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THE DETERMINANT OF A SPECIAL FIVE-DIAGONAL MATRIX AND THE FIBONACCI POLYNOMIALS

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Abstract: Many mathematicians investigated in papers various types of integer matrices the entries of which satisfy a second order recurrence. Some of the authors used methods leading to obtain real or complex factorizations of the Fibonacci or the Lucas numbers. Civciv (2008) computed the determinant of a five-diagonal matrix with the Fibonacci numbers as its entries. His result is given more generally and completely in this paper. It is showed that the determinant of a matrix, the entries of which are the Gibonacci numbers, is related to the values of the Fibonacci polynomial. Calculations are done by using the eigenvalues of the given matrix.

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1. Introduction

The study of matrices with entries given as the Fibonacci and related numbers has a long history. Some application problems are often solved using graphs or

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digraphs associated with this type of matrices (see more details in [6]). One of the main purposes of the investigation of matrices with the Fibonacci and the Lucas numbers is to derive various factorizations of these numbers. The well-known Fibonacci numbers F_n and Lucas numbers L_n are defined as terms of the sequences given by the same recurrence with the different initial terms. Concretely, $F_{n+2} = F_{n+1} + F_n, F_0 = 0, F_1 = 1$ or $L_{n+2} = L_{n+1} + L_n, L_0 = 2, L_1 = 1$, respectively. Cahill et al. [2] studied certain families of tridiagonal matrices and their connection to these sequences and derived the following complex factorizations

$$F_n = \prod_{j=1}^{n-1} \left(1 - 2i\cos\frac{j\pi}{n}\right), \quad n \geq 2, \tag{1}$$

and

$$L_n = \prod_{j=1}^{n} \left(1 - 2i\cos\frac{(2j-1)\pi}{2n}\right), \quad n \geq 1. \tag{2}$$

They proved the factorizations by considering in what way these numbers can be connected to the Chebyshev polynomials by determinants of suitable tridiagonal matrices.

The Fibonacci-like numbers U_n are given by the second-order recurrence $U_{n+2} = pU_{n+1} - qU_n$ for arbitrary integer parameters p, q different from 0, with $U_0 = 0$, $U_1 = 1$. Some results on the factorization of the Fibonacci-like numbers and their squares are given in [7]. These factorizations were found using the circulant matrices, their determinants and eigenvalues. Then

$$U_n = \prod_{j=1}^{n-1} \left(p - 2\sqrt{q}\cos\frac{j\pi}{n}\right), \quad n \geq 2,$$

and

$$U_n^2 = \prod_{j=1}^{n-1} \left(p^2 - 2q - 2q\cos\frac{2j\pi}{n}\right), \quad n \geq 2.$$

The Lucas-like numbers V_n are defined by the same recurrence as the numbers U_n with $V_0 = 2, V_1 = p$. However, no similar factorizations for the Lucas-like numbers were found by using the determinant of circulant matrices.

Some classes of polynomials can be defined by the Fibonacci-like recurrence relations. Catalan studied polynomials $f_n(x)$ called the Fibonacci polynomials. He defined these polynomials by the recurrence relation $f_{n+2}(x) = xf_{n+1}(x) +$

$f_n(x)$, where $f_0(x) = 0$, $f_1(x) = 1$. There is an explicit formula for them

$$f_n(x) = \frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)},$$

where $\alpha(x) = \dfrac{x + \sqrt{x^2+4}}{2}, \beta(x) = \dfrac{x - \sqrt{x^2+4}}{2}$.

If x is a positive integer then the numbers $f_n(x)$ are sometimes called the Fibonacci numbers of the x-th order. It is obvious that $f_n(1) = F_n$ are the common Fibonacci numbers, $f_n(2) = P_n$ are the well-known Pell numbers and so on.

The Fibonacci polynomials can be factored (see e.g. [6], p. 478) as $f_n(x) = \prod_{j=1}^{n-1} \left(x - 2i \cos \frac{j\pi}{n}\right)$ and factorization (1) of F_n also follows from it.

Civciv [4] investigated the following $k \times k$ five-diagonal matrix

$$A_k^{(n)} = \begin{pmatrix} 1-F_nF_{n-1} & F_{n+1} & F_nF_{n-1} & \cdots & \cdots \\ -F_{n+1} & 1-2F_nF_{n-1} & F_{n+1} & \cdots & \cdots \\ F_nF_{n-1} & -F_{n+1} & 1-2F_nF_{n-1} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & 1-2F_nF_{n-1} & F_{n+1} \\ \cdots & \cdots & \cdots & -F_{n+1} & 1-F_nF_{n-1} \end{pmatrix}.$$

