A PRESENTATION OF $M_{2,5}$

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Abstract: Let $M_{2,5}$ be a free metabelian nilpotent Lie algebra of rank 2 and nilpotency class 4. We show that $M_{2,5}$ admits a minimal presentation whose set of defining relators is the set of all basic commutators of length 5.

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1. Introduction

Let F be a free Lie algebra. For any group G and epimorphism $\pi : F \rightarrow G$ let $S \subseteq G$ be such that $\ker \pi$ is the ideal of F generated by S and let $\pi(X) = Y$: then π together with Y and S determines a set of generators and defining relators for G . In symbols, $G = \langle X \mid S \rangle$ denotes a presentation of G with set of generators X and set of relators S . When X and S are both finite, G is called a finitely presented Lie algebra and such a presentation is called minimal if S is the smallest subset with this property. Presentations and minimal presentations play an important role in the theory of groups and algebras. In [4] and [5], D. L. Johnson defined presentations of groups. Wamsley and Searby emphasized minimal presentations of groups. In [3], Wamsley and Searby investigated minimal presentations for metabelian groups by using any outer extension of

a finite cyclic p – group by a finite abelian p – group. Then, Wamsley examined minimal presentations for finite groups in terms of homological theory in [6]. Different from Wamsley and Searby, Moghaddam approached minimal presentations of free groups in [8]. Moghaddam investigated minimal presentations of free metabelian nilpotent groups, in terms of basic commutators. Also, Moghaddam represented finite presentation of $M_{m,n}$. Gerdt and Korniyak examined this presentations of groups in Lie algebras. Gerdt and Korniyak defined an algorithm for analysis of the structure of finitely presented Lie algebras in [7]. For more information about this topic can be reached in [1], [2], [4] and [9]. In this notes, we give a presentation of a free metabelian nilpotent Lie algebra of rank 2 and nilpotency class 4.

2. Preliminaries

Let F be a free Lie Algebra generated by the set $X = \{x, y\}$ over a field K of characteristic 0 and $M_{2,5}$ be a free metabelian Lie algebra of nilpotency class 4 and rank 2. Denote by F' and $\gamma_m(F)$ the derived subalgebra and the m – th term of the lower central series of F respectively. We identify $M_{2,5}$ with $F/\gamma_5(F) + F'$ in the usual way, where F' is the derived subalgebra of F . Consider the Lie algebra defined by the presentation $M_{2,5} = \langle x, y \mid \gamma_5(F) + F' \rangle$. Then $M_{2,5} \cong F/\gamma_5(F) + F'$

We denote the Lie product on F by (ab) , where $a, b \in F$. We need some basic results,

A word of length n is an ordered n – tuples of the elements of X . We write $\ell(u)$ for the length of the word $u \in F$.

Definition 1. A Hall Basis H of F is defined as follows:

- i) $X \subset H$ and X is given an arbitrary total ordering. Let $H_1 = X$.
- ii) H_2 consists of element (xy) such that $x, y \in X$ and $x > y$,

$$H_2 = \{(xy) \mid x, y \in X; x > y\}.$$

- iii) $H_n = \{h = ((uv)w) \mid u, v, w \in \cup_{i=1}^{n-1} H_i; u > v \leq w, \ell(h) = n\}$.

Now, we put $H = \cup_{n=1} H_n$. Then, H is the Hall basis of F constructed on X . We fix the following ordering of the set H :

- 1. X is ordered arbitrarily. (usually consistent with the indices used to denote its letters)

2. Let $u = u_1u_2$, $v = v_1v_2$ be elements of H . If $\text{length}(u) > \text{length}(v)$, put $u > v$. If u and v have the same length, then put $u > v$ if and only if either $u_1 > v_1$ or $u_1 = v_1$ and $u_2 > v_2$. Free generating sets for the lower central terms of a free Lie algebra are given by A. L. Šimel'kin in [10].

