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A NOVEL ARGUMENT THAT $\mathcal{P}(\mathbb{N})$ IS EQUIPOTENT TO $\mathbb{R}$

M. Padraig M.M. McLoughlin
265 Lytle Hall, Department of Mathematics
Kutztown University of Pennsylvania
Kutztown, PA 19530, USA

Abstract: We begin with brief discussion of the claim and some background; we then put forward our pulchrous argument that $\mathcal{P}(\mathbb{N})$ is equipotent to $\mathbb{R}$.

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1. Introduction

Oft one hears or reads that $\mathcal{P}(\mathbb{N})$ is equinumerous to $\mathbb{R}$ and accepts it, begrudgingly, without a rigourous proof (along with some other classic results of cardinal theory). The author has taught mathematics for 30 years and studied mathematics most often under the Moore method or a modified version of the method. He teaches using a modified Moore method and directs undergraduate research using the method. So, being aware that the result $\mathcal{P}(\mathbb{N}) \simeq \mathbb{R}$ was somewhat similar to being aware that $|\mathbb{N}| < |\mathbb{R}|$ (except that $|\mathbb{N}| < |\mathbb{R}|$ was *proven* in the Analysis class he took by a fellow student, Gregory Kuperberg, who created a diagonalisation argument equivalent the Cantor diagonalisation

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argument).[1]

2. Background Definitions

We need to define some terms or notation (where the term or the notation is used differently by different authors) for clarity. Let $U = \mathbb{R}$ throughout. We denote the cardinal naturals as $\mathbb{N}^*$, the ordinal naturals as $\mathbb{N}$, an initial segment of the cardinal naturals as $\mathbb{N}^*_k = \{0, 1, 2, 3, \dots, (k-1), k\}$, and an initial segment of the ordinal naturals as $\mathbb{N}_k = \{1, 2, 3, \dots, (k-1), k\}$. We use the terminology segment for $(a, b) = \{x | a < x < b\}$ and interval for $[a, b] = \{x | a \leq x \leq b\}$.

We note that for $U = \mathbb{R}$ it is the case that the interval $[0, 1]$ consists of all reals that are expressible in decimal form such that it is an array of digits of the form $0.d_1 d_2 d_3 d_4 \dots d_{n-1} d_n d_{n+1} \dots$ where $d_i \in \mathbb{N}^*_9 \ \forall i \in \mathbb{N}$ and the real number is $\sum\limits_{i \in \mathbb{N}} d_i \cdot 10^{-i}$. Such decimal expressions are non-unique with regard to repeating nines.

We use the usual idea of relation and function. Let U be a well defined universe while D and C are sets and f be a well defined relation from D to C. Recall the relation f is a well defined function from D to C *if and only if (iff)* $D \neq \emptyset$; $\forall\, x \in D \;\exists\, y \in C \;\ni\; (x, y) \in f$; and, $x \in D$, $j \in C$, $k \in C$ with $(x, j) \in f \;\wedge\; (x, k) \in f$ implies $j = k$. The set D is defined as the domain of f and is denoted as $dom(f)$. The set C is defined as the codomain of f and is denoted as $cod(f)$. The set $ran(f)$ is defined as the range of f and it is the the set of all elements, q, in C such that there exists and element, p, in D such that $(p, q) \in f$. We also include the definition of the **corange** of the function f:

Definition 2.1. Let U be a well defined universe while D and C are sets. Let f be a well defined function from D to C. The set $cor(f)$ is defined as the corange of f and it is the the set of all elements, x, in D such that there exists and element, y, in C such that $(x, y) \in f$.

3. Background Theorems

Theorem 3.1. *Let A and B be non-empty sets. A is equinumerous with B ($|A| = |B|$) iff there exists a well defined function $f : A \to B$ such that f is*

[1] I now know Kuperberg's argument was equivalent the Cantor diagonalisation argument; but, we *did not know that nor did he* when he did it in 1982 or so in Coke Reed's class at Auburn University. I suppose if one wishes a text to review the ideas one should refer to [1].

bijective.

