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MINIMAL TWO-VALUED ECCENTRIC SEQUENCES

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Abstract: An eccentric sequence of a connected finite graph is a sequence of eccentricities of its vertices. An eccentric sequence is called minimal if it has no proper eccentric subsequence with the same number of distinct eccentricities. We survey known results concerning minimal two-valued eccentric sequences and describe a new infinite class of these sequences. Also a conjecture on all minimal two-valued eccentric sequences is proposed.

AMS Subject Classification: 05C12
Key Words: cycle, eccentricity, eccentric sequence, minimal eccentric sequence

1. Introduction

One concept that pervades all of graph theory is that of distance. Among important notions defined through distance there is the eccentricity of a vertex of a connected graph. The list of the eccentricities (in nondecreasing order) of the vertices of a connected finite graph G is the eccentric sequence of G. An eccentric sequence is called minimal if it has no proper eccentric subsequence with the same number of distinct eccentricities. Characterization of eccentric sequences is considered to be an important problem in graph theory (see Problem 1 in [3]). This problem is extremely difficult. It seems that even finding all

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minimal eccentric sequences with least eccentricity five is a very difficult problem. Eccentric sequences have an important role in aplication of graph theory (see, for instance, [4], [9] and [13]). It is not excluded that eccentric sequences could have a distinguished role in describing reality on the fundamental level.

A survey of known results relating to the topic of the paper is given in its second section. The main results are given in the third section. In that section a new infinite class of minimal eccentric sequences with two values is described and also a new proof for known sequences of this type is given. A conjecture at the end of the third section completes a description of all minimal two-valued eccentric sequences. Since the proof of Theorem 3.2 is rather complicated, proofs of several needed lemmas are presented in the last section of the paper.

For a connected graph G, the *distance* $d_G(u, v)$ or briefly $d(u, v)$ between vertices u and v is the length of a shortest path joining them. The distance between a vertex $u \in V(G)$ and a subgraph H of G will be denoted by $d_G(u, H)$, i.e. $d_G(u, H) = \min\{d_G(u, v); v \in V(H)\}$. The *eccentricity* $e_G(u)$ (briefly $e(u)$) of a vertex $u \in V(G)$ is the distance to a vertex farthest from u in G, i.e. $e_G(u) = \max\{d_G(u, v); v \in V(G)\}$. The *radius* $\operatorname{rad} G$ of G is the minimum eccentricity among the vertices of G while the *diameter* $\operatorname{diam} G$ of G is the maximum eccentricity. The *eccentricity sequence* $\operatorname{es}(G)$ of G is the list of the eccentricities of its vertices in nondecreasing order. Since there are often many vertices having the same ecentricity, we will simplify the sequence by listing it as $e_1^{m_1}, e_2^{m_2}, \dots, e_k^{m_k}$, i.e. $\operatorname{es}(G) = (e_1^{m_1}, e_2^{m_2}, \dots, e_k^{m_k})$. An eccentric sequence is called *minimal* if it has no proper eccentric subsequence with the same number of distinct eccentricities ([11]). Let C be a cycle of G. A vertex u of the cycle C is called *C-excited* (in G) if $e_G(u) > e_C(u)$. The number of C-excited vertices is denoted by $\operatorname{exc}_G C$, i.e. $\operatorname{exc}_G C = |\{u \in V(C); e_G(u) > e_C(u)\}|$ (see [7]). A cycle C in G is called *geodesic* if for any two vertices of C their distance in C equals their distance in G. The *circumference* $c(G)$ of G is the length of any longest cycle in G. A connected unicyclic graph G with the cycle C is called a *sun-graph* (see [7]) if $\deg_G(u) \le 3$ for $u \in V(C)$ and $\deg_G(u) \le 2$ for $u \in V(G) \setminus V(C)$. A $u - v$ path P in a sun-graph G is called a *ray* if $V(P) \cap V(C) = \{u\}$ and $\deg_G(v) = 1$.

2. Survey

In this section we recall the known results relating to the topic of this paper (see also [1], [2] and [5]).

Theorem 2.1. *(see [10])*

a) *A sequence of positive integers is eccentric if and only if some subsequence of it with the same number of distinct members is eccentric.*

b) *If* $\mathrm{es}(G) = (e_1^{m_1}, e_2^{m_2}, \ldots, e_k^{m_k})$ *and* $k \geq 2$ *then* $e_i = e_{i-1} + 1$ *and* $m_i \geq 2$ *for all* $i \in \{2, 3, \ldots, k\}$.

Theorem 2.2. *[7] Let G be a graph with* $\mathrm{rad}\, G = r$, $\mathrm{diam}\, G \leq 2r - 2$, *on at most $3r - 2$ vertices. Then G contains a geodesic cycle of length $2r$ or $2r + 1$.*

Lemma 2.3. *[7] Let C be a cycle of a connected graph G and $|V(G) \setminus V(C)| = k$. Then*

a) $\mathrm{exc}_G(C) \leq 2k - 1$ *if the length of C is even and $k \geq 1$,*

b) $\mathrm{exc}_G(C) \leq 2k$ *if the length of C is odd,*

c) $\mathrm{exc}_G(C) \leq 2k - l$ *if the length of C is even and* $l = |\{u \in V(G); d_G(u, C) = 1\}|$.

