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WEAK AND STRONG CONVERGENCE OF FIXED POINTS OF DEMICONTRACTIVE MAPPINGS IN SMOOTH BANACH SPACES

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Abstract: In this paper, it is proved that the three-step iteration process with error terms in the sense of Xu associated with demicontractive mappings converge weakly to a fixed point of T. Also, if K is compact, then the convergence is strong.

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1. Introduction and Preliminaries

The symbols $D(T)$ and $F(T)$ stand for the domain and the set of fixed points of T (for a single-valued mapping $T : X \to X$, $x \in X$ is called a *fixed point* of

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T if $Tx = x$).

Let $T : D(T) \subset H \to H$ be a a mapping, where H is a Hilbert space.

Definition 1.1. (see [3], [8]) (1) A mapping T is said to be *strictly pseudocontractive* if there exists a constant $k \in [0, 1)$ such that, for all x, $y \in D(T)$,

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|x - y - (Tx - Ty)\|^2.$$

(2) If $F(T) \neq \emptyset$ and there exists a constant $k \in [0, 1)$ such that, for all $x \in D(T)$ and $x^* \in F(T)$,

$$\|Tx - x^*\|^2 \leq \|x - x^*\|^2 + k\|x - Tx\|^2, \tag{1.1}$$

then T is said to be *demicontractive.*

It is easy to see that a strictly pseudocontractive mapping with a fixed point is demicontractive. Thus the class of demicontractive mappings properly includes the class of strictly pseudocontractive mappings with fixed points. The important class of quasi-nonexpansive mappings (where a mapping T is said to be *quasi-nonexpansive* if $F(T) \neq \emptyset$ and

$$\|Tx - x^*\| \leq \|x - x^*\|$$

for all $x \in D(T)$ and $x^* \in F(T)$) is also a subclass of this class of demicontractive mappings.

Remark 1.2. If we set $\lambda = \frac{(1-k)}{2}$, then it is routine to see that, in Hilbert spaces, (1.1) is equivalent to the condition: there exists a constant $\lambda > 0$ such that, for all $x \in D(T)$ and $x^* \in F(T)$,

$$\langle x - Tx, x - x^* \rangle \geq \lambda\|x - Tx\|^2, \tag{1.2}$$

which is the condition introduced by Maruster [15]. Thus the class of nonlinear mappings introduced in 1977 by Hicks and Kubicek [8] and Maruster [15] independently coincide in Hilbert spaces. It is easy to observe from (1.1) that

$$\begin{aligned}\|Tx - x^*\| &\leq \|x - x^*\| + \sqrt{k}\|x - Tx\| \\ &\leq (1 + \sqrt{k})\|x - x^*\| + \sqrt{k}\|Tx - x^*\|,\end{aligned}$$

$$(1 - \sqrt{k})\|Tx - x^*\| \leq (1 + \sqrt{k})\|x - x^*\|,$$

$$\|Tx - x^*\| \leq \left(\frac{1 + \sqrt{k}}{1 - \sqrt{k}}\right)\|x - x^*\| = L\|x - x^*\|,$$

where

$$L = \frac{1 + \sqrt{k}}{1 - \sqrt{k}},$$

and from (1.2) that

$$\|x - x^*\| \geq \lambda \|x - Tx\| \geq \lambda(\|Tx - x^*\| - \|x - x^*\|)$$

and so

$$\|Tx - x^*\| \leq L\|x - x^*\|,$$

where $L = 1 + \lambda^{-1}$. Several authors have studied this class of nonlinear mappings (see, for example, ([1-24, 26-27]) and convergence theorems have been established for the iteration processes of the Mann-type (see, for example, [14]).

Let K be a nonempty subset of an arbitrary Banach space X and X^* be its dual space.

Definition 1.3. The *modulus of smoothness* of X is the function $\rho_X : [0, \infty) \to [0, \infty)$ defined by

$$\rho_X(\tau) := \frac{1}{2} \sup\{\|x + y\| + \|x - y\| - 2 : \|x\| \leq 1, \ \|y\| \leq \tau\}.$$

For any $q > 1$, X is called *q-uniformly smooth* if there exists a constant $c > 0$ such that $\rho_X(\tau) \leq c\tau^q$ and X is called *uniformly smooth* if $\lim_{\tau \to 0} \frac{\rho_X(\tau)}{\tau} = 0$.

Clearly, every q-uniformly smooth Banach space is uniformly smooth. Moreover, it is well known that Hilbert spaces are 2-uniformly smooth while

$$L_p(\text{or } \ell_p) \text{ or } W_m^p \text{ is } \begin{cases} p\text{-uniformly smooth} & \text{if } 1 < p \leq 2, \\ 2\text{-uniformly smooth} & \text{if } p \geq 2. \end{cases}$$

Definition 1.4. Let X^* denote the dual space of X and $J_p : X \to 2^{X^*}$ denote the *generalized duality mapping* defined by

$$J_p(x) := \{f^* \in X^* : \langle x, f^* \rangle = \|x\|^p, \ \|f^*\| = \|x\|^{p-1}\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing.

It is known that J_p is single-valued (denoted by j_p) and Lipschitz Hölder-continuous with constant $L_* > 0$ if X is p-uniformly smooth. That is,

$$\|j_p(x) - j_p(y)\| \leq L_* \|x - y\|^{p-1} \tag{1.3}$$

for all $x, y \in X$. Moreover, for all $x \in X$ with $x \neq 0$, $J_p(x) = \|x\|^{p-2} J_2(x)$, where $J_p = J_2$ is the normalized duality mapping.