Theorem 2. *The set C_m defined as*

$$C_m = \{x = (a_1a_2) \mid x, a_1, a_2 \in H; \ell(x) \geq m; \ell(a_2) < m\}$$

is a set of free generators for $\gamma_m(F)$.

We construct a Hall basis H^{C_m} on C_m for the Lie algebra $\gamma_m(F)$, by forming products of elements of C_m . If h is a product of elements from C_m , we will refer $C_m - \ell$ and $X - \ell$ of h meaning the number of letters used from C_m or X respectively. C_m is a subset of H , so it can be given an order which coincides with the order in H . Let

$$\begin{aligned} H_1^{C_m} &= C_m. \\ H_2^{C_m} &= \{x = (a_1a_2) : a_1, a_2 \in C_m; C_m - \ell(x) = 2\}. \end{aligned}$$

We now give $H_2^{C_m}$ an order as follows: Let $h, g \in H_2^{C_m}$, where $h = h_1h_2$ and $g = g_1g_2$, $h_1, h_2, g_1, g_2 \in C_m$. If $X - \ell(h)$ is less than $X - \ell(g)$, put $h < g$. Suppose h and g have the same $X - \ell(h)$. Then put $h < g$ if and only if either $h_1 < g_1$ or $h_1 = g_1$ and $h_2 < g_2$. Suppose $H_1^{C_m}, \dots, H_{n-1}^{C_m}$ are defined and ordered. We put,

$$\begin{aligned} H_n^{C_m} &= \{x = ((a_1a_2)a_3) \mid C_m - \ell(x) = n; \\ &\quad a_1, a_2, a_3, (a_1a_2) \in \bigcup_{i=1}^{n-1} H_i^{C_m}; a_1 > a_2, a_2 \leq a_3\}. \end{aligned}$$

where the inequality signs refer to the ordering in $\bigcup_{i=1}^{n-1} H_i^{C_m}$.

3. A Presentation of $M_{2,5}$

Let F be a free Lie algebra generated by the set $X = \{x, y\}$ over a field K of characteristic zero. If $c \leq 4$ a free 2-generator nilpotent Lie algebra of class c is also metabelian. $M_{2,6}$ is the Lie algebra with the smallest nilpotent class that is not metabelian.

In this section, we shall give a presentation of $M_{2,5}$. We have that $M_{2,5} = \langle X \mid \gamma_5(F) + F \rangle$. Now, we state the following elementary lemma which will

be useful in many calculations. In the rest of the paper we always assume that $x > y$. We denote the element $\underbrace{(((bx)x)\dots)x}_{s\text{-times}}$ by (bx^s) , where $b, x \in F$.

Lemma 3. *Let a, b are monomials in F , then*

$$(a(bx^s)) = \sum_{k=0}^s (-1)^{s-k} \binom{s}{k} (((ax^{s-k})b)x^k).$$

Proof. Let us prove the Lemma by induction on s . For $s = 1$,

$$\begin{aligned} (a(bx)) &= -(b(xa)) - (x(ab)) \\ &= -((ax)b) + ((ab)x) \\ &= \sum_{k=0}^1 (-1)^{1-k} \binom{1}{k} (((ax^{1-k})b)x^k). \end{aligned}$$

Assume that the Lemma it is true for $s - 1$. Thus,

$$(a(bx^{s-1})) = \sum_{k=0}^{s-1} (-1)^{s-k-1} \binom{s-1}{k} (((ax^{s-k-1})b)x^k).$$

