THE DETERMINANT OF A SPECIAL FIVE-DIAGONAL MATRIX AND THE FIBONACCI POLYNOMIALS

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Abstract: Many mathematicians investigated in papers various types of integer matrices the entries of which satisfy a second order recurrence. Some of the authors used methods leading to obtain real or complex factorizations of the Fibonacci or the Lucas numbers. Civciv (2008) computed the determinant of a five-diagonal matrix with the Fibonacci numbers as its entries. His result is given more generally and completely in this paper. It is showed that the determinant of a matrix, the entries of which are the Gibonacci numbers, is related to the values of the Fibonacci polynomial. Calculations are done by using the eigenvalues of the given matrix.

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1. Introduction

The study of matrices with entries given as the Fibonacci and related numbers has a long history. Some application problems are often solved using graphs or
digraphs associated with this type of matrices (see more details in [6]). One of the main purposes of the investigation of matrices with the Fibonacci and the Lucas numbers is to derive various factorizations of these numbers. The well-known Fibonacci numbers \( F_n \) and Lucas numbers \( L_n \) are defined as terms of the sequences given by the same recurrence with the different initial terms. Concretely, \( F_{n+2} = F_{n+1} + F_n, F_0 = 0, F_1 = 1 \) or \( L_{n+2} = L_{n+1} + L_n, L_0 = 2, L_1 = 1 \), respectively. Cahill et al. [2] studied certain families of tridiagonal matrices and their connection to these sequences and derived the following complex factorizations

\[
F_n = \prod_{j=1}^{n-1} \left( 1 - 2i \cos \frac{j\pi}{n} \right), \quad n \geq 2, \tag{1}
\]

and

\[
L_n = \prod_{j=1}^{n} \left( 1 - 2i \cos \frac{(2j - 1)\pi}{2n} \right), \quad n \geq 1. \tag{2}
\]

They proved the factorizations by considering in what way these numbers can be connected to the Chebyshev polynomials by determinants of suitable tridiagonal matrices.

The Fibonacci-like numbers \( U_n \) are given by the second-order recurrence \( U_{n+2} = pU_{n+1} - qU_n \) for arbitrary integer parameters \( p, q \) different from 0, with \( U_0 = 0, U_1 = 1 \). Some results on the factorization of the Fibonacci-like numbers and their squares are given in [7]. These factorizations were found using the circulant matrices, their determinants and eigenvalues. Then

\[
U_n = \prod_{j=1}^{n-1} \left( p - 2\sqrt{q} \cos \frac{j\pi}{n} \right), \quad n \geq 2,
\]

and

\[
U_n^2 = \prod_{j=1}^{n-1} \left( p^2 - 2q - 2q \cos \frac{2j\pi}{n} \right), \quad n \geq 2.
\]

The Lucas-like numbers \( V_n \) are defined by the same recurrence as the numbers \( U_n \) with \( V_0 = 2, V_1 = p \). However, no similar factorizations for the Lucas-like numbers were found by using the determinant of circulant matrices.

Some classes of polynomials can be defined by the Fibonacci-like recurrence relations. Catalan studied polynomials \( f_n(x) \) called the Fibonacci polynomials. He defined these polynomials by the recurrence relation \( f_{n+2}(x) = xf_{n+1}(x) + \)
\[ f_n(x), \text{ where } f_0(x) = 0, f_1(x) = 1. \] There is an explicit formula for them

\[ f_n(x) = \frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)}, \]

where \( \alpha(x) = \frac{x + \sqrt{x^2 + 4}}{2} \), \( \beta(x) = \frac{x - \sqrt{x^2 + 4}}{2} \).

If \( x \) is a positive integer then the numbers \( f_n(x) \) are sometimes called the Fibonacci numbers of the \( x \)-th order. It is obvious that \( f_n(1) = F_n \) are the common Fibonacci numbers, \( f_n(2) = P_n \) are the well-known Pell numbers and so on.

