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LIE THEORETIC GENERATING RELATIONS OF LAGUERRE 2D POLYNOMIALS

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Abstract: In this paper, we follow the approach of Miller (see [1]) and obtain generating relations of Laguerre 2D polynomials (L2DP) by extending the realization $\uparrow_{\omega,\mu}$ to study multiplier representations of a Lie group G(0,1). Certain (known or new) generating relations for the polynomials related to L2DP are obtained as special cases.

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1. Introduction

Recently, Wunsche (see [3, 4, 5, 6]) introduced Laguerre 2D polynomials (L2DP) $L_{m,n}(z,z^*)$ and discussed their properties and their explicit representations.These polynomials are very effective mean for the representations of many results in quantum optics (quasi probabilities in Fock-state basis, ordering problems, moments) and more over in other regions of physics.

The L2DP $L_{m,n}(z,z^*)$, $(m,n = 0,1,2,...)$ are defined as polynomials of two independent variables (z,z^*) which in application are generally a pair of complex conjugated variables in the following way Wunsche (see [4], p. 3181,

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equation (2.1))

$$L_{m,n}(z,z^*) \equiv \exp(-\frac{\partial^2}{\partial z \partial z^*}) z^m z^{*n} = \sum_{j=0}^{m,n} \frac{(-1)^j m! n!}{j!(m-j)!(n-j)!} z^{m-j} z^{*n-j} \quad (1)$$

The representation of the Laguerre 2D polynomials by the usual Laguerre polynomials $L_n^{\alpha}(u)$ is given by [4], p. 3181, equation (2.5)

$$L_{m,n}(z,z^*) = (-1)^n n! z^{m-n} L_n^{m-n}(zz^*) = (-1)^m m! z^{*n-m} L_m^{n-m}(zz^*). \quad (2)$$

Generating relations involving two variable Laguerre polynomials by using two Laguerre polynomials by using Weisner's (see [7]) group-theoretic method was discussed by Khan and Yasmin, see [8]. Also Khan in [9] derived some generating relations involving L2DP.Here we follow the approach of Miller (see [1]) and obtain generating relations of L2DP by extending the realizations of $\uparrow_{\omega,\mu}$ to multiplier representations of a Lie group G(0,1).

2. Representation $\uparrow_{\omega,\mu}$ of $\ell(0,1)$ and Generating Relations

We note that the following isomorphism (see [1], p. 36)

$$\ell(0,1) \cong L[G(0,1)],$$

where L[G(0,1)] is the Lie algebra of a complex four-dimensional Lie group G(0,1), multiplicative matrix group with elements (see [1], p. 9)

$$g(a,b,c,\tau) = \begin{pmatrix} 1 & ce^{\tau} & a & \tau \\ 0 & e^{\tau} & b & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (3)$$

where $a, b, c, \tau \in C$.

The group G(1,0)is called the complex harmonic oscillator group (see [2], Chapter 10). Abasis for L[G(0,1)] is provided by the matrices (see [1], p. 9)

$$j^+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad j^- = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$
$$j^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \varepsilon = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (4)$$

with commutation relations

$$[j^3, j^{\pm}] = \pm j^{\pm}, [j^+, j^-] = -\varepsilon, [\varepsilon, j^{\pm}] = [\varepsilon, j^3] = 0. \tag{5}$$

The machinery constructed in (see [1], Chapters 1, 2 and 4) will be applied to find realization of the irreducible representation $\uparrow_{\omega,\mu}$ of $\ell(0,1)$ where $\omega,\mu \in C$ such that $\mu \neq 0$. The spectrum S of $\uparrow_{\omega,\mu}$ is the set

$$S = \{-\omega + k,\ k\text{-anon negative integer}\}.$$

In particular we are looking for the function $f_{m,n}(z, z^*, p, s) = Z_{m,n}(z, z^*)p^m s^n$ such that

$$\begin{gathered} J^3 f_{m,n} = m f_{m,n}, E f_{m,n} = \mu f_{m,n}, \\ J^+ = \mu f_{m+1,n}, J^- f_{m,n} = (m+\omega) f_{m-1,n}, \\ C_{0,1} f_{m,n} = (J^+ J^- - E J^3) f_{m,n} = \mu\omega f_{m,n} \end{gathered} \tag{6}$$

for all $m \in S$. The commutation relations satisfied by the operators $J^{\pm}, J^3, E$ are

$$[J^3, J^{\pm}] = \pm J^{\pm}, [J^+, J^-] = -E, [J^{\pm}, E] = [J^3, E] = 0 \tag{7}$$

The number of possible solutions of equation (2.5) is tremendous. We assume that these operators take the form

$$\begin{aligned} J^+ &= p(z - \frac{\partial}{\partial z^*}) \\ J^- &= \frac{1}{p}\frac{\partial}{\partial z} \\ J^3 &= p\frac{\partial}{\partial p} \\ E &= 1 \end{aligned} \tag{8}$$

and note these operators satisfy the commutation relations (7).