Theorem 2.4. *[8] Let $r \geq 3$ and G be a graph with* $\mathrm{es}(G) = (r^\alpha, (r+1)^\beta)$. *Then*

a) *there exists a block B of G which contains all cut-vertices of G and moreover with the property that for every $u \in V(G) \setminus V(B)$ it holds $d_G(u, B) = 1$,*

b) *for circumference of G and for the block B from the previous it holds $c(G) \geq c(B) \geq 2r - 2$,*

c) *if $c(G) < 2r$ then $\alpha \geq 2r - 2$.*

Theorem 2.5. *[11] The minimal eccentric sequences with least eccentricity at most two are precisely:* 0; 1^2; $1, 2^2$; 2^4; $2^2, 3^2$; $2, 3^6$; $2, 3^2, 4^2$.

Theorem 2.6. *[6] There are exactly 13 minimal eccentric sequences with least eccentricity three, namely:* 3^6; $3^5, 4^2$; $3^4, 4^4$; $3^3, 4^6$; $3^2, 4^8$; $3, 4^{10}$; $3, 4^2, 5^{12}$; $3, 4^3, 5^9$; $3, 4^4, 5^7$; $3, 4^5, 5^4$; $3, 4^7, 5^2$; $3^2, 4^2, 5^2$; $3, 4^2, 5^2, 6^2$.

Theorem 2.7. *[8] There are exactly 7 minimal eccentric sequences of type* $4^\alpha, 5^\beta$, *namely:* $4^7, 5^2$; $4^6, 5^4$; $4^5, 5^6$; $4^4, 5^8$; $4^3, 5^9$; $4^2, 5^{12}$; $4, 5^{14}$.

From the last three theorems it is evident that all minimal two-valued eccentric sequences with less eccentricity at most four are known. The following theorem describes minimal two-valued eccentric sequences with less eccentricity $r \geq 5$.

Theorem 2.8. *[7]*

a) *The sequences* $5^9, 6^2$; $5^8, 6^4$; $5^7, 6^6$; $5^6, 6^8$; $5^5, 6^9$ *are minimal eccentric sequences.*

b) *Al minimal eccentric sequences of type* $r^\alpha, (r+1)^\beta$ *for* $r \geq 6$ *and* $\alpha + \beta \leq \frac{8r+5}{3}$ *are:*
$r^{2r-i}, (r+1)^{2i}$, $i = 1, 2$,
$r^{2r-2i+1}, (r+1)^{3i}$, $i = 2, 3, \ldots, \lfloor\frac{2r+1}{3}\rfloor$,
$r^{2r-2i}, (r+1)^{3i+2}$, $i = 2, 3, \ldots, \lfloor\frac{2r-1}{3}\rfloor$.

Note that all minimal one-valued eccentric sequences are also known. These sequences are 0 and r^{2r} for $r \geq 1$ (see[12],[7]).

3. Main Results

The following lemma is a generalization of Lemma 2.3.

Lemma 3.1. *Let* G *be a connected graph which contains a cycle* C *and* G' *be a unicyclic connected subgraph of* G *with the cycle* C. *Then*

$$\mathrm{exc}_G(C) \leq \mathrm{exc}_{G'}(C) + 2.(|V(G)| - |V(G')|).$$

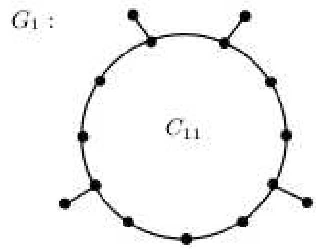
Proof. Let $k = |V(G)| - |V(G')|$. Consider a sequence $G_0 = G'$, $G_1, \ldots, G_k$ of subgraphs of G such that $G_{i+1} = G_i + uv$ $(i < k)$, where $u \in V(G_i)$, $v \in V(G_{i+1}) \setminus V(G)$, $uv \in E(G)$. Obviously, $\mathrm{exc}_{G_{i+1}}(C) \leq \mathrm{exc}_{G_i}(C) + 2$. Since $\mathrm{exc}_G(C) \leq \mathrm{exc}_{G_k}(C)$, we get $\mathrm{exc}_G(C) \leq \mathrm{exc}_{G'}(C) + 2k = \mathrm{exc}_{G'}(C) + 2(|V(G)| - |V(G')|)$. □

It is easy to see (it is sufficient to consider a suitable subgraph G' of G) that the cases a), b), c) in Lemma 2.3 are consequences of Lemma 3.1.