Theorem 3.2. *Let A and B be non-empty sets. $|A| \leq |B|$ iff there exists a well defined function $f : A \to B$ such that f is injective.*

Theorem 3.3. *Let A and B be non-empty sets. $|A| \geq |B|$ iff there exists a well defined function $g : A \to B$ such that f is surjective.*

Theorem 3.4. *Let α and β be cardinal numbers. It is the case that $\alpha < \beta$, $\alpha = \beta$, or $\alpha > \beta$ and moreover no two different conditions can be true simultaneously.*[2]

Theorem 3.5. *Let $U = \mathbb{R}$. Let $x \in [0,1]$. There exists a binary expansion form, call it* **binimal** *form,*[3] *for x such that it is an array of digits of the form $0.b_1b_2b_3b_4\dots b_{n-1}b_nb_{n+1}\dots$ where $b_i \in \mathbb{N}_1^* \ \forall\, i \in \mathbb{N}$ and $x = \sum_{i\in\mathbb{N}} b_i \cdot 2^{-i}$.*

Theorem 3.6. *Let $U = \mathbb{R}$. Let $x \in [0,1]$. There exists a ternary expansion form, call it* **trinimal** *form, for x such that it is an array of digits of the form $0.t_1t_2t_3t_4\dots t_{n-1}t_nt_{n+1}\dots$ where $t_i \in \mathbb{N}_2^* \ \forall\, i \in \mathbb{N}$ and $x = \sum_{i\in\mathbb{N}} t_i \cdot 3^{-i}$.*

We note that real numbers that are expressed in binimal form are non-unique with regard to repeating ones and that real numbers that are expressed in trinimal form are non-unique with regard to repeating twos.

4. Results

Two rudimentary lemmas are needed:

Lemma 4.1. *The segment $(0,1)$ is equinumerous with the interval $[0,1]$.*

Proof. Assume the premises. Define $f : (0,1) \longrightarrow [0,1]$ such that $f(x) = x$. It follows easily that f is a well defined function that is injective so, $|(0,1)| \leq |[0,1]|$.

Define $g : [0,1] \longrightarrow (0,1)$ such that $g(x) = \frac{x}{2} + \frac{1}{4}$. It follows quite nicely that g is a well defined function that is injective so, $|[0,1]| \leq |(0,1)|$ Hence, $|(0,1)| = |[0,1]|$. □

and:

[2]Trichotomy holds for cardinal numbers.

[3]We have seen in texts reals that are expressed in binimal form as binary decimal. We opine such is quite incorrect since 'deci' indicates tens.

Lemma 4.2. *The segment* $(0,1)$ *is equinumerous with* $\mathbb{R}$.

Proof. Assume the premises. Define $f : \mathbb{R} \longrightarrow (0,1)$ such that

$$f(x) = \frac{1}{\pi} \cdot \arctan(x) + \frac{1}{2}.$$

It follows in a facile manner that f is a well defined function that is bijective between $\mathbb{R}$ and the segment $(0,1)$ $\square$

Theorem 4.1. *The interval* $[0,1]$ *is equinumerous with* $\mathcal{P}(\mathbb{N})$.

Proof. Assume the premises. The interval $[0,1]$ consists of all reals that are expressible in binimal form such that it is an array of digits of the form $0.b_1b_2b_3b_4 \ldots b_{n-1}b_nb_{n+1} \ldots$ where $b_i \in \mathbb{N}_1^* \ \forall\ i \in \mathbb{N}$. We note that the expression is unique up to repeating ones.

Define $f : \mathcal{P}(\mathbb{N}) \longrightarrow [0,1]$ where

$$f(A) = \begin{cases} 0 & A = \emptyset \\ 1 & A = \mathbb{N} \\ 0.b_1b_2b_3b_4 \ldots b_{n-1}b_nb_{n+1} \ldots & A \subset \mathbb{N} \wedge \\ & b_i = 1 \ni i \in A; b_i = 0 \ni i \notin A \end{cases}$$

We note in binimal expansion $0 = 0.\overline{0} = 0.b_1b_2b_3b_4 \ldots b_{n-1}b_nb_{n+1} \ldots \ni b_i = 0 \ \forall\ i \in \mathbb{N}$; $1 = 0.\overline{1} = 0.b_1b_2b_3b_4 \ldots b_{n-1}b_nb_{n+1} \ldots \ni b_i = 1 \ \forall i \in \mathbb{N}$; and, $x \in (0,1) \Longrightarrow \exists j \in \mathbb{N} \ni b_j = 1 \wedge \exists k \in \mathbb{N} \ni b_k = 0$ (obviously $j \neq k$).

Clearly $f \subseteq \mathcal{P}(\mathbb{N}) \times [0,1]$ so it is a well defined relation from $\mathcal{P}(\mathbb{N})$ to $[0,1]$. Thus, $cor(f) \subseteq dom(f) \wedge ran(f) \subseteq cod(f)$.

Note $\mathcal{P}(\mathbb{N}) \neq \emptyset$.

