ; x = 0, 2 \\
0 \quad ; x = 1, 3, 4 
\end{cases}
\]

is an N-Ideal of X.

**Definition 3.3.** An N-structure \((X, \eta)\) on a BF-algebra X is said to be a N-structured Closed Ideal (NC-Ideal) of X if

(i) \(\eta(x) \leq \eta(x * y) \lor \eta(y)\) and (ii) \(\eta(0 * x) \leq \eta(x) \forall x, y \in X\).

**Example 3.4.** Consider the BF-algebra \(X = \{0, 1, 2, 3\}\) with the Cayley table given below.

\[
\begin{array}{|c|c|c|c|c|}
\hline
* & 0 & 1 & 2 & 3 \\
\hline
0 & 0 & 1 & 2 & 3 \\
1 & 1 & 0 & 3 & 2 \\
2 & 2 & 3 & 0 & 1 \\
3 & 3 & 2 & 1 & 0 \\
\hline
\end{array}
\]

The N-structure, \((X, \eta)\) defined by \(\eta(x) = \begin{cases} 
-0.4 \quad ; x = 0 \\
-0.3 \quad ; x = 1 \quad \text{is an NC-Ideal} \\
-0.1 \quad ; x = 2, 3 
\end{cases}\)

of X.

**Proposition 3.5.** Every NC-ideal is an N-ideal.

The converse of the Proposition 3.5 is not true, in general, as seen from the following.

**Example 3.6.** Consider the BF-algebra \(X = \{0, 1, 2, 3\}\) with the Cayley
The N-structure, \((X, \eta)\) defined by
\[
\eta(x) = \begin{cases} 
-0.4 & ; x = 0 \\
-0.3 & ; x = 1 \\
-0.1 & ; x = 2, 3 
\end{cases}
\]
is N-Ideal but not an NC-Ideal of \(X\), since \(\eta(0 \ast 1) > \eta(1)\).

**Proposition 3.7.** If \((X, \eta)\) is N-Ideal of \(X\) with \(x \leq y\) for any \(x, y \in X\) then \(\eta(x) \leq \eta(y)\). (i.e) \(\eta\) is order-preserving.

*Proof.* Let \(x, y \in X\) such that \(x \leq y\). Then \(x \ast y = 0\). Thus
\[
\eta(x) \leq \eta(x \ast y) \vee \eta(y) = \eta(0) \vee \eta(y) = \eta(y).
\]

**Proposition 3.8.** If \((X, \eta)\) is N-Ideal of \(X\) with \(x \ast y \leq z\) for any \(x, y, z \in X\) then \(\eta(x) \leq \eta(y) \vee \eta(z)\).

*Proof.* Let \(x, y, z \in X\) such that \(x \ast y \leq z\). Then \((x \ast y) \ast z = 0\).

Thus \(\eta(x) \leq \eta(x \ast y) \vee \eta(y) = (\eta(x \ast y) \ast z) \vee \eta(y)\)
\[
= (\eta(0) \vee \eta(z)) \vee \eta(y) \\
= \eta(y) \vee \eta(z)
\]

Using induction on \(n\) and from the above propositions we have,

**Theorem 3.9.** If \((X, \eta)\) is N-Ideal of \(X\) then for any \(x, a_1, a_2, \ldots, a_n \in X\) and \((\cdots ((x \ast a_1) \ast a_2) \ast \cdots)) \ast a_n = 0\) implies \(\eta(x) \leq \eta(a_1) \vee \eta(a_2) \vee \cdots \vee \eta(a_n)\).

**Theorem 3.10.** Any N-Ideal of \(X\) is an N-subalgebra of \(X\).

The converse of the theorem 3.10 is not be true in general, as seen from the following

**Example 3.11.** For the N-subalgebra in example 2.9. Take \(x = 2\) and \(y = 4\). We have \(\eta(2) = -0.2\) and \(\eta(2 \ast 4) \vee \eta(4) = \eta(3) \vee \eta(4) = -0.8 \vee -0.8 = -0.8\) \(\Rightarrow \eta(x) > \eta(x \ast y) \vee \eta(y)\) and so it is not N-Ideal.