Minimal eccentric sequences from the case (i) of the following theorem are known, except of the sequence $5^4, 6^{11}$, since 2004 (see Theorem 2.8). However, the assumptions in Theorem 3.2 are formulated more transparently than in Theorem 2.8 and we also give a new proof that the eccentric sequences from the case (i) are minimal.

Theorem 3.2. *All minimal eccentric sequences of type* $r^\alpha, (r+1)^\beta$ *for* $r \geq 5$ *and* $\alpha \geq \frac{r+3}{2}$ *are:*

(i) $r^{2r-i}, (r+1)^{2i}$, $i = 1, 2$,
$r^{2r-2i+1}, (r+1)^{3i}$, $i = 2, 3, \ldots, \lfloor\frac{2r+1}{3}\rfloor$,
$r^{2r-2i}, (r+1)^{3i+2}$, $i = 2, 3, \ldots, \lfloor\frac{2r-1}{3}\rfloor$,



$\mathrm{es}(G_1) = (5^4, 6^{11})$

Figure 1

(ii) $r^{\alpha}, (r+1)^{4r-3\alpha+2}$, *if* $r = 9$ *or* $r \geq 11$, *and* $\frac{r+3}{2} \leq \alpha \leq \frac{2r}{3}$.

Proof. a) First we show that each of the given sequences is eccentric.

A sequence $r^{2r-i}, (r+1)^{2i}$ for $i = 1, 2$ is realizable through a sun-graph with the cycle C_{2r} of length $2r$ and with i rays of length one. A sequence $r^{2r-2i+1}, (r+1)^{3i}$ for $i = 2, 3, \ldots, \lfloor \frac{2r+1}{3} \rfloor$ is realizable through any sun-graph with the cycle C_{2r+1},with i rays of length one and such that $\mathrm{exc}_G\, C_{2r+1} = 2i$ and no C_{2r+1}-excited vertex is a cut-vertex of G (we recommend to see Figures 3.11 - 3.13 in [7]). Analogously, a sequence $r^{2r-2i}, (r+1)^{3i+2}$ for $i = 2, 3, \ldots, \lfloor \frac{2r-1}{3} \rfloor$ is realizable through any sun-graph G with the cycle C_{2r+1}, with $i+1$ rays of length one and such that $\mathrm{exc}_G\, C_{2r+1} = 2i+1$ and no C_{2r+1}-excited vertex is a cut-vertex of G (see Figures 3.15 - 3.17 in [7]). For instance the sequence $5^4, 6^{11}$ is realizable through the graph G_1 in Figure 1.

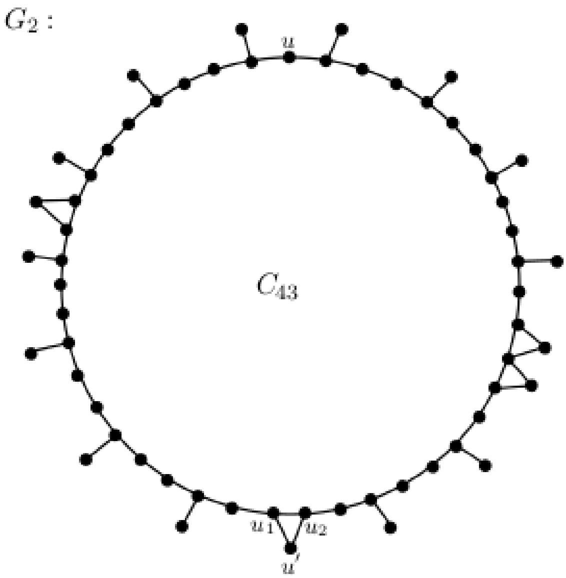
A sequence of type $r^{\alpha}, (r+1)^{4r-3\alpha+2}$ is realizable through a graph G, which can be constructed in two steps as follows:

(1°) We construct a sun-graph G' with the cycle C_{2r+1}, with α rays of length one and such that $\mathrm{exc}_{G'}\, C_{2r+1} = 2\alpha$ and no C_{2r+1}-excited vertex is a cut-vertex of G'.

(2°) We add $2r+1-3\alpha$ new vertices and $2(2r+1-3\alpha)$ new edges to the graph G' by the following rule. If a vertex u of G' is not a cut-vertex of G', $e_{G'}(u) = r$ and $d_{G'}(u, u_1) = d_{G'}(u, u_2) = r$ for $u_1, u_2 \in V(C_{2r+1})$, $u_1 \neq u_2$, then we add a new vertex u' and two edges $u'u_1$, $u'u_2$. For example, Figure 2 shows the graph G_2 with eccentric sequence $21^{13}, 22^{47}$.

b) We show that each of the considered eccentric sequences is minimal.

The sequence $5^4, 6^{11}$ is minimal according to Lemma 4.1. In what follows we will suppose that $\alpha \neq 4$ for $r = 5$.