He expressed the determinant of $A_k^{(n)}$ in the form

$$\det A_k^{(n)} = \begin{cases} \prod_{j=1}^{k} \left(1 - 2iF_{n+2} \cos \frac{j\pi}{k+1} - 4F_nF_{n-1} \cos^2 \frac{j\pi}{k+1}\right), & n \text{ odd}, \\ \prod_{j=1}^{k} \left(1 - 2iF_{n-2} \cos \frac{j\pi}{k+1} + 4F_nF_{n-1} \cos^2 \frac{j\pi}{k+1}\right), & n \text{ even}. \end{cases}$$

But this result is imprecise with respect to a small mistake at the end of derivation. The correct relation can be expressed in the form

$$\det A_k^{(n)} = \prod_{j=1}^{k} \left(1 - 2iF_{n+1} \cos \frac{j\pi}{k+1} - 4F_nF_{n-1} \cos^2 \frac{j\pi}{k+1}\right) \tag{3}$$

as we will show in the next section of this paper.

Some of the following ideas are based on one of our previous contributions [8].

2. The Main Results

There are many connections between the determinants of tridiagonal matrices and the Fibonacci numbers or numbers which are given by their generalization. Some five-diagonal matrices and their determinants have also this property. We can investigate a generalization of the Civciv's matrix $A_k^{(n)}$. The entries of the new matrix are the generalized Fibonacci numbers G_n, sometimes called the Gibonacci numbers. The Gibonacci sequence satisfies the Fibonacci recurrence $G_{n+2} = G_{n+1} + G_n$, but its initial terms can be arbitrary integers G_0, G_1, where at least one of them is different from 0.

Theorem 1. *Let M_k be a five-diagonal square matrix of an order $k \geq 3$ given as*

$$M_k = \begin{pmatrix} 1-G_nG_{n-1} & G_{n+1} & G_nG_{n-1} & \cdots & \cdots \\ -G_{n+1} & 1-2G_nG_{n-1} & G_{n+1} & \cdots & \cdots \\ G_nG_{n-1} & -G_{n+1} & 1-2G_nG_{n-1} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & 1-2G_nG_{n-1} & G_{n+1} \\ \cdots & \cdots & \cdots & -G_{n+1} & 1-G_nG_{n-1} \end{pmatrix}.$$

Then

$$\det M_k = \prod_{j=1}^{k} \left(1 - 2iG_{n+1}\cos\frac{j\pi}{k+1} - 4G_nG_{n-1}\cos^2\frac{j\pi}{k+1}\right). \tag{4}$$

Proof. It is easy to see that $M_k = P_kQ_k$, where P_k, Q_k are the following three-diagonal square matrices of an order k

$$P_k = \begin{pmatrix} 1 & G_n & 0 & \cdots & \cdots \\ -G_n & 1 & G_n & \cdots & \cdots \\ 0 & -G_n & 1 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & 1 & G_n \\ \cdots & \cdots & \cdots & -G_n & 1 \end{pmatrix},$$

$$Q_k = \begin{pmatrix} 1 & G_{n-1} & 0 & \cdots & \cdots \\ -G_{n-1} & 1 & G_{n-1} & \cdots & \cdots \\ 0 & -G_{n-1} & 1 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & 1 & G_{n-1} \\ \cdots & \cdots & \cdots & -G_{n-1} & 1 \end{pmatrix}.$$

It means that $\det M_k = \det P_k \det Q_k$. As P_k, Q_k are tridiagional matrices the following recurrences of the second order hold for their determinants ($k \geq 3$)

$$\det P_k = \det P_{k-1} + G_n^2 \det P_{k-2}, \det P_1 = 1, \det P_2 = 1 + G_n^2,$$
$$\det Q_k = \det Q_{k-1} + G_{n-1}^2 \det Q_{k-2}, \det Q_1 = 1, \det Q_2 = 1 + G_{n-1}^2.$$

It is easy to see that $\det P_k$ is also equal to $\det \overline{P_k}$, where

$$\overline{P_k} = \begin{pmatrix} 1 & iG_n & 0 & \cdots & \cdots \\ iG_n & 1 & iG_n & \cdots & \cdots \\ 0 & iG_n & 1 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & 1 & iG_n \\ \cdots & \cdots & \cdots & iG_n & 1 \end{pmatrix}$$

as the sequence of $\det \overline{P_k}$ satisfies the same reccurence as the sequence of $\det P_k$. The numbers G_n are only changed to G_{n-1} for $\det Q_k = \det \overline{Q_k}$.

We know that the determinant of a square matrix can be expressed as the product of its eigenvalues. We can write

$$\overline{P_k} = I + iG_n \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots \\ 1 & 0 & 1 & \cdots & \cdots \\ 0 & 1 & 0 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & 0 & 1 \\ \cdots & \cdots & \cdots & 1 & 0 \end{pmatrix} = I + iG_n E,$$

where I is the identity matrix of an order k.