Lemma 1.5. (see [25]) *If X is a real q-uniformly smooth Banach space, then there exists positive constant C_q such that, for all $x, y \in X$,*

$$\|x+y\|^q \leq \|x\|^q + q\langle y, j_q(x)\rangle + C_q\|y\|^q. \tag{1.4}$$

Remark 1.6. ([16]) In [4], Chidume extended the condition (1.2) to arbitrary real Banach spaces X. If X is q-uniformly smooth, then the condition (1.2) becomes

$$\begin{aligned}\langle x-Tx, j_q(x-x^*)\rangle &\geq \|x-x^*\|^{q-2}\langle x-Tx, j(x-x^*)\rangle \\ &\geq \|x-x^*\|^{q-2}\lambda\|x-Tx\|^2 \\ &\geq \lambda^{q-1}\|x-Tx\|^q,\end{aligned}$$

which implies that

$$\langle Tx-x^*, j_q(x-x^*)\rangle \leq \|x-x^*\|^q - \lambda^{q-1}\|x-Tx\|^q. \tag{1.5}$$

Lemma 1.7. (see [13], [22]) *Suppose that $\{\rho_n\}$ and $\{\sigma_n\}$ are two sequences of nonnegative numbers such that, for some real number $n_0 \geq 1$,*

$$\rho_{n+1} \leq \rho_n + \sigma_n$$

for all $n \geq n_0$.

(a) If $\sum_{n\geq 0}\sigma_n < \infty$, then $\lim_{n\to\infty}\rho_n$ exists;

(b) If $\sum_{n\geq 0}\sigma_n < \infty$ and $\{\rho_n\}$ has a subsequence converging to zero, then $\lim_{n\to\infty}\rho_n = 0$.

Definition 1.8. A mapping $T : K \to X$ is said to be *demiclosed* at a point $z \in X$ if the weak convergence of $\{x_n\}$ in K to some point $p \in K$ and the strong convergence of $\{Tx_n\}$ to z implies that $Tp = z$.

Definition 1.9. A mapping $T : K \to X$ is said to be *demicompact* at a point $z \in K$ if, for any bounded sequence $\{x_n\}$ in K such that $(I-T)x_n \to z$ as $n \to \infty$, then there exist a subsequence $\{x_{n_j}\}$ and a point $p \in K$ such that $x_{n_j} \to p$ as $j \to \infty$ and $(I-T)p = z$.

Let K be a nonempty closed convex subset of a real q-uniformly smooth Banach space and $T : K \to K$ be a demicontractive mapping with $(I-T)$ demiclosed at $0 \in K$. In this paper, it is proved that the three-step iteration process with error terms in the sense of Xu [26] associated with demicontractive mappings converge weakly to a fixed point of T. Also, if K is compact, then the convergence is strong.

2. Main Results

We now prove our main results.

Theorem 2.1. *Let K be a nonempty closed convex subset of a real q-uniformly smooth Banach space X and $T : K \to K$ be a demicontractive mapping such that $(I-T)$ is demiclosed at $0 \in K$. Let $\{v_n\}$, $\{u_n\}$ and $\{w_n\}$ be three bounded sequences in K. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$, $\{c'_n\}$, $\{a''_n\}$, $\{b''_n\}$ and $\{c''_n\}$ be the real sequences in $[0,1]$ satisfying*

(i) $a_n + b_n + c_n = a'_n + b'_n + c'_n = a''_n + b''_n + c''_n = 1$ *for all* $n \geq 0$;

(ii) $\sum_{n\geq 0} b_n = \infty$ *and* $\lim_{n\to\infty} b_n = 0$;

(iii) $\sum_{n\geq 0} {b'_n}^s < \infty$, $\sum_{n\geq 0} {b''_n}^s < \infty$, $\sum_{n\geq 0} c_n^s < \infty$, $\sum_{n\geq 0} {c'_n}^s < \infty$ *and* $\sum_{n\geq 0} {c''_n}^s < \infty$, *where* $s = \min\{1, q-1\}$.

Then the sequence $\{x_n\}$ generated from an arbitrary $x_0 \in K$ by

$$\begin{cases} x_{n+1} = a_n x_n + b_n T y_n + c_n v_n, \\ y_n = a'_n x_n + b'_n T z_n + c'_n u_n, \\ z_n = a''_n x_n + b''_n T x_n + c''_n w_n \end{cases} \tag{2.1}$$

for all $n \geq 0$ converges weakly to a fixed point of T.

Proof. Let $x^* \in K$ be a fixed point of T. Then, using (1.4), we have the following estimates:

$$\begin{aligned} &\|b''_n(Tx_n - x^*) + c''_n(w_n - x^*)\|^q \\ &\leq {b''_n}^q \|Tx_n - x^*\|^q + C_q {c''_n}^q \|w_n - x^*\|^q \\ &\quad + q {b''_n}^{q-1} c''_n \|w_n - x^*\| \|Tx_n - x^*\|^{q-1} \\ &\leq L^q {b''_n}^q \|x_n - x^*\|^q + C_q {c''_n}^q \|w_n - x^*\|^q \\ &\quad + q L^{q-1} {b''_n}^{q-1} c''_n \|w_n - x^*\| \|x_n - x^*\|^{q-1} \end{aligned}$$

and

$$\begin{aligned} &\|j_q(x_n - x^*) - j_q(z_n - x^*)\| \\ &\leq L_* \|x_n - z_n\|^{q-1} \\ &\leq L_* [[(1+L) b''_n + c''_n] \|x_n - x^*\| + c''_n \|w_n - x^*\|]^{q-1}. \end{aligned}$$