The Fibonacci polynomials can be factored (see e.g. [6], p. 478) as

\[ f_n(x) = x^{n-1} \prod_{j=1}^{n} \left( 1 - 2iF_n \cos \frac{j\pi}{k+1} - 4F_n F_{n-1} \cos^2 \frac{j\pi}{k+1} \right), \quad n \text{ odd}, \]

\[ \prod_{j=1}^{k} \left( 1 - 2iF_{n-2} \cos \frac{j\pi}{k+1} + 4F_n F_{n-1} \cos^2 \frac{j\pi}{k+1} \right), \quad n \text{ even}. \]

But this result is imprecise with respect to a small mistake at the end of derivation. The correct relation can be expressed in the form

\[ \det A_k^{(n)} = \prod_{j=1}^{k} \left( 1 - 2iF_{n+1} \cos \frac{j\pi}{k+1} - 4F_n F_{n-1} \cos^2 \frac{j\pi}{k+1} \right) \quad (3) \]

as we will show in the next section of this paper.

Some of the following ideas are based on one of our previous contributions [8].
2. The Main Results

There are many connections between the determinants of tridiagonal matrices and the Fibonacci numbers or numbers which are given by their generalization. Some five-diagonal matrices and their determinants have also this property. We can investigate a generalization of the Civciv’s matrix $A_k^{(n)}$. The entries of the new matrix are the generalized Fibonacci numbers $G_n$, sometimes called the Gibonacci numbers. The Gibonacci sequence satisfies the Fibonacci recurrence $G_{n+2} = G_{n+1} + G_n$, but its initial terms can be arbitrary integers $G_0, G_1$, where at least one of them is different from 0.

**Theorem 1.** Let $M_k$ be a five-diagonal square matrix of an order $k \geq 3$ given as

$$M_k = \begin{pmatrix}
1 & G_n & 0 & \cdots & \cdots \\
-2G_n & 1 & G_n & \cdots & \cdots \\
G_n & -2G_n & 1 & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
G_n & -2G_n & 1 & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix}.$$ 

Then

$$\det M_k = \prod_{j=1}^{k} \left( 1 - 2iG_{n+1} \cos \frac{j\pi}{k+1} - 4G_n G_{n-1} \cos^2 \frac{j\pi}{k+1} \right). \quad (4)$$

**Proof.** It is easy to see that $M_k = P_k Q_k$, where $P_k, Q_k$ are the following three-diagonal square matrices of an order $k$

$$P_k = \begin{pmatrix}
1 & G_n & 0 & \cdots & \cdots \\
-2G_n & 1 & G_n & \cdots & \cdots \\
G_n & -2G_n & 1 & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
G_n & -2G_n & 1 & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix},$$

$$Q_k = \begin{pmatrix}
1 & G_{n-1} & 0 & \cdots & \cdots \\
-2G_{n-1} & 1 & G_{n-1} & \cdots & \cdots \\
G_{n-1} & -2G_{n-1} & 1 & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
G_{n-1} & -2G_{n-1} & 1 & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix}.$$
It means that \( \det M_k = \det P_k \det Q_k \). As \( P_k, Q_k \) are tridiagonal matrices the following recurrences of the second order hold for their determinants \((k \geq 3)\)

\[
\begin{align*}
\det P_k &= \det P_{k-1} + G_n^2 \det P_{k-2}, \quad \det P_1 = 1, \quad \det P_2 = 1 + G_n^2, \\
\det Q_k &= \det Q_{k-1} + G_{n-1}^2 \det Q_{k-2}, \quad \det Q_1 = 1, \quad \det Q_2 = 1 + G_{n-1}^2.
\end{align*}
\]

It is easy to see that \( \det P_k \) is also equal to \( \det \overline{P_k} \), where

\[
\overline{P_k} = \begin{pmatrix}
1 & iG_n & 0 & \cdots & \cdots \\
iG_n & 1 & iG_n & \cdots & \cdots \\
0 & iG_n & 1 & \cdots & \cdots \\
\vdots & \vdots & \vdots & \ddots & \ddots \\
\vdots & \vdots & \vdots & \ddots & \ddots \\
\cdots & \cdots & \cdots & \cdots & 1 & iG_n \\
\cdots & \cdots & \cdots & \cdots & iG_n & 1
\end{pmatrix}
\]

as the sequence of \( \det \overline{P_k} \) satisfies the same recurrence as the sequence of \( \det P_k \). The numbers \( G_n \) are only changed to \( G_{n-1} \) for \( \det Q_k = \det \overline{Q_k} \).