$\mathrm{es}(G_2) = (21^{13}, 22^{47})$

Figure 2

First we show that $\alpha + \beta \le 3r - 1$ for each sequence $r^\alpha, (r+1)^\beta$, except the sequence $5^4, 6^{11}$.
If a considered sequence is from the case (i), then it is easy to see that $\alpha + \beta \le 2r + \lfloor \frac{2r-1}{3} \rfloor + 2$. We get $\alpha + \beta \le 2r + \frac{2r-1}{3} + 2 = \frac{8r+5}{3}$ and $\frac{8r+5}{3} \le 3r - 1$ for $r \ge 8$. Further it is easy to verify that $\alpha + \beta \le 3r - 1$ also for $r \in \{5, 6, 7\}$.
If a considered sequence is from the case (ii), then $\alpha \ge \frac{r+3}{2}$ and we get $\alpha + \beta = 4r - 2\alpha + 2 \le 3r - 1$.

Now we are going to show that any considered sequence, except $5^4, 6^{11}$, is minimal. Suppose, contrary to our claim, that $r^\alpha, (r+1)^\beta$ is not a minimal eccentric sequence. Then there exists a graph G' such that $\mathrm{es}(G') = (r^{\alpha'}, (r+1)^{\beta'})$, $\alpha' + \beta' = \alpha + \beta - 1$, $\alpha' \le \alpha$ and $\beta' \le \beta$. Since $|V(G')| = \alpha' + \beta' \le 3r - 2$, G' contains a geodesic cycle C_{2r} or C_{2r+1} (see Theorem 2.2).
Further we distinguish five cases.

(1) $\alpha = 2r - 1$
In this case $G' = C_{2r}$ and $\mathrm{es}(G') = (r^{2r})$, a contradiction.

(2) $\alpha = 2r - 2$
In this case we have $|V(G')| = 2r+1$. If $G' = C_{2r+1}$ then $\mathrm{es}(G') = (r^{2r+1})$,

a contradiction. If G' contains C_{2r} then (by Lemma 2.3a) $\mathrm{exc}_{G'}(C_{2r}) \leq 1$ and so $\alpha' > \alpha$, a contradiction.

(3) $\alpha = 2r - 2i + 1$
In this case we have $|V(G')| = 2r + i$. If G' contains C_{2r} then (by Lemma 4.3) $\mathrm{exc}_{G'}(C_{2r}) \leq i$ and so $\alpha' \geq 2r - i$. Since $i \geq 2$, we get $\alpha' > 2r - 2i + 1 = \alpha$, a contradiction. If G' contains C_{2r+1} then, by Lemma 2.3b, $\mathrm{exc}_{G'}(C_{2r+1}) \leq 2(i-1)$. Hence $\alpha' \geq 2r+1-2(i-1) = 2r-2i+3 > \alpha$, a contradiction.

(4) $\alpha = 2r - 2i$
In this case we have $|V(G')| = 2r + i + 1$. If G' contains C_{2r} then $\mathrm{exc}_{G'}(C_{2r}) \leq i+1$ (see Lemma 4.3). Hence $\alpha' \geq 2r-(i+1) > 2r-2i = \alpha$, a contradiction. If G' contains C_{2r+1} then $\mathrm{exc}_{G'}(C_{2r+1}) \leq 2i$ (see Lemma 2.3b). Hence $\alpha' \geq 2r + 1 - 2i > 2r - 2i = \alpha$, and again we have a contradiction.

(5) $\frac{r+3}{2} \leq \alpha \leq \frac{2r}{3}$
In this case $|V(G')| = 4r - 2\alpha + 1$. If G' contains C_{2r} then $\mathrm{exc}_{G'}(C_{2r}) \leq 2r - 2\alpha + 1$ (see Lemma 4.3). Hence $\alpha' \geq 2r - (2r - 2\alpha + 1) = 2\alpha - 1$. Since $\alpha \geq \frac{r+3}{2} \geq 6$, we get $\alpha' > \alpha$, a contradiction. If G' contains C_{2r+1} then $\mathrm{exc}_{G'}(C_{2r+1}) \leq |V(G')| - (2r+1) + \alpha' = 2r - 2\alpha + \alpha'$ (see Lemma 4.3). We get $\alpha' \geq 2r + 1 - (2r - 2\alpha + \alpha') = 2\alpha - \alpha' + 1$ and so we have $2\alpha' \geq 2\alpha + 1$. Hence $\alpha' > \alpha$, a contradiction.

c) We show that there is no other minimal eccentric sequence of type $r^\alpha, (r+1)^\beta$ for $r \geq 5$ and $\alpha \geq \frac{r+3}{2}$.

Since $r^{2r-1}, (r+1)^2$ is a minimal eccentric sequence, it follows that $r^\alpha, (r+1)^\beta$ for $\alpha \geq 2r$ is not a minimal eccentric sequence (see Theorem 2.1). It is easy to check that for each α satisfying inequalities $\frac{r+3}{2} \leq \alpha \leq 2r - 1$ we already have a minimal eccentric sequence of type $r^\alpha, (r+1)^\beta$. It is clear that if $\beta_1 \neq \beta_2$ then at most one of the two sequences $r^\alpha, (r+1)^{\beta_1}$ and $r^\alpha, (r+1)^{\beta_2}$ is a minimal eccentric sequence.