Using (1.3) and (1.4), we have

$$
\begin{aligned}
&\|z_n - x^*\|^q \\
&\le a_n''^q\|x_n - x^*\|^q + qa_n''^{q-1}b_n''\langle Tx_n - x^*, j_q(x_n - x^*)\rangle \\
&\quad + qa_n''^{q-1}c_n''\langle w_n - x^*, j_q(x_n - x^*)\rangle \\
&\quad + C_q\|b_n''(Tx_n - x^*) + c_n''(w_n - x^*)\|^q \\
&\le a_n''^q\|x_n - x^*\|^q + qa_n''^{q-1}b_n''(\|x_n - x^*\|^q - \lambda^{q-1}\|x_n - Tx_n\|^q) \\
&\quad + qa_n''^{q-1}c_n''\|w_n - x^*\|\|x_n - x^*\|^{q-1} + C_q[L^q b_n''^q\|x_n - x^*\|^q \\
&\quad + C_q c_n''^q\|w_n - x^*\|^q + qL^{q-1}b_n''^{q-1}c_n''\|w_n - x^*\|\|x_n - x^*\|^{q-1}] \\
&= (a_n''^q + qa_n''^{q-1}b_n'' + C_qL^q b_n''^q)\|x_n - x^*\|^q \\
&\quad - q\lambda^{q-1}a_n''^{q-1}b_n''\|x_n - Tx_n\|^q \\
&\quad + qc_n''(a_n''^{q-1} + C_qL^{q-1}b_n''^{q-1})\|w_n - x^*\|\|x_n - x^*\|^{q-1} \\
&\quad + C_q^2 c_n''^q\|w_n - x^*\|^q \\
&\le \|x_n - x^*\|^q - q\lambda^{q-1}a_n''^{q-1}b_n''\|x_n - Tx_n\|^q + C_q^2c_n''^q\|w_n - x^*\|^q \\
&\quad + qc_n''(a_n''^{q-1} + C_qL^{q-1}b_n''^{q-1})\|w_n - x^*\|\|x_n - x^*\|^{q-1}.
\end{aligned}
$$

The last inequality [16] follows from the fact that, for all $x, y \in [0, 1]$,

$$
\begin{aligned}
f(x, y) &= (1 - x - y)^q + q(1 - x - y)^{q-1}x + C_qL^qx^q \\
&\le (1 - x)^q + q(1 - x)^{q-1}x + C_qL^qx^q \\
&\le 1.
\end{aligned}
$$

Indeed, $g(x) = (1 - x)^q + q(1 - x)^{q-1}x + C_qL^qx^q$ is monotone increasing on $[0, 1]$. Therefore, $g(x) \le g(1) = C_qL_q$. Unfortunately, C_qL_q need not be less than or equal to 1. However, since $\sum_{n\ge0} b_n'' < \infty$, $\lim_{n\to\infty} b_n'' = 0$ and so, for all n sufficiently large, $g(b_n'') < 1$. Observe that

$$
\begin{aligned}
\|z_n - x^*\| &\le \|z_n - x_n\| + \|x_n - x^*\| \\
&\le [(1 + L)b_n'' + c_n'' + 1]\|x_n - x^*\| + c_n''\|w_n - x^*\|,
\end{aligned}
$$

$$
\begin{aligned}
&\|b_n'(Tz_n - x^*) + c_n'(u_n - x^*)\|^q \\
&\le b_n'^q\|Tz_n - x^*\|^q + C_qc_n'^q\|u_n - x^*\|^q \\
&\quad + qb_n'^{q-1}c_n'\|u_n - x^*\|\|Tz_n - x^*\|^{q-1} \\
&\le L^qb_n'^q\|z_n - x^*\|^q + C_qc_n'^q\|u_n - x^*\|^q \\
&\quad + qL^{q-1}b_n'^{q-1}c_n'\|u_n - x^*\|\|z_n - x^*\|^{q-1}
\end{aligned}
$$