We can assume that $\omega = 0$ and $\mu = 1$ without any loss of generality for the theory of special functions. In terms of the functions $Z_{m,n}(z, z^*)$ relation (6) become

$$\begin{gathered} (z - \frac{\partial}{\partial z^*}) Z_{m,n}(z, z^*) = Z_{m+1,n}(z, z^*) \\ (\frac{\partial}{\partial z}) Z_{m,n}(z, z^*) = m Z_{m-1,n}(z, z^*) \\ (-\frac{\partial^2}{\partial z^* \partial z} + z\frac{\partial}{\partial z} - m) Z_{m,n}(z, z^*) = 0 \\ m = 0, 1, 2, ... \end{gathered} \tag{9}$$

Again, if we take the function $f_{m,n}(z,z^*,p,s) = Z_{m,n}(z,z^*)p^m s^n$ such that

$$\begin{aligned} J^{3'} f_{m,n} &= n f_{m,n}, E' f_{m,n} = \mu f_{m,n}, \\ J^{+'} &= \mu f_{m,n+1}, J^{-'} f_{m,n} = (n+\omega) f_{m,n-1}, \\ C'_{0,1} f_{m,n} &= (J^{+'} J^{-'} - E' J^{3'}) f_{m,n} = \mu\omega f_{m,n} \end{aligned} \tag{10}$$

for all $n \in S$,then the differential operators $J^{\pm'}, J^{3'}, E'$ are given by

$$\begin{aligned} J^{+'} &= s(z^* - \frac{\partial}{\partial z}) \\ J^{-'} &= \frac{1}{s}\frac{\partial}{\partial z^*} \\ J^{3'} &= s\frac{\partial}{\partial s} \\ E' &= 1 \end{aligned} \tag{11}$$

and satisfy the commutation relations identical to (7).

Just like before taking $\omega = 0$ and $\mu = 1$, relation (10) become

$$\begin{aligned} (z^* - \frac{\partial}{\partial z}) Z_{m,n}(z,z^*) &= Z_{m,n+1}(z,z^*) \\ (\frac{\partial}{\partial z^*}) Z_{m,n}(z,z^*) &= n Z_{m,n-1}(z,z^*) \\ (-\frac{\partial^2}{\partial z^* \partial z} + z^* \frac{\partial}{\partial z^*} - n) Z_{m,n}(z,z^*) &= 0 \\ n &= 0,1,2,... \end{aligned} \tag{12}$$

We see from (9) and (12) that $Z_{m,n}(z,z^*) = L_{m,n}(z,z^*)$ where $L_{m,n}(z,z^*)$ is given by (1). The functions

$$f_{m,n}(z,z^*,p,s) = L_{m,n}(z,z^*)p^m s^n, \quad m \in S,$$

form a basis for a realization of the representation $\uparrow_{0,1}$ of $\ell(0,1)$.This realization of $\ell(0,1)$ can be extended to a local multiplier representation $T(g)$,$g \in G(0,1)$ defined on F the space of all functions analytic in a neighborhood of the point $(z^0, z^{*0}, p^0, s^0) = (1,1,1,1)$.

Using operators (8),the multiplier representation (see [1], p. 17) takes the form

$$\begin{aligned} [T(\exp a\varepsilon)f](z,z^*,p,s) &= \exp(a) f((z,z^*,p,s) \\ [T(\exp bj^+)f](z,z^*,p,s) &= \exp(bzp) f((z,z^* - bp,p,s) \\ [T(\exp cj^-)f](z,z^*,p,s) &= f(z + \frac{c}{p}, z^*,p,s) \\ [T(\exp \tau j^3)f](z,z^*,p,s) &= f(z,z^*,pe^\tau,s) \end{aligned} \tag{13}$$

for $f \in F$.If $g \in G(0,1)$ has parameters (a,b,c,τ), then

$$T(g) = T(\exp a\varepsilon)T(\exp bj^{+})T(\exp cj^{-})T(\exp \tau j^{3})$$

and therefore we obtain

$$[T(g)f](z,z^{*},p,s) = \exp(a+bzp)f(z+\frac{c}{p}, z^{*}-bp, pe^{\tau}, s) \tag{14}$$

