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MORE ON THE DIOPHANTINE EQUATION $2^x + 3^y = z^2$

Banyat Sroysang

Department of Mathematics and Statistics
Faculty of Science and Technology
Thammasat University
Rangsit Center
Pathumthani 12121, THAILAND

Abstract: In this paper, we present a completed proof that $(0, 1, 2)$, $(3, 0, 3)$ and $(4, 2, 5)$ are only three solutions for the Diophantine equation $2^x + 3^y = z^2$ where x, y and z are non-negative integers.

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1. Introduction

In 2007, Acu [1] showed that $(3, 0, 3)$ and $(2, 1, 3)$ are only two solutions (x, y, z) for the Diophantine equation $2^x + 5^y = z^2$ where x, y and z are non-negative integers.

In 2011, Suvarnamani [9] present that $(0, 1, 2)$, $(3, 0, 3)$ and $(4, 2, 5)$ are only three solutions for the Diophantine equation $2^x + 3^y = z^2$ where x, y and z are non-negative integers. In this proof, $z - 2^{k+\frac{1}{2}} = p^u$ where k, u are non-negative integers and p is a positive odd prime number. This is impossible. We note that z and p^u are integers but $2^{k+\frac{1}{2}}$ is not since $2^{k+\frac{1}{2}} = (2^k)\sqrt{2}$.

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In this paper, we present a completed proof that $(0,1,2)$, $(3,0,3)$ and $(4,2,5)$ are only three solutions for the Diophantine equation $2^x + 3^y = z^2$ where x, y and z are non-negative integers. For related papers, we list them as follows.

In 2011, Suvarnamani, Singta and Chotchaisthit [10] showed in 2011 that the two Diophantine equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$ have no non-negative integer solution.

In 2012, Peker and Cenberci [5] suggested that the Diophantine equation $8^x + 19^y = z^2$ has no non-negative integer solution. In fact, $8^1 + 19^0 = 9 = 3^2$. However, we [7] showed that $(1,0,3)$ is a unique solution (x,y,z) for the Diophantine equation $8^x + 19^y = z^2$ where x, y and z are non-negative integers. Moreover, we [6, 8] proved that (i) $(1,0,3)$ is a unique solution (x,y,z) for the Diophantine equation $8^x + 19^y = z^2$ where x, y and z are non-negative integers and (ii) the Diophantine equation $31^x + 32^y = z^2$ has no non-negative integer solution.

In the same year, Chotchaisthit [3] found all non-negative integer solutions for the Diophantine equation of type $4^x + p^y = z^2$ where p is a positive prime number.

2. Preliminaries

The Catalan's conjecture [2] state that $(3,2,2,3)$ is a unique solution (a,b,x,y) for the Diophantine equation $a^x - b^y = 1$ where a, b, x and y are integers with $\min\{a,b,x,y\} > 1$. This was proven in 2004 by Mihailescu [4].

Proposition 2.1. (see [4]) $(3,2,2,3)$ *is a unique solution* (a,b,x,y) *for the Diophantine equation* $a^x - b^y = 1$ *where* a, b, x *and* y *are integers with* $\min\{a,b,x,y\} > 1$.

To prove the main results, we present two Lemmas.

Lemma 2.2. (see [3]) $(0,1,2)$ *and* $(2,2,5)$ *are only two solutions* (k,y,z) *for the Diophantine equation* $4^k + 3^y = z^2$ *where* k, y *and* z *are non-negative integers.*

Lemma 2.3. (see [1]) $(3,3)$ *is a unique solution* (x,z) *for the Diophantine equation* $2^x + 1 = z^2$ *where* x *and* z *are non-negative integers.*

3. Results

In [9], $(0, 1, 2)$, $(3, 0, 3)$ and $(4, 2, 5)$ are only three solutions for the Diophantine equation $2^x + 3^y = z^2$ where x, y and z are non-negative integers. But the proof is not completed. In this section, we present a completed proof.

Theorem 3.1. $(0, 1, 2)$, $(3, 0, 3)$ *and* $(4, 2, 5)$ *are only three solutions for the Diophantine equation* $2^x + 3^y = z^2$ *where* x, y *and* z *are non-negative integers.*

Proof. Let x, y and z be non-negative integers such that $2^x + 3^y = z^2$. We will divide the number x into two cases.

Case 1. x is even. Let $x = 2k$ where k is a non-negative integer. Then $4^k + 3^y = z^2$. By Lemma 2.2, we have $(k, y, z) \in \{(0, 1, 2), (2, 2, 5)\}$. Thus, $(x, y, z) \in \{(0, 1, 2), (4, 2, 5)\}$.

Case 2. x is odd. We will divide the number y into three subcases.