If we take the parts a), b), c) into account, the proof is finished. □

Remark 3.3. In most cases minimal eccentric sequences from Theorem 3.2 are realizable through many different graphs. An exception is for instance the sequence $r^{2r-1}, (r+1)^2$. Note that if a minimal eccentric sequence is realizable through a sun-graph then its rays can in general be deployed in different ways (see the proof of Theorem 3.4 in [7]). If a graph which realizes a minimal eccentric sequence from Theorem 3.2 has, for instance, a subgraph depicted in

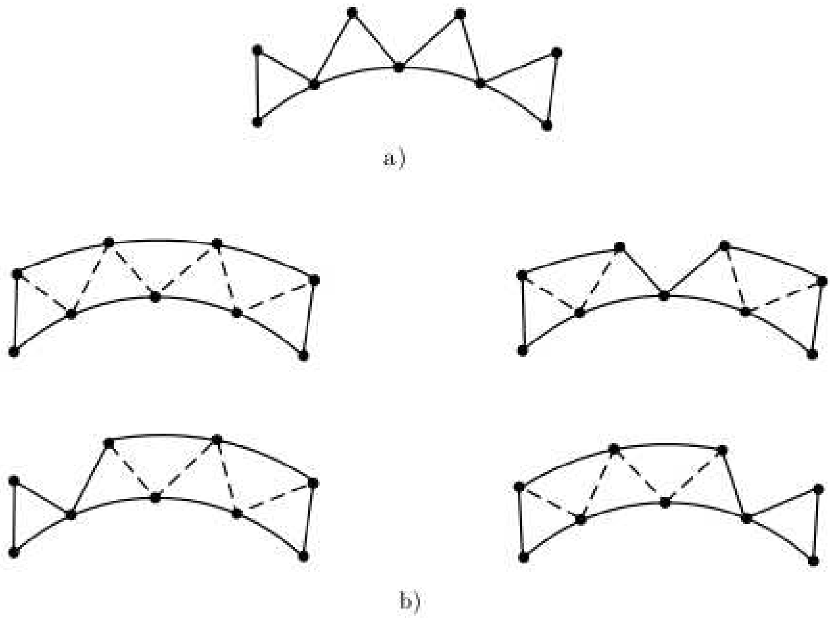


Figure 3

Figure 3a, then this subgraph can be replaced by any of the graphs in Figure 3b containing an arbitrary subset of dashed edges.

Remark 3.4. It follows from Theorem 3.2 that the conjecture about minimal two-valued eccentric sequences given in [8] does not hold. All minimal eccentric sequences of type $r^\alpha, (r+1)^\beta$ are known for $r \in \{1,2,3,4\}$ (see Theorems 2.5, 2.6 and 2.7). If $r \geq 5$ then all minimal eccentric sequences of type $r^\alpha, (r+1)^\beta$ are known for $\alpha \geq \frac{r+3}{2}$ (Theorem 3.2).

Conjecture 3.5. *All minimal eccentric sequences of type $r^\alpha, (r+1)^\beta$ for $r \geq 3$ and $\alpha < \frac{r+3}{2}$ are*

(i) $r^\alpha, (r+1)^{4r-3\alpha+2}, \quad 2 \leq \alpha < \frac{r+3}{2}$,

(ii) $r, (r+1)^{4r-2}$.

Remark 3.6. Note that Conjecture 3.5 holds for $r \in \{3,4\}$ (see Theorems 2.6 and 2.7).

Remark 3.7. Graphs realizing eccentric sequences from the case (i) of Conjecture 3.5 can be found using the procedure described in the proof of