and

$$\begin{aligned}
\|y_n - x^*\|^q &\le a_n'^q\|x_n - x^*\|^q + C_q\|b_n'(Tz_n - x^*) + c_n'(u_n - x^*)\|^q \\
&\quad + qa_n'^{q-1}b_n'\langle Tz_n - x^*, j_q(x_n - x^*)\rangle + qa_n'^{q-1}c_n'\langle u_n - x^*, j_q(x_n - x^*)\rangle \\
&\le a_n'^q\|x_n - x^*\|^q + C_q[L^q b_n'^q\|z_n - x^*\|^q + C_q c_n'^q\|u_n - x^*\|^q \\
&\quad + qL^{q-1}b_n'^{q-1}c_n'\|u_n - x^*\|\|z_n - x^*\|^{q-1}] \\
&\quad + qa_n'^{q-1}b_n'\langle Tz_n - x^*, j_q(z_n - x^*)\rangle \\
&\quad + qa_n'^{q-1}b_n'\langle Tz_n - x^*, j_q(x_n - x^*) - j_q(z_n - x^*)\rangle \\
&\quad + qa_n'^{q-1}c_n'\|u_n - x^*\|\|x_n - x^*\|^{q-1} \\
&\le a_n'^q\|x_n - x^*\|^q + qa_n'^{q-1}b_n'(\|z_n - x^*\|^q - \lambda^{q-1}\|z_n - Tz_n\|^q) \\
&\quad + qa_n'^{q-1}b_n'\|Tz_n - x^*\|\|j_q(x_n - x^*) - j_q(z_n - x^*)\| \\
&\quad + qa_n'^{q-1}c_n'\|u_n - x^*\|\|x_n - x^*\|^{q-1} + C_qL^q b_n'^q\|z_n - x^*\|^q \\
&\quad + C_q^2 c_n'^q\|u_n - x^*\|^q + qC_qL^{q-1}b_n'^{q-1}c_n'\|u_n - x^*\|\|z_n - x^*\|^{q-1} \\
&\le a_n'^q\|x_n - x^*\|^q + qa_n'^{q-1}b_n'\|z_n - x^*\|^q - q\lambda^{q-1}a_n'^{q-1}b_n'\|z_n - Tz_n\|^q \\
&\quad + qLL_*a_n'^{q-1}b_n'[[(1+L)b_n'' + c_n'' + 1]\|x_n - x^*\| + c_n''\|w_n - x^*\|] \\
&\qquad \times [[(1+L)b_n'' + c_n'']\|x_n - x^*\| + c_n''\|w_n - x^*\|]^{q-1} \\
&\quad + qa_n'^{q-1}c_n'\|u_n - x^*\|\|x_n - x^*\|^{q-1} + C_qL^q b_n'^q\|z_n - x^*\|^q \\
&\quad + C_q^2 c_n'^q\|u_n - x^*\|^q + qC_qL^{q-1}b_n'^{q-1}c_n'\|u_n - x^*\|\|z_n - x^*\|^{q-1} \\
&= (a_n'^q + qa_n'^{q-1}b_n' + C_qL^q b_n'^q)\|x_n - x^*\|^q - q\lambda^{q-1}a_n'^{q-1}b_n'\|z_n - Tz_n\|^q \\
&\quad - q\lambda^{q-1}a_n''^{q-1}(qa_n'^{q-1} + C_qL^q b_n'^{q-1})b_n'b_n''\|x_n - Tx_n\|^q \\
&\quad + q(a_n''^{q-1} + C_qL^{q-1}b_n''^{q-1})(qa_n'^{q-1} + C_qL^q b_n'^{q-1})b_n'c_n''\|w_n - x^*\| \\
&\qquad \times \|z_n - x^*\|^{q-1} + C_q^2(qa_n'^{q-1} + C_qL^q b_n'^{q-1})b_n'c_n''^q\|w_n - x^*\|^q \\
&\quad + C_q^2 c_n'^q\|u_n - x^*\|^q + qLL_*a_n'^{q-1}b_n'[[(1+L)b_n'' + c_n'' + 1]\|x_n - x^*\| \\
&\quad + c_n''\|w_n - x^*\|][[(1+L)b_n'' + c_n'']\|x_n - x^*\| + c_n''\|w_n - x^*\|]^{q-1} \\
&\quad + qC_qL^{q-1}b_n'^{q-1}c_n'\|u_n - x^*\|[[(1+L)b_n'' + c_n'' + 1]\|x_n - x^*\| \\
&\quad + c_n''\|w_n - x^*\|]^{q-1} + qa_n'^{q-1}c_n'\|u_n - x^*\|\|x_n - x^*\|^{q-1} \\
&\le \|x_n - x^*\|^q - q\lambda^{q-1}a_n'^{q-1}b_n'\|z_n - Tz_n\|^q \\
&\quad - q\lambda^{q-1}a_n''^{q-1}(qa_n'^{q-1} + C_qL^q b_n'^{q-1})b_n'b_n''\|x_n - Tx_n\|^q \\
&\quad + q(a_n''^{q-1} + C_qL^{q-1}b_n''^{q-1})(qa_n'^{q-1} + C_qL^q b_n'^{q-1})b_n'c_n''\|w_n - x^*\| \\
&\qquad \times \|z_n - x^*\|^{q-1} + C_q^2(qa_n'^{q-1} + C_qL^q b_n'^{q-1})b_n'c_n''^q\|w_n - x^*\|^q \\
&\quad + C_q^2 c_n'^q\|u_n - x^*\|^q + qLL_*a_n'^{q-1}b_n'[[(1+L)b_n'' + c_n'' + 1]\|x_n - x^*\|
\end{aligned}$$

$$+ c_n''\|w_n - x^*\|]][[(1+L)b_n'' + c_n'']\|x_n - x^*\| + c_n''\|w_n - x^*\|]^{q-1}$$
$$+ qC_qL^{q-1}b_n'^{q-1}c_n'\|u_n - x^*\|[[(1+L)b_n'' + c_n'' + 1]\|x_n - x^*\|$$
$$+ c_n''\|w_n - x^*\|]^{q-1} + qa_n'^{q-1}c_n'\|u_n - x^*\|\|x_n - x^*\|^{q-1}.$$

Now, also we have

$$\begin{aligned}\|b_n(Ty_n - x^*) + c_n(v_n - x^*)\|^q &\le b_n^q\|Ty_n - x^*\|^q + C_qc_n^q\|v_n - x^*\|^q \\ &\quad + qb_n^{q-1}c_n\|v_n - x^*\|\|Ty_n - x^*\|^{q-1} \\ &\le L_qb_n^q\|y_n - x^*\|^q + C_qc_n^q\|v_n - x^*\|^q \\ &\quad + qL^{q-1}b_n^{q-1}c_n\|v_n - x^*\|\|y_n - x^*\|^{q-1},\end{aligned}$$

$$\begin{aligned}\|j_q(x_n - x^*) - j_q(y_n - x^*)\| &\le L_*\|x_n - y_n\|^{q-1} \\ &\le L_*[[[1 + L[(1+L)b_n'' + c_n'' + 1]]b_n' + c_n']\|x_n - x^*\| \\ &\quad + Lb_n'c_n''\|w_n - x^*\| + c_n'\|u_n - x^*\|]^{q-1},\end{aligned}$$

