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A NONLINEAR VARIATIONAL INEQUALITY INVOLVING H-MONOTONE OPERATORS

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Abstract: The aim of this paper is to study the existence of solutions and convergence of the perturbed three-step iterative algorithm for a nonlinear variational inequality involving H-monotone and strongly monotone operators.

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Key Words: nonlinear variational inequality, H-monotone operator, strongly monotone operator, resolvent operator technique, perturbed three-step iterative algorithm

1. Introduction

Variational inequality theory has become a very effective and powerful tool for studying a wide range of problems arising in pure and applied sciences which include work on mathematical programming, optimization theory, engineering,

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differential equations, mechanics, contact problems in elasticity, control problems, general equilibrium problems in economics and transportation, etc. It is well known that one of the most interesting and important problems in the variational inequality theory is the development of an efficient iterative algorithm to compute approximate solutions of various variational inequalities and inclusions, see, for example, [1]-[5]. Fang and Huang [2] introduced the definitions of H-monotone operator and its resolvent operator, established the Lipschitz continuity of the resolvent operator, constructed an iterative algorithm, and obtained the existence of solutions for a class of variational inclusions and convergence of the iterative algorithm.

The aim of this paper is to study solvability of a nonlinear variational inequality, which includes the variational inequalities and variational inclusions in [1]-[5] as special cases. Using the resolvent operator technique and Banach fixed point theorem, we prove the existence of solutions and convergence of the perturbed three-step iterative algorithm for the nonlinear variational inequality.

2. Preliminaries

Throughout this paper, we assume that X is a real Hilbert space endowed with a norm $\|\cdot\|$ and an inner product $\langle\cdot,\cdot\rangle$, respectively, 2^X stands for the family of all the nonempty subsets of X, I denotes the identity operator on X and $\mathbb{R} = (-\infty, +\infty)$. Assume that $H, A, B, : X \to X$ and $N, M : X \times X \to X$ are operators and $W : X \to 2^X$ is a multivalued operator. Given $f \in X$, we consider the following problem: find $u \in X$ such that

$$f \in N(A(u), B(u)) + W(u), \tag{2.1}$$

which is called the *nonlinear variational inequality.*

Some special cases of problem (2.1) are as follows:

(1) If $f = 0$ and $N(x, y) = x$ for any $x, y, z \in X$, then problem (2.1) reduces to the following variational inclusion problem: find $u \in X$ such that

$$0 \in A(u) + W(u), \tag{2.2}$$

which was introduced and studied by Fang and Huang [2].

(2) If $W = \partial\psi$, where $\partial\psi$ denotes the subdifferential of a proper convex and lower semi-continuous functional $\psi : X \to \mathbb{R} \cup \{+\infty\}$, then problem (2.2) collapses to finding $u \in X$ such that

$$\langle A(u), v - u\rangle + \psi(v) - \psi(u) \geq 0, \quad \forall v \in X, \tag{2.3}$$

which was introduced and studied by many authors [1]-[4].

(3) If $W = \partial\delta_K$, where δ_K is the indicator function of a nonempty closed and convex subset K of X, then problem (2.2) is equivalent to the following problem studied in [3]: find $u \in K$ such that

$$\langle A(u), v - u\rangle \geq 0, \quad \forall v \in K. \tag{2.4}$$

We now recall and introduce the following definitions and results.

Definition 2.1. Let $N : X \times X \to X$, $g, b, c, H : X \to X$ be operators and $W : X \to 2^X$ be a multivalued operator.