We know that the determinant of a square matrix can be expressed as the product of its eigenvalues. We can write

\[
\overline{P_k} = I + iG_n \begin{pmatrix}
0 & 1 & 0 & \cdots & \cdots \\
1 & 0 & 1 & \cdots & \cdots \\
0 & 1 & 0 & \cdots & \cdots \\
\vdots & \vdots & \vdots & \ddots & \ddots \\
\vdots & \vdots & \vdots & \ddots & \ddots \\
\cdots & \cdots & \cdots & \cdots & 0 & 1 \\
\cdots & \cdots & \cdots & \cdots & 1 & 0
\end{pmatrix} = I + iG_n E,
\]

where \( I \) is the identity matrix of an order \( k \).

The eigenvalues of the square matrix \( E \) are the real numbers \( -2 \cos \frac{j\pi}{k+1} \), \( j = 1, 2, \ldots, k \), (more details e.g. in [2]) and therefore the eigenvalues \( \lambda_j \) of \( \overline{P_k} \) have the form \( \lambda_j = 1 - 2iG_n \cos \frac{j\pi}{k+1} \).

Thus, \( \det P_k = \det \overline{P_k} = \prod_{j=1}^{k} \lambda_j = \prod_{j=1}^{k} \left( 1 - 2iG_n \cos \frac{j\pi}{k+1} \right) \) and similarly
det \( Q_k = \det \overline{Q_k} = \prod_{j=1}^{k} \left( 1 - 2iG_{n-1} \cos \frac{j\pi}{k+1} \right) \). Then
\[
\det M_k = \det P_k \det Q_k = 
= \prod_{j=1}^{k} \left( 1 - 2iG_n \cos \frac{j\pi}{k+1} \right) \left( 1 - 2iG_{n-1} \cos \frac{j\pi}{k+1} \right) = 
= \prod_{j=1}^{k} \left( 1 - 2i(G_n + G_{n-1}) \cos \frac{j\pi}{k+1} - 4G_nG_{n-1} \cos^2 \frac{j\pi}{k+1} \right) = 
= \prod_{j=1}^{k} \left( 1 - 2iG_{n+1} \cos \frac{j\pi}{k+1} - 4G_nG_{n-1} \cos^2 \frac{j\pi}{k+1} \right)
\]
which completes the proof of identity (4).

**Corollary 2.** The relation
\[
\det M_k = G_{n-1}^k G_n^k f_{k+1} \left( \frac{1}{G_{n-1}} \right) f_{k+1} \left( \frac{1}{G_n} \right)
\]
holds for an arbitrary positive integer \( n \geq 1 \) if \( G_0 \neq 0, G_1 \neq 0 \) and \( f_k(x) \) is the \((k+1)\)-st Fibonacci polynomial.

**Proof.** We can write
\[
\det M_k = \prod_{j=1}^{k} \left( 1 - 2iG_{n-1} \cos \frac{j\pi}{k+1} \right) \prod_{j=1}^{k} \left( 1 - 2iG_n \cos \frac{j\pi}{k+1} \right) = 
= G_{n-1}^k G_n^k \prod_{j=1}^{k} \left( \frac{1}{G_{n-1}} - 2i \cos \frac{j\pi}{k+1} \right) \prod_{j=1}^{k} \left( \frac{1}{G_n} - 2i \cos \frac{j\pi}{k+1} \right) = 
= G_{n-1}^k G_n^k f_{k+1} \left( \frac{1}{G_{n-1}} \right) f_{k+1} \left( \frac{1}{G_n} \right)
\]
with respect to the factorization of the Fibonacci polynomials.

Now, we will consider some special cases.

**Example 3.** Let \( G_n = F_n \), we obtain the Civciv’s determinant \( \det A_k^{(n)} \) from [4]. Then for \( n \geq 2 \) we have
\[
\det A_k^{(n)} = \prod_{j=1}^{k} \left( 1 - 2iF_{n+1} \cos \frac{j\pi}{k+1} - 4F_n F_{n-1} \cos^2 \frac{j\pi}{k+1} \right)
\]
which is relation (3) of the previous section. It follows from (5) that

\[
\det A_n^{(n)} = F_n^{k} F_{n-1}^{k} f_{k+1} \left( \frac{1}{F_{n-1}} \right) f_{k+1} \left( \frac{1}{F_n} \right).
\]

If \( n = 2 \) then the determinant \( \det A_k^{(2)} \) is the same as that of the open Problem 1 in [4]. It means that

\[
\text{det } A_k^{(2)} = \left| \begin{array}{ccccccc}
0 & 2 & 1 & \cdots & \cdots & \cdots & \cdots \\
-2 & -1 & 2 & \cdots & \cdots & \cdots & \cdots \\
1 & -2 & -1 & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \vdots & \vdots & \cdots & \cdots & \cdots & -1 \\
\vdots & \vdots & \vdots & \cdots & \cdots & \cdots & -2 \\
\end{array} \right| = F_1^k F_2^k f_{k+1} (1) f_{k+1} (1) = F_{k+1}^2
\]

and the problem is solved up.