$$\begin{aligned}\|y_n - x^*\| &\le \|y_n - x_n\| + \|x_n - x^*\| \\ &\le [[1 + L[(1+L)b_n'' + c_n'' + 1]]b_n' + c_n' + 1]\|x_n - x^*\| \\ &\quad + Lb_n'c_n''\|w_n - x^*\| + c_n'\|u_n - x^*\|,\end{aligned}$$

$$\begin{aligned}&\|x_{n+1} - x^*\|^q \\ &\le a_n^q\|x_n - x^*\|^q + C_q\|b_n(Ty_n - x^*) + c_n(v_n - x^*)\|^q \\ &\quad + qa_n^{q-1}b_n\langle Ty_n - x^*, j_q(x_n - x^*)\rangle + qa_n^{q-1}c_n\langle v_n - x^*, j_q(x_n - x^*)\rangle \\ &\le a_n^q\|x_n - x^*\|^q + C_q[b_n^q\|Ty_n - x^*\|^q + C_qc_n^q\|v_n - x^*\|^q \\ &\quad + qb_n^{q-1}c_n\|v_n - x^*\|\|Ty_n - x^*\|^{q-1}] \\ &\quad + qa_n^{q-1}b_n\langle Ty_n - x^*, j_q(y_n - x^*)\rangle \\ &\quad + qa_n^{q-1}b_n\langle Ty_n - x^*, j_q(x_n - x^*) - j_q(y_n - x^*)\rangle \\ &\quad + qa_n^{q-1}c_n\|v_n - x^*\|\|x_n - x^*\|^{q-1} \\ &\le a_n^q\|x_n - x^*\|^q + qa_n^{q-1}b_n(\|y_n - x^*\|^q - \lambda^{q-1}\|y_n - Ty_n\|^q) \\ &\quad + qLa_n^{q-1}b_n\|y_n - x^*\|\|j_q(x_n - x^*) - j_q(y_n - x^*)\| \\ &\quad + qa_n^{q-1}c_n\|v_n - x^*\|\|x_n - x^*\|^{q-1} + C_qL^qb_n^q\|y_n - x^*\|^q \\ &\quad + C_q^2c_n^q\|v_n - x^*\|^q + qC_qL^{q-1}b_n^{q-1}c_n\|v_n - x^*\|\|y_n - x^*\|^{q-1}\end{aligned}$$

$$
\begin{aligned}
&\leq (a_n^q + q a_n^{q-1} b_n + C_q L^q b_n^q)\|x_n - x^*\|^q \\
&\quad - q\lambda^{q-1} a_n'^{\,q-1}(q a_n^{q-1} + C_q L^q b_n^{q-1}) b_n b_n' \|z_n - T z_n\|^q \\
&\quad - q\lambda^{q-1} a_n''^{\,q-1}(q a_n'^{\,q-1} + C_q L^q b_n'^{\,q-1})(q a_n^{q-1} + C_q L^q b_n^{q-1}) \\
&\qquad \times b_n b_n' b_n'' \|x_n - T x_n\|^q - q\lambda^{q-1} a_n^{q-1} b_n \|y_n - T y_n\|^q \\
&\quad + q(a_n''^{\,q-1} + C_q L^{q-1} b_n''^{\,q-1})(q a_n'^{\,q-1} + C_q L^q b_n'^{\,q-1}) \\
&\qquad \times (q a_n^{q-1} + C_q L^q b_n^{q-1}) b_n b_n' c_n'' \|w_n - x^*\| \|x_n - x^*\|^{q-1} \\
&\quad + C_q^2 (q a_n'^{\,q-1} + C_q L^q b_n'^{\,q-1})(q a_n^{q-1} + C_q L^q b_n^{q-1}) b_n b_n' c_n''^{\,q} \|w_n - x^*\|^q \\
&\quad + C_q^2 (q a_n^{q-1} + C_q L^q b_n^{q-1}) b_n c_n'^{\,q} \|u_n - x^*\|^q \\
&\quad + q(q a_n^{q-1} + C_q L^q b_n^{q-1}) a_n'^{\,q-1} b_n c_n' \|u_n - x^*\| \|x_n - x^*\|^{q-1} \\
&\quad + q a_n^{q-1} c_n \|v_n - x^*\| . \|x_n - x^*\|^{q-1} + C_q^2 c_n^q \|v_n - x^*\|^q \\
&\quad + q L L_* (q a_n^{q-1} + C_q L^q b_n^{q-1}) a_n'^{\,q-1} b_n b_n' ([(L+1) b_n'' + c_n''] \|x_n - x^*\| \\
&\quad + c_n'' \|w_n - x^*\|)^{q-1} ([(L+1) b_n'' + c_n'' + 1] \|x_n - x^*\| + c_n'' \|w_n - x^*\|) \\
&\quad + q C_q L^{q-1} (q a_n^{q-1} + C_q L^q b_n^{q-1}) b_n'^{\,q-1} b_n c_n' \|u_n - x^*\| \\
&\qquad \times ([(L+1) b_n'' + c_n'' + 1] \|x_n - x^*\| + c_n'' \|w_n - x^*\|)^{q-1} \\
&\quad + q L L_* a_n^{q-1} b_n ([1 + L[(L+1) b_n'' + c_n'' + 1] b_n' + c_n' + 1] \|x_n - x^*\| \\
&\quad + L b_n' c_n'' \|w_n - x^*\| + c_n' \|u_n - x^*\|) \\
&\qquad \times ([1 + L[(L+1) b_n'' + c_n'' + 1] b_n' + c_n'] \|x_n - x^*\| + L b_n' c_n'' \|w_n - x^*\| \\
&\quad + c_n' \|u_n - x^*\|)^{q-1} + q C_q L^{q-1} b_n^{q-1} c_n \|v_n - x^*\| \\
&\qquad \times ([1 + L[(L+1) b_n'' + c_n'' + 1] b_n' + c_n' + 1] \|x_n - x^*\| \\
&\quad + L b_n' c_n'' \|w_n - x^*\| + c_n' \|u_n - x^*\|)^{q-1}.
\end{aligned}
$$