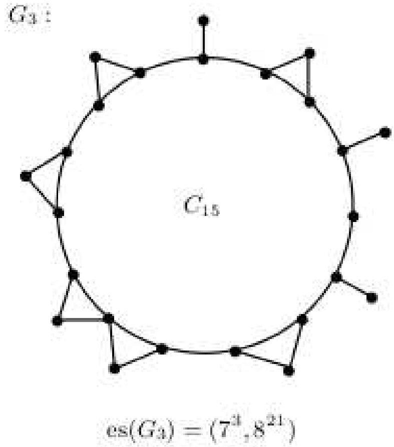


Figure 4

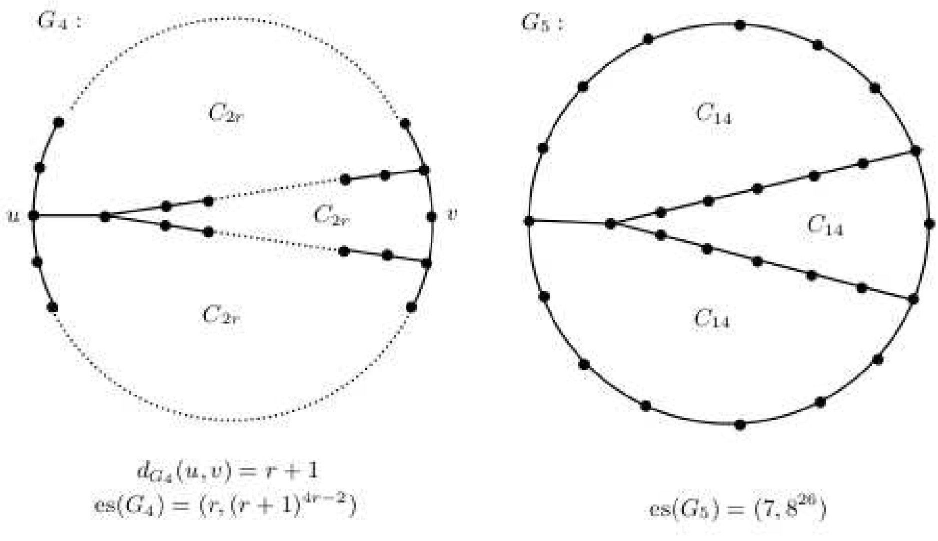


Figure 5

Theorem 3.2 (see the steps (1°), (2°)). For instance the graph G_3 with $\mathrm{es}(G_3) = (7^3, 8^{21})$ is shown in Figure 4.

The eccentric sequence $r, (r+1)^{4r-2}$ (the case (ii) of Conjecture 3.5) is realizable through a graph G_4 in Figure 5 (e.g. the graph G_5 with $\mathrm{es}(G_5) = (7, 8^{26})$ is shown in Figure 5, too).

4. Proofs of Auxiliary Results

In this section we give proofs of results used in the previous section.

Lemma 4.1. *The sequence* $5^4, 6^{11}$ *is a minimal eccentric sequence.*

Proof. The sequence $5^4, 6^{11}$ is eccentric (see Figure 1). Now we are going to show that it is minimal.
Let G be a graph for which $\mathrm{es}(G) = (5^\alpha, 6^\beta)$, $\alpha + \beta = 14$, $\alpha \leq 4$ and $\beta \leq 11$. By Theorem 2.4c G contains a cycle C_m, $m \geq 10$. We distinguish three cases.

(1) $m = 11$
By Lemma 2.3b we get $\mathrm{exc}_G(C_{11}) \leq 6$ and it implies $\alpha \geq 5$, a contradiction.

(2) $m = 10$
By Lemma 2.3c we have $l = |\{v \in V(G); d(v, C_{10}) = 1\}| \leq 2$. It is easy to see that the case $l = 1$ cannot occur. If $l = 2$ then $\mathrm{exc}_G(C_{10}) \leq 6$ (see Lemma 2.3c). It follows that $e_G(u) = 6$ for every vertex $u \in V(G) \setminus V(C_{10})$ and consequently u is not a cut-vertex of G. Now it is easy to check that $\alpha \geq 6$, a contradiction.

(3) $m \in \{12, 13, 14\}$
According to the cases (1), (2) we can assume that G contains neither C_{10} nor C_{11}. Obviously, C_m is not a geodesic cycle. Now it is easy to verify that $\mathrm{rad}\, G < 5$, a contradiction.

□

Now consider graphs H_1, H_2 schematically depicted in Figure 6 with the properties $d_{C_m}(u_1, u_2) = r$, $d_{C_m}(v_1, v_2) + d_{C_m}(v_2, v_3) + d_{C_m}(v_3, v_1) = m$.

Lemma 4.2. *Let G be a graph with radius r and with at most $3r - 2$ vertices. Let C_m, $m \in \{2r, 2r + 1\}$, be a geodesic cycle of G. Then neither H_1 nor H_2 is a subgraph of G.*

Proof. a) Let H_1 be a subgraph of G (note that $d_{C_m}(u_1, u_2) = r$). Since C_m is a geodesic cycle, the graph G has at least $3r - 1$ vertices, a contradiction.

b) Let H_2 be a subgraph of G and H_1 be not a subgraph of G.
We have $|V(H_2) \setminus V(C_m)| = x_1 + x_2 + x_3 + 1$ (see Figure 6). Since C_m is a