(a1) g is said to be *monotone* if

$$\langle g(x) - g(y), x - y\rangle \geq 0, \quad \forall x, y \in X;$$

(a2) g is said to be *strictly monotone* if g is monotone and

$$\langle g(x) - g(y), x - y\rangle = 0$$

if and only if $x = y$;

(a3) W is said to be *monotone* if

$$\langle x - y, u - v\rangle \geq 0, \quad \forall u, v \in X,\ x \in Wu,\ y \in Wv;$$

(a4) W is said to be *maximal monotone* if W is monotone and $(I+\rho W)(X) = X$ for any $\rho > 0$;

(a5) W is said to be *H-monotone* if W is monotone and $(H + \rho H)(X) = X$ for any $\rho > 0$;

(a6) g is said to be *s-Lipschitz continuous* and *t-strongly monotone* if there exist positive constants s and t satisfying, respectively,

$$\|g(x) - g(y)\| \leq s\|x - y\|, \quad \langle g(x) - g(y), x - y\rangle \geq t\|x - y\|^2, \quad \forall x, y \in X;$$

(a7) b is called *s-strongly monotone* with respect to H and the first argument of N if there exists a positive constant s satisfying

$$\langle N(b(x), u) - N(b(y), u), H(x) - H(y)\rangle \geq s\|x - y\|^2, \quad \forall x, y, u \in X;$$

(a8) N is called *s-Lipschitz continuous* with respect to the first argument if there exists a positive constant s satisfying

$$\|N(x, u) - N(y, u)\| \leq s\|x - y\|, \quad \forall x, y, u \in X.$$

Similarly we can define the Lipschitz continuity of N with respect to the second argument. It is known that a maximal monotone operator need not be H-monotone for some H, and if W is H-monotone and H is strictly monotone, then W is maximal monotone.

Definition 2.2. (see [2]) Let $H : X \to X$ be a strictly monotone operator and $W : X \to 2^X$ be an H-monotone operator. For any given $\rho > 0$, the *resolvent operator* $R^H_{W,\rho} : X \to X$ is defined by

$$R^H_{W,\rho}(x) = (H + \rho W)^{-1}(x), \quad \forall x \in X.$$

Lemma 2.3. (see [6]) *Let* $\{a_n\}_{n\geq 0}$, $\{b_n\}_{n\geq 0}$ *and* $\{c_n\}_{n\geq 0}$ *be nonnegative sequences satisfying*

$$a_{n+1} \leq (1 - t_n)a_n + t_n b_n + c_n, \quad \forall n \geq 0,$$

where $\{t_n\}_{n\geq 0} \subset [0, 1]$, $\sum_{n=0}^{\infty} t_n = \infty$, $\lim_{n\to\infty} b_n = 0$ *and* $\sum_{n=0}^{\infty} c_n < \infty$. *Then* $\lim_{n\to\infty} a_n = 0$.

Lemma 2.4. (see [2]) *Let* $H : X \to X$ *be* r*-strongly monotone and* $W : X \to 2^X$ *be* H*-monotone. Then the resolvent operator* $R^H_{W,\rho} : X \to X$ *is* r^{-1}*-Lipschitz continuous.*

3. Main Results

Lemma 3.1. *Let* $H : X \to X$ *be a strictly monotone operator,* $W : X \to 2^X$ *be an* H*-monotone operator,* t *and* ρ *be two positive constants. Then the following statements are equivalent:*

(b1) *the nonlinear variational inequality* (2.1) *possesses a solution* $u \in X$*;*

(b2) *there exists* $u \in X$ *satisfying*

$$u = R^H_{W,\rho}(H(u) - \rho N(A(u), B(u)) + \rho f); \tag{3.1}$$

(b3) *the mapping* $G : X \to X$ *defined by*

$$G(x) = (1 - t)x + tR^H_{W,\rho}(H(x) - \rho N(A(x), B(x)) + \rho f), \quad \forall x \in X \tag{3.2}$$

has a fixed point $u \in X$.

Proof. Observe that (b1) holds if and only if

$$H(u) - \rho N(A(u), B(u)) + \rho f \in (H + \rho W)(u),$$

which is equivalent to (3.1) by the definition of resolvent operator. It follows from (3.2) and $t > 0$ that G has a fixed point $u \in X$ if and only if (3.1) holds. This completes the proof. □

Based on Lemma 3.1, we suggest the following perturbed three-step iterative algorithm for the nonlinear variational inequality (2.1).