Subcase 2.1. $y = 0$. Then $2^x + 1 = z^2$. By Lemma 2.3, we have $x = 3$ and $z = 3$. Thus, $(x, y, z) = (3, 0, 3)$.

Subcase 2.2. y is even and $y > 0$. Let $y = 2l$ where l is a positive integer. Then $z^2 - 3^{2l} = 2^x$. Then $(z - 3^l)(z + 3^l) = 2^x$. Thus, $z - 3^l = 2^u$ where u is a non-negative integer. Then $z + 3^l = 2^{x-u}$. Thus, $2(3^l) = 2^{x-u} - 2^u = 2^u(2^{x-2u} - 1)$. Then $u = 1$. Then $3^l = 2^{x-2} - 1$. Then $2^{x-2} - 3^l = 1$. Since $l \geq 1$, we obtain that $2^{x-2} = 3^l + 1 \geq 3^1 + 1 = 4$. Then $x \geq 4$. By Proposition 2.1, we have $l = 1$. Then $2^{x-2} = 4$. Then $x = 4$. This is a contradiction.

Subcase 2.3. y is odd. Note that z is also odd. Then $z^2 \equiv 1 \pmod 4$. Let $y = 2l + 1$ where l is a non-negative integer. Then $z^2 = 2(4^k) + 3(9^l) \equiv 3 \pmod 4$. This is a contradiction.

Therefore, $(0, 1, 2)$, $(3, 0, 3)$ and $(4, 2, 5)$ are only three solutions for the Diophantine equation $2^x + 3^y = z^2$ where x, y and z are non-negative integers. □

Corollary 3.2. *The Diophantine equation* $2^x + 3^y = w^4$ *has no non-negative integer solution.*

Proof. Suppose that there are non-negative integers x, y and w such that $2^x + 3^y = w^4$. Let $z = w^2$. Then $2^x + 3^y = z^2$. By Theorem 3.1, we have $(x, y, z) \in \{(0, 1, 2), (3, 0, 3), (4, 2, 5)\}$. Then $w^2 = z \in \{2, 3, 5\}$. This is a contradiction. Hence, the equation $2^x + 3^y = w^4$ has no non-negative integer solution. □

Corollary 3.3. $(1, 0, 3)$ *is a unique solution for the Diophantine equation* $8^m + 9^n = z^2$ *where* m, n *and* z *are non-negative integers.*

Proof. Let m, n and z be non-negative integers such that $8^m + 9^n = z^2$. Let $x = 3m$ and $y = 2m$. Then $2^x + 3^y = z^2$. By Theorem 3.1, we have $(x, y, z) \in \{(0, 1, 2), (3, 0, 3), (4, 2, 5)\}$. Then $m = 1, n = 0$ and $z = 3$. Hence, $(1, 0, 3)$, is a unique solution for the equation $8^m + 9^n = z^2$ where m, n and z are non-negative integers. □

Corollary 3.4. *The Diophantine equation* $8^m + 9^n = w^4$ *has no non-negative integer solution.*

Proof. Suppose that there are non-negative integers m, n and w such that $8^m + 9^n = w^4$. Let $z = w^2$. Then $8^m + 9^n = z^2$. By Corollary 3.3, we have $(m, n, z) = (1, 0, 3)$. Then $w^2 = z = 3$. This is a contradiction. Hence, the equation $8^m + 9^n = w^4$ has no non-negative integer solution. □

Corollary 3.5. $(1, 1, 5)$ *is a unique solution for the Diophantine equation* $9^m + 16^n = z^2$ *where* m, n *and* z *are non-negative integers.*

Proof. Let m, n and z be non-negative integers such that $9^m + 16^n = z^2$. Let $x = 4n$ and $y = 2m$. Then $2^x + 3^y = z^2$. By Theorem 3.1, we have $(x, y, z) \in \{(0, 1, 2), (3, 0, 3), (4, 2, 5)\}$. Then $m = 1, n = 1$ and $z = 5$. Hence, $(1, 1, 5)$, is a unique solution for the equation $9^m + 16^n = z^2$ where m, n and z are non-negative integers. □

Corollary 3.6. *The Diophantine equation* $9^m + 16^n = w^4$ *has no non-negative integer solution.*

Proof. Suppose that there are non-negative integers m, n and w such that $9^m + 16^n = w^4$. Let $z = w^2$. Then $9^m + 16^n = z^2$. By Corollary 3.5, we have $(m, n, z) = (1, 1, 5)$. Then $w^2 = z = 5$. This is a contradiction. Hence, the equation $9^m + 16^n = w^4$ has no non-negative integer solution. □

4. Open Problem

Let p and q be positive prime numbers. We may ask for the set of all solutions (x, y, z) for the Diophantine equation $p^x + q^y = z^2$ where x, y and z are non-negative integers.

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