Let

$$M = \max\Big\{ \sup_{n\geq 0}\{\|u_n - x^*\|\}, \sup_{n\geq 0}\{\|v_n - x^*\|\}, \sup_{n\geq 0}\{\|w_n - x^*\|\} \Big\}.$$

Now, we consider the following two cases:

Case 1: For all $n \geq 0$, $\|x_n - x^*\| \geq M$. Then we have

$$\|x_{n+1} - x^*\|^q \leq (1 + \gamma_n)\|x_n - x^*\|^q,$$

where

$$\gamma_n = f(b_n'^{\,s}, b_n''^{\,s}, c_n^s, c_n'^{\,s}, c_n''^{\,s}), \quad \sum_{n=1}^{\infty} \gamma_n < \infty.$$

Case 2: For all $n \geq 0$, $\|x_n - x^*\| \leq M$. Then we know that, for a constant $M_1 \geq M^q$,

$$\|x_{n+1} - x^*\|^q \leq \|x_n - x^*\|^q + M_1 \gamma_n.$$

Thus, according to Lemma 1.7, $\lim_{n\to\infty}\|x_n - x^*\|$ exists and so the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, $\{Tx_n\}$, $\{Ty_n\}$ and $\{Tz_n\}$ are bounded. Furthermore, for a positive constant M_0, we have

$$q\lambda^{q-1}a_n^{q-1}b_n\|y_n - Ty_n\|^q \leq \|x_n - x^*\|^q - \|x_{n+1} - x^*\|^q + M_0\gamma_n.$$

Thus, iterating downwards, we have

$$q\lambda^{q-1}\sum_{n\geq 0} a_n^{q-1}b_n\|y_n - Ty_n\|^q \leq \|x_0 - x^*\|^q + M_0\sum_{n\geq 0}\gamma_n < \infty$$

and so, from the hypothesis,

$$\liminf_{n\to\infty}\|y_n - Ty_n\| = 0$$

and so there exists a subsequence $\{y_{n_j}\} \subset \{y_n\}$ such that

$$y_{n_j} \rightharpoonup p, \quad y_{n_j} - Ty_{n_j} \to 0$$

as $j \to \infty$. Since $(I - T)$ is demiclosed at $0 \in K$ and K is weakly closed, it follows that $p \in F(T)$. Thus, for any $f^* \in X^*$, we have

$$f^*(y_{n_j} - p) \to 0$$

as $j \to \infty$. Now, we consider

$$\begin{aligned} f^*(x_{n_j} - p) &= f^*(x_{n_j} - y_{n_j}) + f^*(y_{n_j} - p) \\ &= b'_{n_j}f^*(x_{n_j} - Tz_{n_j}) + c'_{n_j}f^*(x_{n_j} - u_{n_j}) + f^*(y_{n_j} - p) \end{aligned}$$

and so

$$\begin{aligned} \lim_{j\to\infty}|f^*(x_{n_j} - p)| &\leq \lim_{j\to\infty}[b'_{n_j}|f^*(x_{n_j} - Tz_{n_j})| + c'_{n_j}|f^*(x_{n_j} - u_{n_j})| \\ &\quad + |f^*(y_{n_j} - p)|] \\ &= 0. \end{aligned}$$

Thus we have $x_{n_j} \rightharpoonup p$ as $j \to \infty$. Observe that, for a constant $M_0 > 0$,

$$\begin{aligned} \|x_n - Ty_n\| &\leq \|x_n - y_n\| + \|y_n - Ty_n\| \\ &\leq \|y_n - Ty_n\| + b'_n\|x_n - Tz_n\| + c'_n\|x_n - u_n\| \\ &\leq \|y_n - Ty_n\| + M_0(b'_n + c'_n). \end{aligned}$$

Hence we have

$$\lim_{j\to\infty} \|x_{n_j} - Ty_{n_j}\| = 0.$$

We now claim that, for all $k \geq 0$,

$$f^*(x_{n_j+k} - p) \to 0$$

as $j \to \infty$. Suppose that the claim is true for some $k = m$. Then, from

$$\begin{aligned} f^*(y_{n_j+m} - p) = b'_{n_j+m} f^*(Tz_{n_j+m} - x_{n_j+m}) \\ + c'_{n_j+m} f^*(u_{n_j+m} - x_{n_j+m}) + f^*(x_{n_j+m} - p), \end{aligned}$$

we see that

$$y_{n_j+m} \rightharpoonup p$$

as $j \to \infty$ and so, additionally, from

$$\begin{aligned} f^*(x_{n_j+m+1} - p) = b_{n_j+m} f^*(Ty_{n_j+m} - x_{n_j+m}) \\ + c_{n_j+m} f^*(v_{n_j+m} - x_{n_j+m}) + f^*(x_{n_j+m} - p), \end{aligned}$$

it follows that

$$\begin{aligned} &|f^*(x_{n_j+m+1} - p)| \\ &\leq Mc_{n_j+m} + |f^*(x_{n_j+m} - p)| + \|f^*\| \|x_{n_j+m} - Ty_{n_j+m}\| \\ &\to 0 \end{aligned}$$

as $j \to \infty$. Since the claim is trivially true for $k = 0$, it follows from the inductive hypothesis that the claim holds for all $k > 0$. Hence $\{x_n\}$ converges weakly to a fixed point of T. This completes the proof. □