The eigenvalues of the square matrix E are the real numbers $-2\cos\frac{j\pi}{k+1}$, $j = 1, 2, ..., k$, (more details e.g. in [2]) and therefore the eigenvalues λ_j of $\overline{P_k}$ have the form $\lambda_j = 1 - 2iG_n \cos\frac{j\pi}{k+1}$.

Thus, $\det P_k = \det \overline{P_k} = \prod_{j=1}^{k} \lambda_j = \prod_{j=1}^{k} \left(1 - 2iG_n \cos\frac{j\pi}{k+1}\right)$ and similarly

$\det Q_k = \det \overline{Q_k} = \prod_{j=1}^{k}\left(1 - 2iG_{n-1}\cos\frac{j\pi}{k+1}\right)$. Then

$$\begin{aligned}\det M_k &= \det P_k \det Q_k = \\ &= \prod_{j=1}^{k}\left(1 - 2iG_n\cos\frac{j\pi}{k+1}\right)\left(1 - 2iG_{n-1}\cos\frac{j\pi}{k+1}\right) = \\ &= \prod_{j=1}^{k}\left(1 - 2i\left(G_n + G_{n-1}\right)\cos\frac{j\pi}{k+1} - 4G_nG_{n-1}\cos^2\frac{j\pi}{k+1}\right) = \\ &= \prod_{j=1}^{k}\left(1 - 2iG_{n+1}\cos\frac{j\pi}{k+1} - 4G_nG_{n-1}\cos^2\frac{j\pi}{k+1}\right)\end{aligned}$$

which completes the proof of identity (4). □

Corollary 2. *The relation*

$$\det M_k = G_{n-1}^k G_n^k f_{k+1}\left(\frac{1}{G_{n-1}}\right) f_{k+1}\left(\frac{1}{G_n}\right) \tag{5}$$

holds for an arbitrary positive integer $n \geq 1$ if $G_0 \neq 0$, $G_1 \neq 0$ and $f_{k+1}(x)$ is the $(k+1)$-st Fibonacci polynomial.

Proof. We can write

$$\begin{aligned}\det M_k &= \prod_{j=1}^{k}\left(1 - 2iG_{n-1}\cos\frac{j\pi}{k+1}\right)\prod_{j=1}^{k}\left(1 - 2iG_n\cos\frac{j\pi}{k+1}\right) = \\ &= G_{n-1}^k G_n^k \prod_{j=1}^{k}\left(\frac{1}{G_{n-1}} - 2i\cos\frac{j\pi}{k+1}\right)\prod_{j=1}^{k}\left(\frac{1}{G_n} - 2i\cos\frac{j\pi}{k+1}\right) = \\ &= G_{n-1}^k G_n^k f_{k+1}\left(\frac{1}{G_{n-1}}\right) f_{k+1}\left(\frac{1}{G_n}\right)\end{aligned}$$

with respect to the factorization of the Fibonacci polynomials. □

Now, we will consider some special cases.

Example 3. *Let $G_n = F_n$, we obtain the Civciv's determinant $\det A_k^{(n)}$ from [4]. Then for $n \geq 2$ we have*

$$\det A_k^{(n)} = \prod_{j=1}^{k}\left(1 - 2iF_{n+1}\cos\frac{j\pi}{k+1} - 4F_nF_{n-1}\cos^2\frac{j\pi}{k+1}\right)$$

which is relation (3) *of the previous section. It follows from* (5) *that*

$$\det A_k^{(n)} = F_{n-1}^k F_n^k f_{k+1}\left(\frac{1}{F_{n-1}}\right) f_{k+1}\left(\frac{1}{F_n}\right).$$

If $n = 2$ then the determinant $\det A_k^{(2)}$ is the same as that of the open Problem 1 in [4]. It means that

$$\det A_k^{(2)} = \begin{vmatrix} 0 & 2 & 1 & \cdots & \cdots \\ -2 & -1 & 2 & \cdots & \cdots \\ 1 & -2 & -1 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & -1 & 2 \\ \cdots & \cdots & \cdots & -2 & 0 \end{vmatrix} = F_1^k F_2^k f_{k+1}(1) f_{k+1}(1) = F_{k+1}^2$$

and the problem is solved up.