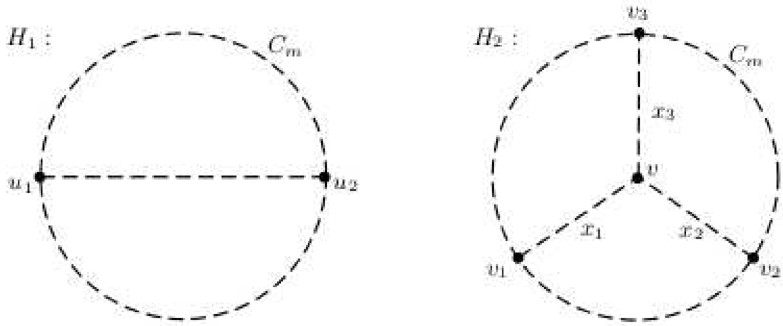


Figure 6

geodesic cycle, we have $(x_1+1+x_2)+(x_2+1+x_3)+(x_3+1+x_1) \geq m-3$. Hence $x_1+x_2+x_3 \geq \frac{m}{2}-3$. If $m=2r+1$ then $x_1+x_2+x_3 > r-3$ and we get $|V(G)| > (2r+1)+(r-3)+1 = 3r-1$, a contradiction. If $m=2r$ then $x_1+x_2+x_3 \geq r-3$. If $x_1+x_2+x_3 > r-3$ or $|V(G)| > |V(H_2)|$, then $|V(G)| > 3r-2$, a contradiction. So we can assume that $x_1+x_2+x_3 = r-3$ and $|V(H_2)| = |V(G)|$. Now we have $r-1 \geq d_{C_{2r}}(v_i,v_j) = d_G(v_i,v_j) = x_i+x_j+2$ for $i \neq j$. Hence $e_G(v) \leq r-1$, a contradiction. □

Lemma 4.3. *Let G be a graph with $\mathrm{es}(G) = (r^\alpha, (r+1)^\beta)$, $r \geq 3$ and $\alpha+\beta \leq 3r-2$. Let C_m, $m \in \{2r, 2r+1\}$, be a geodesic cycle of G. Then*

(i) $\mathrm{exc}_G(C_{2r}) \leq |V(G)| - 2r$,

(ii) $\mathrm{exc}_G(C_{2r+1}) \leq \min\{|V(G)| - (2r+1) + \alpha, 2(|V(G)| - (2r+1))\}$.

Proof. Consider a vertex $w \in V(G) \setminus V(C_m)$. Let W be a component of $G - V(C_m)$ such that $w \in V(W)$. Further let $V_1 = \{x \in V(C_m); d_G(x,W) = 1\}$ and let $a = \max\{d_G(x,y); x,y \in V_1\}$ (note that $d_{C_m}(x,y) = d_G(x,y)$ for $x,y \in V(C_m)$). Let $u,v \in V_1$ be vertices with $d_G(u,v) = a$. Then $a < r$ and for any vertex $x \in V_1$ it holds $d_{C_m}(u,x) + d_{C_m}(v,x) = a$ (see Lemma 4.2).

Let $u \overset{W}{—} v$ denote a path such that all its vertices except u, v belong to $V(W)$. Since C_m is a geodesic cycle of G, the length of the considered path is at least a and so $|V(W)| \geq a-1$. Let $a_1 = \max\{d_G(u,x); x \in V(W)\}$ and $a_2 = \max\{d_G(v,x); x \in V(W)\}$. Obviously, $a_i \geq a-1$, $i \in \{1,2\}$. Now we are going to show that $a_i \leq a+1$. Suppose, contrary to our claim, that there exists a vertex $x \in V(W)$ for which $d_G(u,x) \geq a+2$. Hence $d_G(v,x) \geq 2$ (since $d_G(u,v) = a$). Now it is easy to verify that there is a vertex $y \in V(C_m)$ for

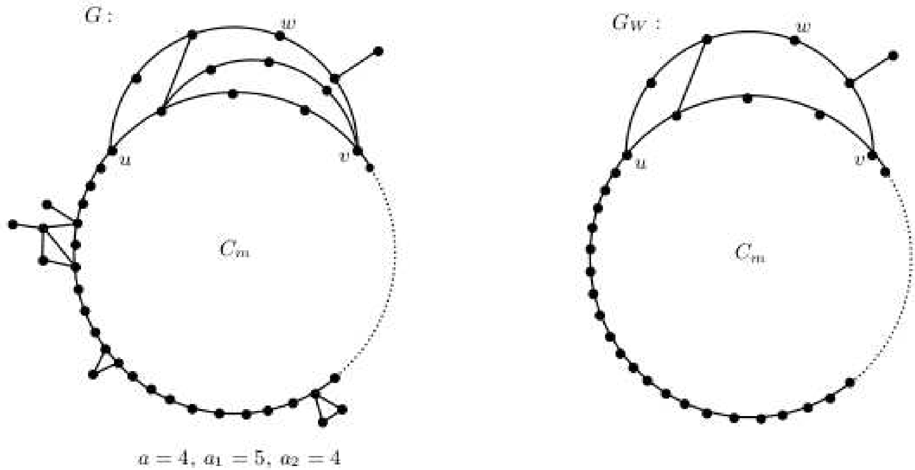


Figure 7

which $d_G(x, y) \geq r + 2$, a contradiction. In what follows we can assume that $a - 1 \leq a_i \leq a + 1$, $i \in \{1, 2\}$.