We will show that it is true for s ,

$$\begin{aligned} (a(bx^s)) &= (a((bx^{s-1})x)) \\ &= -((bx^{s-1})(xa)) - (x(a(bx^{s-1}))) \\ &= -((ax)(bx^{s-1})) + ((a(bx^{s-1}))x) \\ &= -((ax)(bx^{s-1})) + \left(\sum_{k=0}^{s-1} (-1)^{s-k-1} \binom{s-1}{k} (((ax^{s-k-1})b)x^k) \right) x \\ &= \sum_{k=0}^{s-1} (-1)^{s-k} \binom{s-1}{k} (((ax^{s-k})b)x^k) \\ &\quad + \sum_{k=0}^{s-1} (-1)^{s-k-1} \binom{s-1}{k} (((ax^{s-k-1})b)x^{k+1}) \\ &= \sum_{k=0}^{s-2} (-1)^{s-k-1} \binom{s-1}{k+1} (((ax^{s-k-1})b)x^{k+1}) \end{aligned}$$

$$\begin{aligned}
& +(-1)^s \binom{s-1}{0} ((ax^s)b) \\
& + \sum_{k=0}^{s-1} (-1)^{s-k-1} \binom{s-1}{k} (((ax^{s-k-1})b)x^{k+1}) \\
= & \sum_{k=0}^{s-2} (-1)^{s-k-1} \binom{s-1}{k+1} (((ax^{s-k-1})b)x^{k+1}) \\
& +(-1)^s \binom{s-1}{0} ((ax^s)b) \\
& + \sum_{k=0}^{s-2} (-1)^{s-k-1} \binom{s-1}{k} (((ax^{s-k-1})b)x^{k+1}) \\
& +(-1)^0 \binom{s-1}{s-1} ((ab)x^s) \\
= & \sum_{k=0}^{s-2} (-1)^{s-k-1} \left(\binom{s-1}{k+1} + \binom{s-1}{k} \right) (((ax^{s-k-1})b)x^{k+1}) \\
& +(-1)^s \binom{s-1}{0} ((ax^s)b) + (-1)^0 \binom{s-1}{s-1} ((ab)x^s) \\
= & \sum_{k=0}^{s-2} (-1)^{s-k-1} \binom{s}{k+1} (((ax^{s-k-1})b)x^{k+1}) \\
& +(-1)^s \binom{s-1}{0} ((ax^s)b) + (-1)^0 \binom{s-1}{s-1} ((ab)x^s) \\
= & \sum_{k=0}^{s-2} (-1)^{s-k} \binom{s}{k} (((ax^{s-k})b)x^k) + (-1)^s \binom{s}{0} ((ax^s)b) \\
& +(-1)^{s-1} \binom{s}{1} (((ax^{s-1})b)x) + (-1)^1 \binom{s}{s-1} (((ax^1)b)x^{s-1}) \\
& +(-1)^s \binom{s-1}{0} ((ax^s)b) + (-1)^0 \binom{s-1}{s-1} ((ab)x^s) \\
= & \sum_{k=0}^s (-1)^{s-k} \binom{s}{k} (((ax^{s-k})b)x^k)
\end{aligned}$$

Lemma 4. *The following monomials belong to the ideal $\langle H_5 \rangle$ of F generated by H_5 .*

- 1) $((((xy)x)x)x) = ((xy)x^3).$
- 2) $((((xy)x)x)y) = (((xy)x^2)y).$
- 3) $((((xy)x)y)y) = (((xy)x)y^2).$
- 4) $((((xy)x)y)x).$
- 5) $((((xy)y)x)x) = (((xy)y)x^2).$
- 6) $((((xy)y)x)y).$
- 7) $((((xy)y)y)y) = ((xy)y^3).$
- 8) $((((xy)y)y)x) = (((xy)y^2)x).$

Proof. By the definition the monomials of 1, 5, 7, 8 are elements of H_5 . Therefore, they are also elements of $\langle H_5 \rangle$. Now, let us show that monomials of 2, 3, 4 and 6 are also elements of $\langle H_5 \rangle$.

2) Using the Jacobi identity we obtain,

$$\begin{aligned}
 (((xy)x)x)y &= -(xy)((xy)x) - (y((xy)x))x \\
 &= ((xy)x)(xy) + (((xy)x)y)x \\
 &= ((xy)x)(xy) + ((-xy)(xy) - ((y(xy))x))x \\
 &= ((xy)x)(xy) + (((xy)y)x)x.
 \end{aligned}$$

Since $((xy)x)(xy)$ and $((((xy)y)x)x)$ are elements of H_5 then $((((xy)x)x)y) \in \langle H_5 \rangle$.