If $n = 3$ then the matrix $A_k^{(3)}$ has this form

$$A_k^{(3)} = \begin{pmatrix} -1 & 3 & 2 & \cdots & \cdots \\ -3 & -3 & 3 & \cdots & \cdots \\ 2 & -3 & -3 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & -3 & 3 \\ \cdots & \cdots & \cdots & -3 & -1 \end{pmatrix}$$

and the following relation holds for its determinant

$$\det A_k^{(3)} = \prod_{j=1}^{k}\left(1 - 2iF_4\cos\frac{j\pi}{k+1} - 4F_3F_2\cos^2\frac{j\pi}{k+1}\right) =$$
$$= F_2^k F_3^k f_{k+1}\left(\frac{1}{F_2}\right) f_{k+1}\left(\frac{1}{F_3}\right) = 2^k f_{k+1}(1) f_{k+1}\left(\frac{1}{2}\right) =$$
$$= 2^k F_{k+1} f_{k+1}\left(\frac{1}{2}\right),$$

where $f_{k+1}\left(\frac{1}{2}\right) = 2\,\dfrac{\left(\frac{1+\sqrt{17}}{4}\right)^{k+1} - \left(\frac{1-\sqrt{17}}{4}\right)^{k+1}}{\sqrt{17}}$.

Example 4. *Let $G_n = L_n$, the following $k \times k$ five-diagonal matrix*

$$B_k^{(n)} = \begin{pmatrix} 1-L_nL_{n-1} & L_{n+1} & L_nL_{n-1} & \cdots & \cdots \\ -L_{n+1} & 1-2L_nL_{n-1} & L_{n+1} & \cdots & \cdots \\ L_nL_{n-1} & -L_{n+1} & 1-2L_nL_{n-1} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & 1-2L_nL_{n-1} & L_{n+1} \\ \cdots & \cdots & \cdots & -L_{n+1} & 1-L_nL_{n-1} \end{pmatrix}$$

has the determinant

$$\det B_k^{(n)} = \prod_{j=1}^{k} \left(1 - 2iL_{n+1}\cos\frac{j\pi}{k+1} - 4L_nL_{n-1}\cos^2\frac{j\pi}{k+1}\right) = $$
$$= L_{n-1}^k L_n^k f_{k+1}\left(\frac{1}{L_{n-1}}\right) f_{k+1}\left(\frac{1}{L_n}\right)$$

using identity (5).

If $n = 2$ then the matrix $B_k^{(2)}$ has this form

$$B_k^{(2)} = \begin{pmatrix} -2 & 4 & 3 & \cdots & \cdots \\ -4 & -5 & 4 & \cdots & \cdots \\ 3 & -4 & -5 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & -5 & 4 \\ \cdots & \cdots & \cdots & -4 & -2 \end{pmatrix}$$

and the following relation holds for its determinant

$$\det B_k^{(2)} = \prod_{j=1}^{k} \left(1 - 2iL_3\cos\frac{j\pi}{k+1} - 4L_2L_1\cos^2\frac{j\pi}{k+1}\right) = $$
$$= L_1^k L_2^k f_{k+1}\left(\frac{1}{L_1}\right) f_{k+1}\left(\frac{1}{L_2}\right) = 3^k f_{k+1}(1)\, f_{k+1}\left(\frac{1}{3}\right) = $$
$$= 3^k F_{k+1} f_{k+1}\left(\frac{1}{3}\right),$$

where $f_{k+1}\left(\frac{1}{3}\right) = 3\,\dfrac{\left(\frac{1+\sqrt{37}}{6}\right)^{k+1} - \left(\frac{1-\sqrt{37}}{6}\right)^{k+1}}{\sqrt{37}}$.

References

[1] N.H. Bong, Fibonacci matrices and matrix representation of Fibonacci numbers, *Southeast Asian Bulletin of Mathematics*, **23**, No. 3 (1999), 357-374.

[2] N.D. Cahill, J.R. D'Errico, J.P. Spence, Complex factorizations of the Fibonacci and Lucas numbers, *Fibonacci Quarterly*, **41**, No. 1 (2003), 13-19.

[3] N.D. Cahill, D. Narayan, Fibonacci and Lucas numbers as tridiagonal matrix determinants, *Fibonacci Quarterly*, **42**, No. 3 (2004), 216-221.

[4] H. Civciv, A note on the determinant of five-diagonal matrices with Fibonacci numbers, *Int. J. Contemp. Math. Sciences*, **3**, No. 9 (2008), 419-424.

[5] E. Kilic, D. Tasci, Negatively subscripted Fibonacci and Lucas numbers and their complex factorizations, *Ars Combinatoria*, **96** (2010), 275-288.

[6] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley & Sons, New York (2001).

[7] J. Seibert, P. Trojovsky, Circulants and the factorization of the Fibonacci-like numbers, *Acta Mathematica Universitatis Ostraviensis*, **14**, No. 1 (2006), 63-70.

[8] J. Seibert, On the determinant of a special matrix with the Gibonacci numbers, In: *Proceedings of the 10th conference APLIMAT 2011*, Bratislava (2011), 159-164.