Theorem 2.2. *Suppose that all the conditions in Theorem 2.1 hold. If K is a compact subset of X, then the sequence $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. Proceeding as in the proof of Theorem 2.1, we can show that

$$\liminf_{n\to\infty} \|y_n - Ty_n\| = 0.$$

This immediately implies that a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ converges strongly to a fixed point of T, say p. From the fact that, for a constant $M_0 \geq 0$,

$$\|x_n - p\| \leq \|x_n - y_n\| + \|y_n - p\|$$

$$\leq b'_n\|x_n - Tz_n\| + c'_n\|x_n - u_n\| + \|y_n - p\|$$
$$\leq M_0(b'_n + c'_n) + \|y_n - p\|,$$

it follows that $\{x_{n_j}\}$ also converges strongly to p. This implies that

$$\liminf_{n\to\infty} \|x_n - p\| = 0.$$

Observe that, from the hypotheses, $\sigma_n = M_0\gamma_n$ is summable and, putting $\Phi_n = \|x_n - p\|^q$, we have

$$\Phi_{n+1} \leq \Phi_n + \sigma_n.$$

Thus, following the approach of [16], for any $\varepsilon > 0$, there exists an integer j_0 sufficiently large such that

$$\Phi_{n_j} \leq \frac{\varepsilon}{4}$$

for all $j \geq j_0$ and there exists another integer n_1 sufficiently large such that

$$\sum_{n\geq n_1} \sigma_n \leq \frac{\varepsilon}{4}$$

since the tail of a summable series is arbitrarily small. Choose j_* sufficiently large such that

$$n_{j_*} \geq \max\{n_{j_0}, N_1\}.$$

Then, for any $k \geq 0$, we have

$$\Phi_{n_{j_*}+k+1} \leq \Phi_{n_{j_*}} + \sum_{r=0}^{k+1} \sigma_{n_{j_*}+r} \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}.$$

Since $\varepsilon > 0$ is arbitrary, it follows that $x_n \to x^*$ strongly as $n \to \infty$. This completes the proof. □

Corollary 2.3. *Let X, K, T, $\{v_n\}$ and $\{u_n\}$ be as in Theorem 2.1. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$ and $\{c'_n\}$ be the real sequences in $[0, 1]$ satisfying*

(i) $a_n + b_n + c_n = a'_n + b'_n + c'_n = 1$ *for all* $n \geq 0$;

(ii) $\sum_{n\geq 0} b_n = \infty$ *and* $\lim_{n\to\infty} b_n = 0$;

(iii) $\sum_{n\geq 0} b'^s_n < \infty$, $\sum_{n\geq 0} c_n < \infty$ *and* $\sum_{n\geq 0} c'^s_n < \infty$, *where* $s = \min\{1, q-1\}$.

Then the sequence $\{x_n\}$ generated from an arbitrary $x_0 \in K$ by

$$\begin{cases} x_{n+1} = a_n x_n + b_n T y_n + c_n v_n, \\ y_n = a'_n x_n + b'_n T x_n + c'_n u_n \end{cases} \tag{2.2}$$

for all $n \geq 0$ converges weakly to a fixed point of T.

Corollary 2.4. *Suppose that all the conditions in Corollary 2.3 hold. If K is a compact subset of X, then the sequence $\{x_n\}$ converges strongly to a fixed point of T.*

Corollary 2.5. *Let X, K, T and $\{v_n\}$ be as in Theorem 2.1. Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be real sequences in $[0,1]$ satisfying*
(i) $a_n + b_n + c_n = 1$ *for all* $n \geq 0$;
(ii) $\sum_{n\geq 0} b_n = \infty$ *and* $\lim_{n\to\infty} b_n = 0$;
(iii) $\sum_{n\geq 0} c_n < \infty$.
Then the sequence $\{x_n\}$ generated from an arbitrary $x_0 \in K$ by

$$x_{n+1} = a_n x_n + b_n T x_n + c_n v_n \tag{2.3}$$

for all $n \geq 0$ converges weakly to a fixed point of T.

Corollary 2.6. *Suppose that all the conditions in Corollary 2.5 hold. If K is a compact subset of X, then the sequence $\{x_n\}$ converges strongly to a fixed point of T.*

Corollary 2.7. *Let X, K, $\{v_n\}$, $\{u_n\}$ and $\{w_n\}$ be as in Theorem 2.1 and $T : K \to K$ is a continuous mapping at a cluster point p of the sequence $\{x_n\}$ defined by (2.1). Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$, $\{c'_n\}$, $\{a''_n\}$, $\{b''_n\}$ and $\{c''_n\}$ be the real sequences in $[0,1]$ satisfying*
(i) $a_n + b_n + c_n = a'_n + b'_n + c'_n = a''_n + b''_n + c''_n = 1$ *for all* $n \geq 0$;
(ii) $0 < \alpha \leq b_n < 1$ *for all* $n \geq 0$;
(iii) $\sum_{n\geq 0} b'^{\,s}_n < \infty$, $\sum_{n\geq 0} b''^{\,s}_n < \infty$, $\sum_{n\geq 0} c_n < \infty$, $\sum_{n\geq 0} c'^{\,s}_n < \infty$ *and* $\sum_{n\geq 0} c''^{\,s}_n < \infty$, *where* $s = \min\{1, q-1\}$.
Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof. Let $\delta_n = q\lambda^{q-1} a_n^{q-1} b_n$. Then, for a positive constant d, we have $\delta_n \geq d > 0$ and so, from

$$q\lambda^{q-1} a_n^{q-1} b_n \|y_n - Ty_n\|^q \leq \|x_n - x^*\|^q - \|x_{n+1} - x^*\|^q + M_1 \gamma_n,$$