3) Using the Jacobi identity we obtain,

$$\begin{aligned}
 (((xy)x)y)y &= ((-xy)(xy) - ((y(xy))x))y \\
 &= (((xy)y)x)y \\
 &= -(xy)((xy)y) - ((y((xy)y))x) \\
 &= ((xy)y)(xy) + (((xy)y)y)x.
 \end{aligned}$$

Since $((xy)y)(xy)$ and $((((xy)y)y)x)$ are elements of H_5 then $((((xy)x)y)y) \in \langle H_5 \rangle$.

4) Using the Jacobi identity we obtain,

$$\begin{aligned}
 (((xy)x)y)x &= ((-xy)(xy) - ((y(xy))x))x \\
 &= (((xy)y)x)x.
 \end{aligned}$$

Since $((((xy)y)x)x) \in H_5$ then $((((xy)x)y)x) \in \langle H_5 \rangle$.

6) Using the Jacobi identity we obtain,

$$\begin{aligned} (((xy)y)x)y) &= -(xy)((xy)y) - ((y((xy)y))x) \\ &= ((xy)y)(xy) + (((xy)y)y)x). \end{aligned}$$

Since $((xy)y)(xy)$ and $((((xy)y)y)x)$ are elements of H_5 then $((((xy)y)x)y) \in \langle H_5 \rangle$.

Theorem 5. *The free metabelian nilpotent Lie algebra $M_{2,5}$ admits the following presentation*

$$M_{2,5} = \langle x, y \mid H_5 \rangle$$

Proof. Since, $F \subset \gamma_5(F)$ then $M_{2,5} = \langle x, y \mid \gamma_5(F) \rangle$. So, it is enough to show that $\gamma_5(F) \in \langle H_5 \rangle$. For every $u \in \gamma_5(F)$, the form of u is $\sum_i \alpha_i \omega_i$, where $\alpha_i \in K, \omega_i \in H^{C_5}$. Since $\omega_i \in H^{C_5}$ then either $\omega_i \in H_5$ or ω_i is the form of $((c_1 c_2) \dots c_k)$ such that $c_j \in C_5$ for $j = 1, 2, \dots, k$. If $\omega_i \in H_5$ then $\omega_i \in \langle H_5 \rangle$. If $\omega_i \notin H_5$ then since the form of ω_i is $((c_1 c_2) \dots c_k)$, each c_j can be written as a product of commutators of the form $((xy)y^i)x^j$ ($i + j \geq 0$). That is, the form of c_j is $c_j = ((a_1 a_2) \dots a_r)$ such that $a_r \in \{(((xy)y^i)x^j) \mid i + j \geq 0\}$ where $a_1 > a_2, a_2 \leq a_3 \leq \dots \leq a_r, \ell(a_r) < 5, r \geq 1, \ell(c_j) \geq 5$.

If $r = 1$, then $c_j = a_1$. Since $\ell(c_j) \geq 5, c_j = a_1 = (((xy)y^i)x^j)$ and $i + j \geq 3$. Hence, $c_j \in \langle H_5 \rangle$.

Now, let $r > 1$. Since $c_j = ((a_1 a_2) \dots a_r)$ and $a_1 > a_2, \ell(a_1) \neq 2$. Then, it is necessary $\ell(a_1) \geq 3$. Let $a_1 = (((xy)y^i)x^j)$. If $i + j \geq 3$, then since $\ell(a_1) \geq 5, c_j = ((a_1 a_2) \dots a_r) \in \langle H_5 \rangle$. Now, we will examine the cases $i + j = 1$ and $i + j = 2$. Since $\ell(a_1) \neq 2$ then $i + j \geq 1$. Considering all the possibilities we need to prove the following cases when $i + j = 1$ and $i + j = 2$:

- (a) $i = 0, j = 1$,
- (b) $i = 1, j = 0$,
- (c) $i = 2, j = 0$,
- (d) $i = 1, j = 1$,
- (e) $i = 0, j = 2$.

