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ON THE DIOPHANTINE EQUATION $7^x + 8^y = z^2$

Banyat Sroysang

Department of Mathematics and Statistics
Faculty of Science and Technology
Thammasat University, Rangsit Center
Pathumthani 12121, THAILAND

Abstract: In this paper, we show that the Diophantine equation $7^x + 8^y = z^2$ has a unique non-negative integer solution. The solution (x, y, z) is $(0, 1, 3)$.

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1. Introduction

In 2012, Sroysang [5] showed that $(1, 0, 2)$ is a unique solution (x, y, z) for the Diophantine equation $3^x + 5^y = z^2$ where x, y and z are non-negative integers. This proved by the Catalan's conjecture [2]: $(3, 2, 2, 3)$ is a unique solution (a, b, x, y) for the Diophantine equation $a^x - b^y = 1$ where a, b, x and y are integers with $\min\{a, b, x, y\} > 1$. The Catalan's conjecture was proven by Mihailescu [4] in 2004. In this paper, we will show by the Catalan's conjecture that $(0, 1, 3)$ is a unique solution (x, y, z) for the Diophantine equation $7^x + 8^y = z^2$ where x, y and z are non-negative integers. For related papers, we list them as follows. In 2007, Acu [1] proved that $(3, 0, 3)$ and $(2, 1, 3)$ are only two solutions (x, y, z) for the Diophantine equation $2^x + 5^y = z^2$ where x, y and z are non-negative integers. In 2011, Suvarnamani, Singta and Chotchaisthit [8] proved that the two Diophantine equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$ have no non-negative integer solution. In 2012, Chotchaisthit [3] found all non-

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negative integer solutions for the Diophantine equation of type $4^x + p^y = z^2$ where p is a positive prime number. In the same year, Sroysang [6] showed that $(1, 0, 3)$ is a unique solution (x, y, z) for the Diophantine equation $8^x + 19^y = z^2$ where x, y and z are non-negative integers. Moreover, he [7] proved that the Diophantine equation $31^x + 32^y = z^2$ has no non-negative integer solution.

2. Preliminaries

Proposition 2.1. [4] (**the Catalan's conjecture**) $(3, 2, 2, 3)$ *is a unique solution* (a, b, x, y) *for the Diophantine equation* $a^x - b^y = 1$ *where* a, b, x *and* y *are integers with* $\min\{a, b, x, y\} > 1$.

Lemma 2.2. *The Diophantine equation* $7^x + 1 = z^2$ *has no non-negative integer solution.*

Proof. Suppose that there are non-negative integers x and z such that $7^x + 1 = z^2$. If $x = 0$, then $z^2 = 2$ which is impossible. Then $x \geq 1$. Thus, $z^2 = 7^x + 1 \geq 7^1 + 1 = 8$. Then $z \geq 3$. Now, we consider on the equation $z^2 - 7^x = 1$. By Proposition 2.1, we have $x = 1$. Then $z^2 = 8$. This is a contradiction. Hence, the equation $7^x + 1 = z^2$ has no non-negative integer solution. □

Lemma 2.3. $(1, 3)$ *is a unique solution* (y, z) *for the Diophantine equation* $1 + 8^y = z^2$ *where* y *and* z *are non-negative integers.*

Proof. Let y and z be non-negative integers such that $1 + 8^y = z^2$. If $y = 0$, then $z^2 = 2$ which is impossible. Then $y \geq 1$. Thus, $z^2 = 8^y + 1 \geq 8^1 + 1 = 9$. Then $z \geq 3$. Now, we consider on the equation $z^2 - 8^y = 1$. By Proposition 2.1, we have $y = 1$. Then $z = 3$. Hence, $(1, 3)$ is a unique solution (y, z) for the equation $1 + 8^y = z^2$ where y and z are non-negative integers. □

By Lemma 2.3, we can see that $7^0 + 8^1 = 3^2$. In the next section, we will present the main result.

3. Results

Theorem 3.1. $(0, 1, 3)$ *is a unique solution* (x, y, z) *for the Diophantine equation* $7^x + 8^y = z^2$ *where* x, y *and* z *are non-negative integers.*

Proof. Let x, y and z be non-negative integers such that $7^x + 8^y = z^2$. By Lemma 2.2, we have $y \geq 1$. Now, we divide the number x into two cases.

Case $x = 0$. By Lemma 2.3, we have $y = 1$ and $z = 3$.

Case $x \geq 1$. Note that z is odd. Then $z^2 \equiv 1 \pmod 4$. This implies that $7^x \equiv 1 \pmod 4$. Thus, x is even. Let $x = 2k$ where k is a positive integer. Then $z^2 - 7^{2k} = 2^{3y}$. Then $(z - 7^k)(z + 7^k) = 2^{3y}$. Thus, $z - 7^k = 2^u$ where u is a non-negative integer. Then $z + 7^k = 2^{3y-u}$. Thus, $2(7^k) = 2^{3y-u} - 2^u = 2^u(2^{3y-2u} - 1)$. It follows that $u = 1$. Then $7^k = 2^{3y-2} - 1$. Thus, $2^{3y-2} - 7^k = 1$. Since $k \geq 1$, we have $3y - 2 \geq 3$. By Proposition 2.1, we have $k = 1$. Then $2^{3y-2} = 8$. Then $3y - 2 = 3$. Thus, $y = \frac{5}{3}$. This is a contradiction.

Therefore, $(0, 1, 3)$ is a unique solution (x, y, z) for the equation $7^x + 8^y = z^2$ where x, y and z are non-negative integers. □

Corollary 3.2. *The Diophantine equation $7^x + 8^y = w^4$ has no non-negative integer solution.*

Proof. Suppose that there are non-negative integers x, y and w such that $7^x + 8^y = w^4$. Let $z = w^2$. Then $7^x + 8^y = z^2$. By Theorem 3.1, we have $(x, y, z) = (0, 1, 3)$. Then $w^2 = z = 3$. This is a contradiction. Hence, the equation $7^x + 8^y = w^4$ has no non-negative integer solution. □

4. Open Problem

We note in our results that 7 is odd prime number and $8 - 7 = 1$. Let p be a positive odd prime number. We may ask for the set of all solutions (x, y, z) for the Diophantine equation $p^x + (p+1)^y = z^2$ where x, y and z are non-negative integers.

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