The following gives a sufficient condition for an N-subalgebra to be an N-ideal
Theorem 3.12. Let \((X, \eta)\) be an N-subalgebra of \(X\) with \(x \ast y \leq z, \forall x, y, z \in X.\)

(1) If \(\eta(x) \leq \eta(y) \vee \eta(z) \forall x, y, z \in X\) then \((X, \eta)\) is an N-Ideal of \(X.\)

(2) If \(\eta(x) \leq \eta(x \ast y) \vee \eta(y) \forall x, y \in X\) then \((X, \eta)\) is an NC-Ideal of \(X.\)

Proof. Let \((X, \eta)\) be an N-subalgebra and \(x \ast y \leq z, \forall x, y, z \in X.\)

Proof (1). For an N-subalgebra of \(X,\) we have \(\eta(0) \leq \eta(x).\)

Also since \(x \ast (x \ast y) \leq y\) and from the hypothesis,
\(\eta(x) \leq \eta(x \ast y) \vee \eta(y).\) Hence \((X, \eta)\) is an N-Ideal of \(X.\)

Proof (2). For any \(x \in X,\) we have,
\(\eta(0 \ast x) \leq \eta(0) \vee \eta(x) = \eta(x \ast x) \vee \eta(x) \leq (\eta(x) \vee \eta(x)) \vee \eta(x) = \eta(x).\) Hence by hypothesis, \((X, \eta)\) is an NC-Ideal of \(X.\)

Theorem 3.13. If \((X, \eta)\) is an NC-Ideal of \(X,\) then \(K = \{x \in X \mid \eta(x) = \eta(0)\}\)

is an Ideal of \(X.\)

Proof. Clearly \(0 \in K\) and hence \(K \neq \phi.\) Let \(x \ast y\) and \(y \in K.\)

\(\Rightarrow \eta(x \ast y) = \eta(y) = 0. \Rightarrow \eta(x) \leq \eta(x \ast y) \vee \eta(y) = 0 \vee 0 = 0.\)

But \(\eta(0) \leq \eta(x) \Rightarrow \eta(x) = 0,\) showing that \(x \in K,\) completes the proof.

The following theorem shows the arbitrary union of family of NC-ideals of \(X\) is also an NC-ideal of \(X.\)

Theorem 3.14. Let \(\{\eta_i; \ i \in I\}\) be the family of NC-Ideals of \(X.\) Then \(\bigcup_i \eta_i\) NC-Ideal of \(X.\)

Proof. Let \(x\) and \(y \in X.\)

Since \(\{\eta_i; i \in I\}\) be the family of NC-Ideals of \(X,\) \(\forall i \in I,\) we have (i) \(\eta_i(x) \leq \eta_i(x \ast y) \vee \eta_i(y)\) and (ii) \(\eta_i(0 \ast x) \leq \eta_i(x) \forall x, y \in X.\)

Now \(\bigcup_i \eta_i(x) = \sup \{\eta_i(x); \ i \in I\} \leq \sup \{\eta_i(x \ast y) \vee \eta_i(y); \ i \in I\}\)

\[\begin{align*}
&= \sup \{\eta_i(x \ast y) \ i \in I\} \vee \sup \{\eta_i(y) \ i \in I\} \\
&= \bigcup_i \eta_i(x \ast y) \vee \bigcup_i \eta_i(y) \\
\end{align*}\]

And \(\bigcup_i \eta_i(0 \ast x) = \sup \{\eta_i(0 \ast x); i \in I\} \leq \sup \{\eta_i(x); i \in I\} = \bigcup_i \eta_i(x)\)

Hence \(\bigcup_i \eta_i\) NC-Ideal of \(X.\)

4. Conclusion

In this paper we extend the concepts of N-ideal of a BF-algebra and some of the properties have been discussed. In the paper “On BF-algebras”, Andrzej
Walendziak (see [1]) proved that every BF-algebra is a BG-algebra. Hence all the results proved in this paper can easily verify for a BG-algebra and it can be extended for different ideal structures in a BF-algebra.

References