Proof of (a). If $i = 0, j = 1$, then $a_1 = ((xy)x)$. Since $a_1 > a_2$, we consider the following two cases:

Case 1. If $a_2 = (xy)$, then $c_j = ((((((xy)x)(xy))a_3) \dots a_r))$. Hence, $(a_1 a_2) = ((xy)x)(xy)$ is element of H_5 . Hence, $c_j \in \langle H_5 \rangle$.

Case 2. If $a_2 = ((xy)y)$, then $c_j = ((((((xy)x)((xy)y))a_3) \dots a_r))$. Hence,

$$(a_1 a_2) = ((xy)x)((xy)y)$$

$$\begin{aligned}
&= ((xy)x)(xy^2) \\
&= \sum_{k=0}^2 (-1)^{2-k} \binom{2}{k} (((((xy)x)y^{2-k})x)y^k) \\
&= (-1)^2 \binom{2}{0} (((((xy)x)y^2)x) + (-1)^1 \binom{2}{1} (((((xy)x)y^1)x)y) \\
&\quad + (-1)^0 \binom{2}{2} (((((xy)x)x)y^2) \\
&= (((((xy)x)y)y)x) - 2(((xy)x)y)x)y - (((((xy)x)x)y)y)
\end{aligned}$$

By Lemma 4, every term in the above sum are elements of $\langle H_5 \rangle$. Then, $c_j \in \langle H_5 \rangle$.

Proof of (b). If $i = 1, j = 0$, then $a_1 = ((xy)y)$. Since $a_1 > a_2$, then $a_2 = (xy)$. Hence, $(a_1 a_2) = ((xy)y)(xy)$ is an element of H_5 . So, $c_j = (((((xy)y)(xy))a_3) \dots a_r)$ is an element of $\langle H_5 \rangle$.

Proof of (c). If $i = 2, j = 0$, then $a_1 = (((xy)y)y)$. Since $a_1 > a_2$ we consider the following three cases:

Case 1. If $a_2 = (xy)$, then $c_j = (((((((xy)y)y)(xy))a_3) \dots a_r)$. Hence,

$$\begin{aligned}
(a_1 a_2) &= (((xy)y)y)(xy) \\
&= ((xy^3)(xy)) \\
&= -((((xy)y)y)y)x) + (((((xy)y)y)x)y).
\end{aligned}$$

It is clear that the first term in the above sum belongs to $\langle H_5 \rangle$. By Lemma 4, the second term is also an element of $\langle H_5 \rangle$. Hence, $c_j \in \langle H_5 \rangle$.

Case 2. If $a_2 = ((xy)y)$, then

$$c_j = (((((((xy)y)y)((xy)y))a_3) \dots a_r).$$

Hence,

$$\begin{aligned}
(a_1 a_2) &= (((xy)y)y)((xy)y) \\
&= (xy^3)(xy^2) \\
&= \sum_{k=0}^2 (-1)^{2-k} \binom{2}{k} (((((xy^3)y^{2-k})x)y^k) \\
&= (-1)^2 \binom{2}{0} (((xy^3)y^2)x) + (-1)^1 \binom{2}{1} (((((xy^3)y^1)x)y^1)
\end{aligned}$$

$$\begin{aligned}
& +(-1)^0 \binom{2}{2} (((xy^3)x)y^2) \\
= & ((((((xy)y)y)y)y)x) - 2((((((xy)y)y)y)x)y) \\
& +((((((xy)y)y)x)y)y)).
\end{aligned}$$

It is clear that, every term in the above sum belongs to $\langle H_5 \rangle$. Hence, $c_j \in \langle H_5 \rangle$.