The matrix element of $T(g)$ with respect to the analytic basis $f_{m,n}(z,z^{*},p,s) = L_{m,n}(z,z^{*})p^{m}s^{n}$ are the function $A_{lk}(g)$ uniquely determined by $\uparrow_{\omega,\mu}$ of $\ell(0,1)$ and we obtain relations

$$[T(g)f_{k,n}](z,z^{*},p,s) = \sum_{l=0}^{\infty} A_{lk}(g)f_{l,n}(z,z^{*},p,s), k=0,1,2,...$$

which simplify to the identity

$$\exp(a+\tau k+bzp)L_{k,n}(z+\frac{c}{p}, z^{*}-bp) = \sum_{l=0}^{\infty} A_{lk}(g)L_{l,n}(z,z^{*})p^{l-k}, \tag{15}$$

where $k=0,1,2,...$, and the matrix element $A_{lk}(g)$ are given by (see [1], p. 87, equation (4.26))

$$A_{lk}(g) = \exp(a+\tau k)c^{k-l}L_{l}^{k-l}(-bc), k,l \geq 0. \tag{16}$$

Substituting (16) into (15), we obtain the generating relation

$$\begin{gathered}\exp(bzp)L_{k,n}(z+\frac{c}{p}, z^{*}-bp) = \sum_{l=0}^{\infty} c^{k-l}L_{l}^{k-l}(-bc)L_{l,n}(z,z^{*})p^{l-k}, \\ b,c,p \in C, n,k=0,1,2,...\end{gathered} \tag{17}$$

Again taking the operators (11) and proceeding exactly as before,

$$\begin{gathered}\exp(b'zs)L_{m,q}(z-b's, z^{*}+\frac{c'}{s}) = \sum_{i=0}^{\infty} (c')^{q-i}L_{i}^{q-i}(-bc)L_{m,i}(z,z^{*})s^{i-q}, \\ b', c', s \in C, m,q=0,1,2,...\end{gathered} \tag{18}$$

3. Applications

We consider some special cases of the generating relations obtained in previous section, which yield many new and known relations for the polynomials related to L2DP.

I. Making use of (2) in (17) we get

$$\exp(bzp)(1-\frac{bp}{z^*})^{n-k}L_k^{n-k}((z+\frac{c}{p})(z^*-bp))k! = \sum_{l=0}^{\infty} c^{k-l}L_l^{k-l}(-bc)L_l^{n-l}(zz^*)(z^*)^{k-l}(-p)^{l-k}l!, \quad b,c,p\in C, n,k=0,1,2,... \tag{19}$$

Now in particular, taking $z=1$ and replacing z^* by u, p by $-t$ and n by q in (19) we get a result of Miller (see [1], (4.94), p. 112).

II. Again making use of (2) in (18) we get

$$\exp(b'zs)(z^*+\frac{c'}{s})^{q-m}L_m^{q-m}((z-b's)(z^*+\frac{c'}{s})) = \sum_{i=0}^{\infty}(c')^{q-i}L_i^{q-i}(-bc)(z^*)^{i-m}L_m^{i-m}(zz^*)s^{i-q}, \quad b',c',s\in C, m,q=0,1,2,... \tag{20}$$

Again in particular taking $s=1$ and replacing z by b_1, z^* by c_1, b' by $-b_2$, c' by c_2, m by l, q by $l+n$ and r by j in (20) we get a result of Miller (see [1], (4.28), p. 88).

4. Conclusion

We have considered the problem of framing L2DP $L_{m,m}(z,z^*)$ into the context of the representation $\uparrow_{\omega,\mu}$ of the Lie algebra $\ell(0,1)$ of the complex harmonic group $G(0,1)$. Generating relations involving L2DP are obtained by using Millers technique. Some relations for the products of Laguerre polynomials and identities of Miller are also obtained as special cases.

Further,we observe that these operators $J^-, J^+, J^{-'}J^{+'}$ and $I=1$ satisfy the following commutation relations

$$[J^-,J^+]=[J^{-'},J^{+'}]=I,[J^-,J^{+'}]=[J^{-'},J^+]=0,$$

$$[J^-, J^{-'}] = [J^+, J^{+'}] = 0.$$

These relations imply that the five operators $J^-, J^+, J^{-'}J^{+'}, I$ are closed with regard to the commutation relations. Therefore, they form a realization of an abstract five-dimensional Lie algebra which is the Lie algebra of the Heisenberg-Weyl group $W(2, R)$ or to its complex extension $W(2, C)$ for a two-mode system. Thus L2DP form a certain basis for this realization of the Heisenberg-Weyl algebra or to its complex extension $\omega(2, C)$ (see [3] and the references therein). By the quadratic combinations of the basic operators $J^-, J^+, J^{-'}J^{+'}, I$, we can form ten more operators, which form several Lie Algebras.

The study of the L2DP for applications as well as for its connections with various Lie algebras is an interesting problem for further research.

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