Algorithm 3.2. *Let $A, B, H, H_n : X \to X$, $N : X \times X \to X$ be operators and $W : X \to 2^X$ be H-monotone and $W_n : X \to 2^X$ be H_n-monotone for each $n \geq 0$. Given $f, u_0 \in X$, the iterative sequence $\{u_n\}_{n\geq 0}$ is defined by, $\forall n \geq 0$,*

$$\begin{aligned} w_n &= (1 - c_n)u_n + c_n R^{H_n}_{W_n,\rho}(H(u_n) - \rho N(A(u_n), B(u_n)) + \rho f) \\ v_n &= (1 - b_n)u_n + b_n R^{H_n}_{W_n,\rho}(H(w_n) - \rho N(A(w_n), B(w_n)) + \rho f) \\ u_{n+1} &= (1 - a_n)u_n + a_n R^{H_n}_{W_n,\rho}(H(v_n) - \rho N(A(v_n), B(u_n)) + \rho f), \end{aligned}$$

where $\{W_n\}_{n\geq 0}$ is a sequence of H_n-monotone operators approximating W in a specific sense and the sequences $\{a_n\}_{n\geq 0}$, $\{b_n\}_{n\geq 0}$ and $\{c_n\}_{n\geq 0}$ are sequences in $[0, 1]$ satisfying certain conditions.

Theorem 3.3. *Let $H : X \to X$ be s-strongly monotone and h-Lipschitz continuous. Let $H_n : X \to X$ be s_n-strongly monotone for each $n \geq 0$. Let $W : X \to 2^X$ be H-monotone and $W_n : X \to 2^X$ be H_n-monotone for each $n \geq 0$. Let $N : X \times X \to X$ be i-Lipschitz continuous and j-Lipschitz continuous with respect to the first and second arguments, respectively. Assume that $A, B : X \to X$ are a-Lipschitz continuous, b-Lipschitz continuous, respectively, A is α-strongly monotone with respect to H and the first argument of N. Let*

$$p = jb, \quad J = (ia)^2 - p^2, \quad K = \alpha - p, \quad L = h^2 - s^2, \tag{3.3}$$

$$\lim_{n\to\infty} s_n = s, \tag{3.4}$$

$$\lim_{n\to\infty} \|R^{H_n}_{W_n,\rho}(x) - R^{H}_{W,\rho}(x)\| = 0, \quad \forall x \in X,\ \rho > 0, \tag{3.5}$$

$$\sum_{n=0}^{\infty} a_n = +\infty. \tag{3.6}$$

Assume that there exists a positive constant ρ satisfying

$$\rho < sp^{-1} \tag{3.7}$$

and one of the following conditions:

$$|\rho - KJ^{-1}| < J^{-1}\sqrt{K^2 - LJ}, \quad J > 0, \quad |K| > \sqrt{LJ}; \tag{3.8}$$

$$|\rho - KJ^{-1}| > -J^{-1}\sqrt{K^2 - LJ}, \quad J < 0. \tag{3.9}$$

Then for any given $f \in X$, the nonlinear variational inequality (2.1) *has a unique solution $u \in X$ and the sequence $\{u_n\}_{n\geq 0}$ defined by Algorithm 3.2 converges strongly to u.*