Case 3. If $a_2 = ((xy)x)$, then

$$c_j = (((((((xy)y)y)((xy)x))a_3) \dots) a_r).$$

Hence,

$$\begin{aligned}
(a_1 a_2) &= (((xy)y)y)((xy)x) \\
&= (xy^3)((xy)x) \\
&= -((((xy)y)y)x)(xy) + (((((xy)y)y)(xy))x).
\end{aligned}$$

From Case 1, the second term belongs to $\langle H_5 \rangle$. By Lemma 4, the first term is also an element of $\langle H_5 \rangle$. Hence, $c_j \in \langle H_5 \rangle$.

Proof of (d). If $i = 1$, $j = 1$, then $a_1 = (((xy)y)x)$. Since $a_1 > a_2$, we consider the following four cases:

Case 1. If $a_2 = (xy)$, then $c_j = (((((((xy)y)x)(xy))a_3) \dots) a_r)$. Hence,

$$\begin{aligned}
(a_1 a_2) &= (((xy)y)x)(xy) \\
&= -((((xy)y)x)y)x) + (((((xy)y)x)x)y).
\end{aligned}$$

It is clear that, the second term in the above sum belongs to $\langle H_5 \rangle$. By Lemma 4, the first term is also an element of $\langle H_5 \rangle$. Hence, $c_j \in \langle H_5 \rangle$.

Case 2. If $a_2 = ((xy)y)$, then

$$c_j = (((((((xy)y)x)((xy)y))a_3) \dots) a_r).$$

Hence,

$$\begin{aligned}
(a_1 a_2) &= (((xy)y)x)((xy)y) \\
&= (((xy)y)x)(xy^2) \\
&= \sum_{k=0}^2 (-1)^{2-k} \binom{2}{k} (((((((xy)y)x)y^{2-k})x)y^k)
\end{aligned}$$

$$\begin{aligned}
&= (-1)^2 \binom{2}{0} (((((xy)y)x)y^2)x) \\
&\quad + (-1)^1 \binom{2}{1} (((((xy)y)x)y^1)x)y^1) \\
&\quad + (-1)^0 \binom{2}{2} (((((xy)y)x)x)y^2) \\
&= (((((xy)y)x)y)y)x) - 2((((xy)y)x)y)x)y \\
&\quad + (((((xy)y)x)x)y)y).
\end{aligned}$$

The third term in the above sum belongs to $\langle H_5 \rangle$. By Lemma 4, the first and the second term are elements of $\langle H_5 \rangle$. Hence, $c_j \in \langle H_5 \rangle$.

Case 3. If $a_2 = ((xy)x)$, then

$$c_j = ((((((xy)y)x)((xy)x))a_3) \dots) a_r).$$

Hence,

$$\begin{aligned}
(a_1 a_2) &= (((xy)y)x)((xy)x) \\
&= -((((xy)y)x)x)(xy) + (((((xy)y)x)(xy))x).
\end{aligned}$$

The first term in the above sum belongs to $\langle H_5 \rangle$. From Case 1, the second term is also an element of $\langle H_5 \rangle$. Hence, $c_j \in \langle H_5 \rangle$.

Case 4. If $a_2 = (((xy)y)y)$, then

$$c_j = ((((((xy)y)x)((xy)y)y))a_3) \dots) a_r).$$

Hence,

$$\begin{aligned}
(a_1 a_2) &= (((xy)y)x)((xy)y)y) \\
&= (((xy)y)x)(xy^3) \\
&= \sum_{k=0}^3 (-1)^{3-k} \binom{3}{k} ((((((xy)y)x)y^{3-k})x)y^k) \\
&= (-1)^3 \binom{3}{0} ((((((xy)y)x)y^3)x) \\
&\quad + (-1)^2 \binom{3}{1} ((((((xy)y)x)y^2)x)y^1)
\end{aligned}$$

$$\begin{aligned}
& +(-1)^1 \binom{3}{2} ((((((xy)y)x)y)x)y^2) \\
& +(-1)^0 \binom{3}{3} ((((((xy)y)x)x)y^3) \\
& = -((((((xy)y)x)y^3)x) + 3((((((xy)y)x)y^2)x)y) \\
& - 3((((((xy)y)x)y)x)y^2) + ((((((xy)y)x)x)y^3)).
\end{aligned}$$