Let G_W denote the subgraph of G induced by $V(C_m) \cup V(W)$ (see Figure 7). Let b be the length of some $u \overset{W}{—} v$ path. Obviously, $b \geq a$ (C_m is a geodesic cycle). Since $|V(G)| \leq 3r - 2$, we have $|V(W)| \leq r - 2$. Let $V^a = \{x \in V(C_m); d_{C_m}(u, x) + d_{C_m}(x, v) = a\}$. Obviously, $|V^a| = a + 1$. If $s \in V^a$ and $t \in V(W)$ then $d_{G_W}(s, t) = d_G(s, t) \leq \lfloor \frac{a+b}{2} \rfloor + ((r-2)-(b-1)) \leq \frac{a+b}{2} + r - b - 1 = \frac{a-b}{2} + r - 1 \leq r - 1$. Hence $e_{G_W}(s) = r$ for every vertex $s \in V^a$.

(i) Let $m = 2r$.
If $a = 0$ then $u = v$ and u is a cut-vertex of G. Hence $\mathrm{exc}_{G_W}(C_{2r}) = 1 \leq |V(W)|$. So we can assume that $a > 0$. Further we distinguish two cases.

(1) $a_i \in \{a - 1, a\}$, $i \in \{1, 2\}$
Obviously, in the cycle C_{2r} there exist at least $a + 1 + 2(r - a) = 2r - a + 1$ vertices with the eccentricity r in G_W. Hence $\mathrm{exc}_{G_W}(C_{2r}) \leq 2r - (2r - a + 1) = a - 1 \leq |V(W)|$.

(2) $\max\{a_1, a_2\} = a + 1$
It is clear that in the cycle C_{2r} there exist at least $(a + 1) + 2(r - (a + 1)) = 2r - a - 1$ vertices with eccentricity r in G_W. Hence $\mathrm{exc}_{G_W}(C_{2r}) \leq 2r - (2r - a - 1) = a + 1$. It is easy to verify (by considering the length of a shortest $u \overset{W}{—} v$ path) that $|V(W)| \geq a + 1$.

From what has already been proved ($\mathrm{exc}_{G_W}(C_{2r}) \leq |V(W)|$), it easily follows that the proof of the case (i) is finished.

(ii) Let $m = 2r + 1$.
If $a = 0$ then $u = v$ and u is a cut-vertex of G. Hence $e_G(u) = r$ and so $\mathrm{exc}_{G_W}(C_{2r+1}) = 2 \leq |V(W)| + 1$. Let $a > 0$. If $|V(W)| = a - 1$ then $\mathrm{exc}_{G_W}(C_{2r+1}) = 0 < |V(W)|$. So we can assume that $|V(W)| \geq a > 0$. We distinguisch three cases.

(1) $a_i \in \{a-1, a\}$, $i \in \{1, 2\}$
In the cycle C_{2r+1} there exist at least $a + 1 + 2(r - a) = 2r - a + 1$ vertices with eccentricity r in G_W. It implies $\mathrm{exc}_G(C_{2r+1}) \leq 2r + 1 - (2r - a + 1) = a \leq |V(W)|$.

(2) $a_i = a$, $a_j = a + 1$, $\{i, j\} = \{1, 2\}$
We have $\mathrm{exc}_{G_W}(C_{2r+1}) \leq (2r+1) - ((a+1) + (r-a) + (r-(a+1))) = a + 1$. It is easy to verify (by considering the length of a shortest $u \overset{W}{—} v$ path) that $|V(W)| \geq a + 1$. We get $\mathrm{exc}_{G_W}(C_{2r+1}) \leq |V(W)|$.

(3) $a_1 = a_2 = a + 1$
Firstly we show that $e_G(u) = e_G(v) = r$. Suppose, contrary to our claim, that e.g. $e_G(u) = r + 1$. Since we know that $e_{G_W}(v) = e_{G_W}(u) = r$, there exists a vertex $x \in V(G) \setminus V(W)$ such that $d_G(u, x) = r + 1$. Obviously, $d_G(v, x) \geq r + 1 - a$. Since $a_2 = a + 1$, there exists a vertex $y \in V(W)$ such that $d_G(v, y) = a + 1$. Now it is easy to verify that $d_G(x, y) \geq r + 2$, a contradition.
Clearly, $|V(W)| \geq a + 1$ and $\mathrm{exc}_{G_W}(C_{2r+1}) \leq (2r + 1) - ((a + 1) + 2(r - a - 1)) = a + 2$. Finally, in the case (3), we have $\mathrm{exc}_{G_W}(C_{2r+1}) \leq |V(W)| + 1$ and $e_G(u) = e_G(v) = r$.

If we take the possibility $a = 0$ and the cases (1), (2), (3) into account we get $\mathrm{exc}_G(C_{2r+1}) \leq |V(G)| - (2r + 1) + \alpha$. According to Lemma 2.3b we have $\mathrm{exc}_G(C_{2r+1}) \leq 2(|V(G)| - (2r + 1))$ and so the proof is finished.

□

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