Proof. Now we show that the mapping G defined by (3.2) has a unique fixed point $u \in X$, where t is a constant in $(0, 1]$. Let x, y be two arbitrary elements in X. Since A is α-strongly monotone with respect to H and the first argument of N, it follows from the Lipschitz continuity of A, C and H and the Lipschitz continuity of N with respect to the first argument, respectively, that

$$\begin{aligned}
&\|H(x) - H(y) - \rho(N(A(x), B(x)) - N(A(y), B(x)))\| \\
&= [\|H(x) - H(y)\|^2 \\
&\quad - 2\rho\langle N(A(x), B(x)) - N(A(y), B(x)), H(x) - H(y)\rangle \\
&\quad + \rho^2\|N(A(x), B(x)) - N(A(y), B(x))\|^2]^{\frac{1}{2}} \\
&\leq [h^2\|x - y\|^2 - 2\rho\alpha\|x - y\|^2 \\
&\quad + \rho^2(\|N(A(x), B(x)) - N(A(y), B(x))\|)^2]^{\frac{1}{2}} \\
&\leq \sqrt{h^2 - 2\rho\alpha + \rho^2(ia)^2}\|x - y\|.
\end{aligned} \tag{3.10}$$

Note that

$$\|N(A(y), B(x)) - N(A(y), B(y))\| \leq jb\|x - y\|. \tag{3.11}$$

By virtue of Lemma 2.4, (3.2), (3.10) and (3.11), we get that

$$\begin{aligned}
&\|G(x) - G(y)\| \\
&\leq (1 - t)\|x - y\| + t\|R^H_{W,\rho}(H(x) - \rho N(A(x), B(x)) + \rho f) \\
&\quad - R^H_{W,\rho}(H(y) - \rho N(A(y), B(y)) + \rho f)\| \\
&\leq (1 - t)\|x - y\| + ts^{-1}\|H(x) - H(y) \\
&\quad - \rho(N(A(x), B(x)) - N(A(y), B(y)))\| \\
&\leq (1 - t)\|x - y\| + ts^{-1}(\|H(x) - H(y) \\
&\quad - \rho(N(A(x), B(x)) - N(A(y), B(x)))\| \\
&\quad + \rho\|N(A(y), B(x)) - N(A(y), B(y))\|) \\
&\leq (1 - t(1 - \theta))\|x - y\|,
\end{aligned} \tag{3.12}$$

where

$$\theta = s^{-1}\big(\sqrt{h^2 - 2\rho\alpha + \rho^2(ia)^2} + \rho p\big) > 0. \tag{3.13}$$

In light of (3.3), (3.7) and (3.13), we infer that

$$\begin{aligned}\theta < 1 \quad &\Longleftrightarrow \quad \sqrt{h^2 - 2\rho\alpha + \rho^2(ia)^2} < s - \rho p \\ &\Longleftrightarrow \quad J\rho^2 - 2K\rho < -L.\end{aligned}$$

In view of one of (3.8) and (3.9), we conclude that $\theta < 1$, that is, G is a contraction mapping. Hence G has a unique fixed point $u \in X$, which together with Lemma 3.1 ensures that the nonlinear variational inequality (2.1) has a unique solution $u \in X$. It follows that, $\forall n \geq 0$,

$$\begin{aligned}u &= (1 - c_n)u + c_n R^H_{W,\rho}(H(u) - \rho N(A(u), B(u)) + \rho f) \\ &= (1 - b_n)u + b_n R^H_{W,\rho}(H(u) - \rho N(A(u), B(u)) + \rho f) \\ &= (1 - a_n)u + a_n R^H_{W,\rho}(H(u) - \rho N(A(u), B(u)) + \rho f),\end{aligned} \tag{3.14}$$

Next we show that $\lim_{n\to\infty} u_n = u$. Let

$$\theta_n = s_n^{-1}\big(\sqrt{h^2 - 2\rho\alpha + \rho^2(ia)^2} + \rho jb\big),$$

$$\begin{aligned}g_n = \|&R^{H_n}_{W_n,\rho}(H(u) - \rho N(A(u), B(u)) + \rho f) \\ &- R^H_{W,\rho}(H(u) - \rho N(A(u), B(u)) + \rho f)\|, \quad \forall n \geq 0.\end{aligned}$$