and, consequently,

$$d \sum_{n\geq 0} \|y_n - Ty_n\|^q \leq \|x_0 - x^*\|^q + M_1 \sum_{n\geq 0} \gamma_n < \infty,$$

we conclude that

$$\lim_{n\to\infty} \|y_n - Ty_n\| = 0$$

and also

$$\lim_{n\to\infty} \|x_n - Ty_n\| = 0.$$

Since p is a cluster point of the sequence $\{x_n\}$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges strongly to p. Then we have

$$\begin{aligned}\|y_{n_j} - p\| &\le a'_{n_j}\|x_{n_j} - p\| + b'_{n_j}\|Tx_{n_j} - p\| + c'_{n_j}\|u_{n_j} - p\| \\ &\to 0\end{aligned}$$

as $j \to \infty$. Hence we have $y_{n_j} \to p$ as $j \to \infty$ and so $Ty_{n_j} \to Tp$ as $j \to \infty$. Since

$$\lim_{j\to\infty} \|y_{n_j} - Ty_{n_j}\| = \|p - Tp\| = 0,$$

it follows that $p \in F(T)$. As in the proof of Theorem 2.1, we obtain

$$\|x_{n+1} - p\|^q \le \|x_n - p\|^q + M_1\gamma_n.$$

Since $M_1 \sum_{n\ge 0} \gamma_n < \infty$, it follows from Lemma 1.7 that $\lim_{n\to\infty} \|x_n - p\|$ exists. Therefore, since $\lim_{j\to\infty} \|x_{n_j} - p\| = 0$, we have $\lim_{n\to\infty} \|x_n - p\| = 0$. This completes the proof. □

Corollary 2.8. *Let X, K, $\{v_n\}$ and $\{u_n\}$ be as in Theorem 2.1 and $T : K \to K$ is a continuous mapping at a cluster point p of the sequence $\{x_n\}$ defined by* (2.2)*. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$ and $\{c'_n\}$ be the real sequences in* $[0, 1]$ *satisfying*

(i) $a_n + b_n + c_n = a'_n + b'_n + c'_n = 1$ *for all* $n \ge 0$;

(ii) $0 < \alpha \le b_n < 1$ *for all* $n \ge 0$;

(iii) $\sum_{n\ge 0} b'^s_n < \infty$, $\sum_{n\ge 0} c_n < \infty$ *and* $\sum_{n\ge 0} c'^s_n < \infty$, *where* $s = \min\{1, q-1\}$.

Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Corollary 2.9. *Let X, K and $\{v_n\}$ be as in Theorem 2.1 and $T : K \to K$ is a continuous mapping at a cluster point p of the sequence $\{x_n\}$ defined by* (2.3)*. Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be the real sequences in* $[0, 1]$ *satisfying*

(i) $a_n + b_n + c_n = 1$ *for all* $n \ge 0$;

(ii) $0 < \alpha \le b_n < 1$ *for all* $n \ge 0$;

(iii) $\sum_{n\ge 0} c_n < \infty$.

Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Corollary 2.10. *In Corollary 2.7, if a mapping $T : K \to K$ is a demicontractive mapping which is demicompact at $0 \in K$, then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. Since $\{y_n\}$ is bounded and the sequence $\{y_n - Ty_n\}$ converges strongly to 0, then, by the demicompactness of T, there exists a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ which converges strongly to a point $p \in F(T)$. From

$$\begin{aligned}\|x_{n_j} - p\| &\leq \|x_{n_j} - y_{n_j}\| + \|y_{n_j} - p\| \\ &\leq \|y_{n_j} - p\| + b'_{n_j}\|x_{n_j} - Tx_{n_j}\| + c'_{n_j}\|x_{n_j} - u_{n_j}\| \\ &\leq \|y_{n_j} - p\| + M_*(b'_{n_j} + c'_{n_j}) \\ &\to 0\end{aligned}$$

as $j \to \infty$, it follows that $\{x_{n_j}\}$ converges strongly to p. The rest now follows as in the proof of Corollary 2.7. This completes the proof. □

Corollary 2.11. *In Corollary 2.8, if $T : K \to K$ is a demicontractive mapping which is demicompact at $0 \in K$, then the sequence $\{x_n\}$ converges strongly to a fixed point of T.*

Corollary 2.12. *In Corollary 2.9, if $T : K \to K$ is a demicontractive mapping which is demicompact at $0 \in K$, then the sequence $\{x_n\}$ converges strongly to a fixed point of T.*

Remark 2.13. ([16]) (1) By setting $c_n = c'_n = c''_n = 0$ in our results, we can show that the usual Mann and Ishikawa iteration methods converge (strongly and weakly, according to the conditions) to a fixed point of T.

(2) Our results do not depend on Opial's condition.

(3) Our results hold, in particular, in the Lebesgue L_p $(p \neq 2)$ spaces.

(4) Suppose that $A : X \to X$ is a continuous linear operator, with zero as an eigenvalue, satisfying the condition that, for all $z \in D(A)$ and a positive constant λ,

$$\langle Az, j(z)\rangle \geq \lambda\|Az\|^2.$$

Suppose, further, that $f \in R(A)$. Then we may apply our results to prove that the three-step, Ishikawa and Mann iteration methods with errors converge strongly to a solution of the equation $Ax = f$.

Example 2.14. ([2]) Let $T : [0, 3] \to [0, 3]$ be defined by

$$Tx = \begin{cases} 2x - 4 & \text{if } x \in [2, 3], \\ 0 & \text{if } x \in [0, 2]. \end{cases}$$

Then T is demicontractive and Lipschitzian but it is not even a pseudocontractive mapping.

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