The fourth term in the above sum belongs to $\langle H_5 \rangle$. By Lemma 4, the first term, the second term and the third term are elements of $\langle H_5 \rangle$. Hence, $c_j \in \langle H_5 \rangle$.

Proof of (e). If $i = 0, j = 2$ then $a_1 = ((xy)x)x$. Since $a_1 > a_2$, we consider the following five cases for a_2 .

Case 1. If $a_2 = (xy)$ then $c_j = (((((((xy)x)x)(xy))a_3) \dots) a_r)$. Hence,

$$\begin{aligned}
(a_1 a_2) &= (((xy)x)x)(xy) \\
&= -((((((xy)x)x)y)x) + (((((((xy)x)x)x)y)).
\end{aligned}$$

The second term in the above sum belongs to $\langle H_5 \rangle$. By Lemma 4, the first term is an elements of $\langle H_5 \rangle$. Hence, $c_j \in \langle H_5 \rangle$.

Case 2. If $a_2 = ((xy)y)$ then

$$c_j = (((((((((xy)x)x)((xy)y))a_3) \dots) a_r).$$

Hence,

$$\begin{aligned}
(a_1 a_2) &= (((xy)x)x)((xy)y) \\
&= (((xy)x)x)(xy^2) \\
&= \sum_{k=0}^2 (-1)^{2-k} \binom{2}{k} (((((((((xy)x)x)y^{2-k})x)y^k) \\
&= (-1)^2 \binom{2}{0} (((((((((xy)x)x)y^2)x) \\
&\quad + (-1)^1 \binom{2}{1} (((((((((xy)x)x)y^1)x)y^1) \\
&\quad + (-1)^0 \binom{2}{2} (((((((((xy)x)x)x)y^2) \\
&= (((((((((xy)x)x)y)y)x) - 2((((((((((xy)x)x)y)x)y) \\
&\quad + (((((((((xy)x)x)x)y)y).
\end{aligned}$$

The third term in the above sum belongs to $\langle H_5 \rangle$. From Case 1, the first term and the second term are elements of $\langle H_5 \rangle$. Hence, $c_j \in \langle H_5 \rangle$.

Case 3. If $a_2 = ((xy)x)$ then

$$c_j = ((((((xy)x)x)((xy)x))a_3) \dots) a_r).$$

Hence,

$$\begin{aligned} (a_1 a_2) &= (((xy)x)x)((xy)x) \\ &= -(((xy)x)x)x(xy) + (((xy)x)x(xy))x. \end{aligned}$$

First term in the above sum belongs to $\langle H_5 \rangle$. From Case 1, the second term is an element of $\langle H_5 \rangle$. Hence, $c_j \in \langle H_5 \rangle$.

Case 4. If $a_2 = ((xy)y)y$ then

$$c_j = ((((((xy)x)x)((xy)y)y))a_3) \dots) a_r).$$

Hence,

$$\begin{aligned} (a_1 a_2) &= (((xy)x)x)((xy)y)y \\ &= (((xy)x)x)(xy^3) \\ &= \sum_{k=0}^3 (-1)^{3-k} \binom{3}{k} ((((((xy)x)x)y^{3-k})x)y^k) \\ &= (-1)^3 \binom{3}{0} ((((((xy)x)x)y^3)x) \\ &\quad + (-1)^2 \binom{3}{1} ((((((xy)x)x)y^2)x)y^1) \\ &\quad + (-1)^1 \binom{3}{2} ((((((xy)x)x)y)x)y^2) \\ &\quad + (-1)^0 \binom{3}{3} ((((((xy)x)x)x)y^3) \\ &= -((((xy)x)x)y^3)x) + 3((((xy)x)x)y^2)x)y \\ &\quad - 3((((xy)x)x)y)x)y^2 + (((((xy)x)x)x)y^3). \end{aligned}$$