If \( n = 3 \) then the matrix \( A_k^{(3)} \) has this form

\[
A_k^{(3)} = \begin{pmatrix}
-1 & 3 & 2 & \cdots & \cdots \\
-3 & -3 & 3 & \cdots & \cdots \\
2 & -3 & -3 & \cdots & \cdots \\
\vdots & \vdots & \vdots & \ddots & \ddots \\
\vdots & \vdots & \vdots & \cdots & \cdots \\
\end{pmatrix}
\]

and the following relation holds for its determinant

\[
\det A_k^{(3)} = \prod_{j=1}^{k} \left( 1 - 2iF_4 \cos \frac{j\pi}{k+1} - 4F_3F_2 \cos^2 \frac{j\pi}{k+1} \right) =
\]

\[
= F_2^k F_3^k f_{k+1} \left( \frac{1}{F_2} \right) f_{k+1} \left( \frac{1}{F_3} \right) = 2^k f_{k+1} (1) f_{k+1} \left( \frac{1}{2} \right) =
\]

\[
= 2^k F_{k+1} f_{k+1} \left( \frac{1}{2} \right),
\]

where \( f_{k+1} \left( \frac{1}{2} \right) = 2 \frac{\left( \frac{1+\sqrt{17}}{4} \right)^{k+1} - \left( \frac{1-\sqrt{17}}{4} \right)^{k+1}}{\sqrt{17}} \).

**Example 4.** Let \( G_n = L_n \), the following \( k \times k \) five-diagonal matrix

\[
B_k^{(n)} = \begin{pmatrix}
1 - L_n L_{n-1} & L_{n+1} & L_n L_{n-1} & \cdots & \cdots \\
-1 & -2L_n L_{n-1} & L_{n+1} & \cdots & \cdots \\
-2L_n L_{n-1} & -1 & -2L_n L_{n-1} & \cdots & \cdots \\
\vdots & \vdots & \vdots & \ddots & \ddots \\
\vdots & \vdots & \vdots & \cdots & \cdots \\
\end{pmatrix}
\]
has the determinant
\[
\det B_k^{(n)} = \prod_{j=1}^{k} \left( 1 - 2iL_{n+1} \cos \frac{j\pi}{k+1} - 4L_{n-1}L_{n+1} \cos^2 \frac{j\pi}{k+1} \right) = \\
= L_{n-1}^k L_n^k f_{k+1} \left( \frac{1}{L_{n-1}} \right) f_{k+1} \left( \frac{1}{L_n} \right)
\]
using identity (5).

If \( n = 2 \) then the matrix \( B_k^{(2)} \) has this form
\[
B_k^{(2)} = \begin{pmatrix}
-2 & 4 & 3 & \cdots & \cdots \\
-4 & -5 & 4 & \cdots & \cdots \\
3 & -4 & -5 & \cdots & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & -5 & 4 \\
\vdots & \vdots & \vdots & \vdots & -4 & -2
\end{pmatrix}
\]
and the following relation holds for its determinant
\[
\det B_k^{(2)} = \prod_{j=1}^{k} \left( 1 - 2iL_3 \cos \frac{j\pi}{k+1} - 4L_2L_1 \cos^2 \frac{j\pi}{k+1} \right) = \\
= L_1^k L_2^k f_{k+1} \left( \frac{1}{L_1} \right) f_{k+1} \left( \frac{1}{L_2} \right) = 3^k f_{k+1} \left( 1 \right) f_{k+1} \left( \frac{1}{3} \right) = \\
= 3^k F_{k+1} f_{k+1} \left( \frac{1}{3} \right),
\]
where \( f_{k+1} \left( \frac{1}{3} \right) = 3 \left( \frac{1+\sqrt{37}}{6} \right)^{k+1} - \left( \frac{1-\sqrt{37}}{6} \right)^{k+1} \frac{1}{\sqrt{37}} \).

References