Suppose $\exists A \in \mathcal{P}(\mathbb{N}) \ni A \notin cor(f)$. This implies that there is a set that is a subset of $\mathbb{N}$ not mapped by f. By the definition of f, $A \neq \emptyset$ and $A \neq \mathbb{N}$; thus, $A \subset \mathbb{N}$. So, $\exists p \in \mathbb{N} \ni p \notin A$. Moreover $\exists q \in \mathbb{N} \ni q \in A$ since $A \neq \emptyset$. Let $P = \{p \in \mathbb{N} \ni p \notin A\}$. Let $Q = \{q \in \mathbb{N} \ni q \notin A\}$. Consider $y = 0.b_1b_2b_3b_4 \ldots b_{n-1}b_nb_{n+1} \ldots$ where $b_i = 0$ if $i \in Q$ and $b_i = 1$ if $i \in P$; $f^{-1}(y) = A$ under the consideration of the relation f, So, f maps A to $y \Longrightarrow A \in cor(f)$.

So, $dom(f) \subseteq cor(f)$ and we have $dom(f) = cor(f)$.

Suppose $\exists A \in \mathcal{P}(\mathbb{N}) \ni \exists y \in [0,1] \wedge \exists z \in [0,1] \ni (A,y) \in f \wedge (A,z) \in f \ni y \neq z$. This implies $\exists i \in \mathbb{N}$ such that for $y = 0.y_1y_2y_3y_4 \ldots$ and $z = 0.z_1z_2z_3z_4 \ldots$ where $y_i \neq z_i$. So, by definition of f it is the case that $i \in A \wedge i \notin A$.
#!

Hence, $(A, y) \in f \wedge (A, z) \in f \Longrightarrow y = z$.

Therefore, f is a well defined function from $\mathcal{P}(\mathbb{N})$ to $[0, 1]$.

Suppose f is not surjective. So, $\exists w \in [0, 1] \ni w \notin ran(f)$.

We know $0 \in ran(f)$ and $1 \in ran(f)$; so, $w \in (0, 1)$. So, let $w = 0.w_1 w_2 w_3 w_4 \ldots w_{n-1} w_n w_{n+1} \ldots$ and we know $\exists j \in \mathbb{N} \ni w_j = 0$ and $\exists k \in \mathbb{N} \ni w_k = 1$. We construct the set $B \subset \mathbb{N}$ where $i \in B$ if $w_i = 1 \wedge i \notin B$ if $w_i = 0$. It follows most beautifully that $B \in cor(f)$ which forces $(B, w) \in f$ which forces $w \in ran(f)$.

#!

Hence, $cod(f) \subseteq ran(f)$ which produces the result that $cod(f) = ran(f)$ and f is indeed surjective.[4]

Thus, $|\mathcal{P}(\mathbb{N})| \geq |[0, 1]|$ by Theorem 3.3.

We now consider the interval $[0, 1]$ in a different form. Note the interval consists of all reals that are expressible in trinimal form such that it is an array of digits of the form $0.t_1 t_2 t_3 t_4 \ldots t_{n-1} t_n t_{n+1} \ldots$ where $t_i \in \mathbb{N}_2^* \; \forall \; i \in \mathbb{N}$. We note that the expression is unique up to repeating twos.

Define $g : \mathcal{P}(\mathbb{N}) \longrightarrow [0, 1]$ where

$$g(A) = \begin{cases} 0 & A = \emptyset \\ 0.\overline{1} & A = \mathbb{N} \\ 0.t_1 t_2 t_3 t_4 \ldots t_{n-1} t_n t_{n+1} \ldots & A \subset \mathbb{N} \wedge \\ & t_i = 1 \ni i \in A; t_i = 0 \ni i \notin A \end{cases}$$

We note in trinimal expansion $0 = 0.\overline{0} = 0.t_1 t_2 t_3 t_4 \ldots t_{n-1} t_n t_{n+1} \ldots \ni b_i = 0 \forall \; i \in \mathbb{N}$; $1 = 0.\overline{2} = 0.t_1 t_2 t_3 t_4 \ldots t_{n-1} t_n t_{n+1} \ldots \ni b_i = 2 \forall \; i \in \mathbb{N}$; and, $x \in (0, 1) \Longrightarrow \exists j \in \mathbb{N} \ni t_j \neq 2 \vee \exists k \in \mathbb{N} \ni t_k \neq 0$.