The fourth term in the above sum belongs to $\langle H_5 \rangle$. By Lemma 4, the first term, the second term and the third term are elements of $\langle H_5 \rangle$. Hence, $c_j \in \langle H_5 \rangle$.

Case 5. If $a_2 = (((xy)y)x)$ then

$$c_j = (((((((xy)x)x)((xy)y)x))a_3) \dots a_r).$$

Hence,

$$\begin{aligned} (a_1 a_2) &= (((xy)x)x)((xy)y)x) \\ &= -((((xy)x)x)((xy)y) + (((((xy)x)x)((xy)y))x)). \end{aligned}$$

The first term in the above sum belongs to $\langle H_5 \rangle$. From Case 2, the second term is an element of $\langle H_5 \rangle$. Hence, $c_j \in \langle H_5 \rangle$. Since, $c_j \in \langle H_5 \rangle$ and $\omega_i = (((c_1 c_2) \dots c_k)$ then $\omega_i \in \langle H_5 \rangle$. Therefore, $u = \sum_i \alpha_i \omega_i \in \langle H_5 \rangle$. So, we obtain $\gamma_5(F) \subset \langle H_5 \rangle$. Hence, $M_{2,5}$ admits the following presentation

$$\begin{aligned} M_{2,5} &= \langle x, y \mid \langle H_5 \rangle \rangle \\ &= \langle x, y \mid H_5 \rangle \end{aligned}$$

The independency of H_5 leads the minimality of the presentation.

References

- [1] Anthony M. Gaglione and Dennis Speilman. (F) and $F/\gamma_{n+1}(F)$ Revisited, Contemporary Mathematics. Volume 109 (1990), 35-41.
- [2] Chander Kanta Gupta and Frank Levin, *Dimension subgroups of free center-by- metabelin groups*. Illinois Journal of Mathematics. Volume 30, Number 2, Summer(1986).
- [3] D.G. Searby and J. W. Wamsley, *Minimal Presentations For Certain Metabelian Groups*. Prceedings of The American Mathematical Society. Volume 32, Number 2, April(1972).
- [4] D. L.Johnson, *Presentation of Groups*. London Mathematical Society Lecture Note Series 22 (1976), Cambridge University Press.
- [5] D. L.Johnson, *Topics in The Theory of Groups Presentations*. London Mathematical Society Lecture Note Series 42 (1980). Cambridge University Press.
- [6] J. W. Wamsley, *Minimal Presentations For Finite Groups* . Bull. London Math. Soc., 5 (1973), 129-144.

- [7] Vladimir P. Gerdt and Vladimir V. Kornyak, *An Algorithm For Analysis Of The Structure Of Finitely Presented Lie Algebras*. Discrete Mathematics and Theoretical Computer Science 1, (1997), 217-228.
- [8] Gholamhossein Iraghi Moghaddam, *Minimal Presentations of Free Metabelian Nilpotent Groups*. The University Of Manitoba Faculty Of Graduate Studies. December(1999).
- [9] N. Jacobson, *Lie algebras*, New York, 1962.
- [10] Smel'kin, A. L. *Free polynilpotent groups* (Three parts)
 - (i) Soviet Math. Dokl. (Engl. trans.) 4, (1963) 950-953.
 - (ii) Izvest. Akad. Nauk S.S.S.R. Ser. Mat. (Russian) 28, (1964), 91-122.; Trans. Amer. Math. Soc. (Engl. trans.) (1966) 270-304
 - (iii) Dokl. Akad. Nauk. S.S.S.R. 169, (1966), 1024-25.