Clearly (3.4) means that $\lim_{n\to\infty} \theta_n = \theta < 1$. Thus there exists a positive integer P satisfying

$$\theta_n < \frac{1}{2}(1 + \theta) < 1, \quad \forall n > P. \tag{3.15}$$

It follows from Lemma 2.4, Algorithm 3.2, (3.14) and (3.15) that for $n > P$

$$\begin{aligned}\|w_n - u\| \leq\ &(1 - c_n)\|u_n - u\| + c_n\|R^{H_n}_{W_n,\rho}(H(u_n) - \rho N(A(u_n), B(u_n)) + \rho f) \\ &- R^H_{W,\rho}(H(u) - \rho N(A(u), B(u)) + \rho f)\| \\ \leq\ &(1 - c_n)\|u_n - u\| + c_n\|R^{H_n}_{W_n,\rho}(H(u_n) - \rho N(A(u_n), B(u_n)) + \rho f) \\ &- R^{H_n}_{W_n,\rho}(H(u) - \rho N(A(u), B(u)) + \rho f)\| \\ &+ c_n\|R^{H_n}_{W_n,\rho}(H(u) - \rho N(A(u), B(u)) + \rho f) \\ &- R^H_{W,\rho}(H(u) - \rho N(A(u), B(u)) + \rho f)\| \\ \leq\ &(1 - c_n)\|u_n - u\| + c_n s_n^{-1}\|H(u_n) - H(u) \\ &- \rho(N(A(u_n), B(u_n)) - N(A(u), B(u)))\| + c_n g_n \\ \leq\ &(1 - c_n)\|u_n - u\| + c_n s_n^{-1}(\|H(u_n) - H(u) \\ &- \rho(N(A(u_n), B(u_n)) - N(A(u), B(u_n)))\|\end{aligned}$$

$$
\begin{aligned}
&+\rho\|N(A(u),B(u_n))-N(A(u),B(u))\|)+c_ng_n\\
&\le (1-c_n)\|u_n-u\|+c_n\theta_n\|u_n-u\|+c_ng_n\\
&\le \|u_n-u\|+c_ng_n.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
\|v_n-u\| &\le (1-b_n)\|u_n-u\|+b_n\theta_n\|w_n-u\|+b_ng_n\\
&\le \|u_n-u\|+2b_ng_n
\end{aligned}
$$

and

$$
\begin{aligned}
\|u_{n+1}-u\| &\le (1-a_n)\|u_n-u\|+a_n\theta_n\|v_n-u\|+a_ng_n\\
&\le (1-(1-\theta_n)a_n)\|u_n-u\|+3a_ng_n\\
&\le \left(1-\frac{1}{2}(1-\theta)a_n\right)\|u_n-u\|+3a_ng_n
\end{aligned}
\tag{3.16}
$$

for $n > P$. It follows from Lemma 2.3, (3.4), (3.5), (3.7) and (3.16) that $\lim_{n\to\infty}\|u_n-u\|=0$. This completes the proof. □

References

[1] S. Adly, Perturbed algorithms and sensitivity analysis for a general class of variational inclusions, *J. Math. Anal. Appl.*, **201** (1996), 609-630.

[2] Y.P. Fang, N.J. Huang, Monotone operator and resolvent operator technique for variational inclusions, *Appl. Math. Comput.*, **145** (2003), 795-803.

[3] D. Kinderlehrer, G. Stampacchia, *An Introduction to Variational Inequalities and their Applications*, SIAM, Philadelphia, 2000.

[4] A.M. Harder, T.L. Hicks, Stability results for fixed point iteration procedures, *Math. Japonica*, **33** (1988), 693-706.

[5] A. Hassouni, A. Moudafi, A perturbed algorithm for variational inclusions, *J. Math. Anal. Appl.*, **18** (1994), 706-712.

[6] L.S. Liu, Ishikawa and Mann iterative processes with erroes for nonlinear strongly accretive mappings in Banach spaces, *J. Math. Anal. Appl.*, **194** (1995), 114-125.