Our argument for g is similar (almost identical) to our argument that f is a well defined function: g is a well defined function from $\mathcal{P}(\mathbb{N})$ to $[0, 1]$ since $\mathcal{P}(\mathbb{N}) \neq \emptyset$; $\forall A \in \mathcal{P}(\mathbb{N}) \; \exists y \in [0, 1] \; \ni \; (A, y) \in g$; and, $A \in \mathcal{P}(\mathbb{N})$, $j \in [0, 1]$, $k \in [0, 1]$ with $(A, j) \in g \; \wedge \; (A, k) \in g$ implies $j = k$.

Suppose g is not injective. So, $\exists B \in \mathcal{P}(\mathbb{N}), C \in \mathcal{P}(\mathbb{N}), m \in [0, 1], \ni (B, m) \in g, (C, m) \in g$ where $B \neq C$. Now, $m \neq 0$ since that forces $B = C$ and $m \neq 1$ since $1 \notin ran(g)$ so it is the case that $m \in (0, 1)$.

So, let $m = 0.m_1 m_2 m_3 m_4 \ldots m_{n-1} m_n m_{n+1} \ldots \ni m_i \in \mathbb{N}_2^* \, \forall \, i \in \mathbb{N}$ and we know $\exists j \in \mathbb{N} \ni m_j = 0$. Since $B \neq C \; \exists k \in \mathbb{N}$ such that $k \in B$ and $k \notin C$ or since $B \neq C \; \exists p \in \mathbb{N}$ such that $p \notin B$ and $p \in C$.

Case 1: $\exists k \in \mathbb{N}$ such that $k \in B$ and $k \notin C$.

[4] It is worth noting that f is not injective which follows directly from the non-uniqueness of binimal expressions because of repeating ones.

So, since $k \in B \Longrightarrow m_k = 1$ and $k \notin C \Longrightarrow m_k = 0$ by definition of g. But this means $(B, m) \in g, (C, m) \in g$ is false.

Case 2: $\exists p \in \mathbb{N}$ such that $p \in C$ and $p \notin B$ is argued precisely like case 1.
Hence, g is injective.[5]
Thus, $|\mathcal{P}(\mathbb{N})| \leq |[0,1]|$ by Theorem 3.2.
So, $|\mathcal{P}(\mathbb{N})| = |[0,1]|$ by Theorem 3.4. □

Corollary 4.1. *$\mathcal{P}(\mathbb{N})$ is equinumerous with $\mathbb{R}$.*

Proof. Assume the premises. Apply lemma 4.1; lemma 4.2; and, theorem 4.1. The result follows. □

5. Discussion

What made this argument so pleasant to construct was that we used similar functions from $\mathcal{P}(\mathbb{N})$ to $[0,1]$ to show $|\mathcal{P}(\mathbb{N})| \leq |[0,1]|$ and $|\mathcal{P}(\mathbb{N})| \geq |[0,1]|$. So, the proof hinges on the expressibility of reals in binimal or trinimal (binary or ternary); the non-uniqueness of the expression of a real; and, the real numbers which are the same in binimal are not the same in trinimal (or decimal - the repeating ones, twos, or nines nuance about real number expansions in binimal, trinimal, or decimal). But, perhaps, even more pleasant is the wonderful property that with $f : \mathcal{P}(\mathbb{N}) \longrightarrow [0,1]$ the repeating ones in the binimal expansion of the reals created the opening to argue the surjectivity of f but for $g : \mathcal{P}(\mathbb{N}) \longrightarrow [0,1]$ since the reals were expressed in trinimal having a place in the expansion where there was a zero and not a one forces the real numbers to be different![6] A solipistic reading of the proof might lead someone to err that f and g map the same subset of $\mathbb{N}$ to the same real number which is, of course, not the case! So, the quite a neat fact about the reals assists in the creation of a rather nifty proof.

In conclusion, it is hoped that this article has piqued the curiosity or imagination of some readers of the *International Journal of Pure and Applied Mathematics* and has provided an interesting, slightly different, and pulchrous argument which produced a familiar result.

[5] It is worth noting that g is not surjective follows directly from the definition g and the fact that $w = 0.\overline{02}$ we have $w \in cod(g)$ but $w \notin ran(g)$.

[6] Such was the pièce de résistance of the argument; e.g., notice $f(\emptyset) = 0$, $g(\emptyset) = 0$, $f(\mathbb{N}) = 1$, and $g(\mathbb{N}) = \frac{1}{3}$ as just one consequence of the definition of f and g.

References

[1] Walter Rudin, *Principles of Real Analysis, International Series in Pure and Applied Mathematics*, McGraw-Hill, New York (1964).