CHAINABILITY IN TOPOLOGICAL SPACES THROUGH CONTINUOUS FUNCTIONS

Vijeta Iyer¹,², Kiran Shrivastava², Priya Choudhary³
1,2,3Department of Mathematics
S.N.G.G.P.G. College
Shivaji Nagar, Bhopal, 462016, INDIA

Abstract: In this paper concept of ε−chainability in topological spaces has been introduced using continuity. Several well known results prove perquisites to results established in the paper. In case of metric spaces the equivalence of ε−chainability and function−ε−chainability is also established in the paper.

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Key Words: ε-chain, uniform−ε-chainability, uniform chainability, ε-chain preserving map, chain preserving map, strongly −ε-chain preserving map, strong chain preserving map.

* For the definitions of ε-chain, uniform-ε-chainability, uniform chainability refer [1].

Throughout this paper [0,∞) has usual metric topology and ε is positive real number unless mentioned otherwise.

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Definition 1. (see [2]) A map $f$ from a metric space $X$ into another metric space $Y$ is said to be an \(\varepsilon\) – chain preserving map if and only if for every \(\varepsilon\) – chain between any two points in $X$ there is an \(\varepsilon\) – chain between images of these two points under $f$ in $Y$ where $\varepsilon > 0$.

Definition 2. (see [2]) A map $f$ from a metric space $X$ into another metric space $Y$ is said to be a strongly \(\varepsilon\) – chain preserving map if and only if for every \(\varepsilon\) – chain between any two points $x_0, x_1, \ldots, x_n$ of $X$, $f(x_0), f(x_1), \ldots, f(x_n)$ is an \(\varepsilon\) – chain between $f(x_0)$ and $f(x_n)$.

Definition 3. (see [2]) A map $f$ from a metric space $X$ into another metric space $Y$ is said to be chain preserving if and only if $f$ is \(\varepsilon\)–chain preserving for every $\varepsilon > 0$.

Definition 4. (see [2]) A map $f$ from a metric space $X$ into another metric space $Y$ is said to be strongly chain preserving if and only if $f$ is strongly $\varepsilon$-chain preserving for every $\varepsilon > 0$.

Definition 5. A topological space $(X, \tau)$ is said to be function – $f$ – \(\varepsilon\) – chainable if for $\varepsilon > 0$ there exists a non-constant continuous function $f : X \rightarrow [0, \infty)$ such that for every pair of elements $x, y$ of $X$, there is a sequence $x = x_0, x_1, \ldots, x_n = y$ of elements in $X$ with

\[ |f(x_i) - f(x_{i-1})| < \varepsilon; \quad 1 \leq i \leq n. \]

Definition 6. Let $(X, \tau)$ be a topological space and let there exist a non-constant continuous function $f : X \rightarrow [0, \infty)$ such that $X$ is function – $f$ – \(\varepsilon\) - chainable for every $\varepsilon > 0$. Then $X$ is said to be function – $f$ – chainable.

Definition 7. Let $(X, \tau)$ be a topological space and $A \subset X$. Then $A$ is said to be function – $f$ – \(\varepsilon\) – chainable if $(A, \tau_A)$ is function – $f$ – \(\varepsilon\)– chainable subspace of $X$.

Definition 8. Let $(X, \tau)$ be a topological space and $A \subset X$. Let there exist a non-constant continuous function $f : A \rightarrow [0, \infty)$ such that $(A, \tau_A)$ is function – $f$ – \(\varepsilon\)-chainable for every $\varepsilon > 0$. Then $A$ is said to be function – $f$ – chainable.

Definition 9. A topological space $(X, \tau)$ is said to be uniformly function – $f$ – \(\varepsilon\) – chainable if for $\varepsilon > 0$ there exists a positive integer $l_\varepsilon(f)$ and a non-constant continuous function $f : X \rightarrow [0, \infty)$ such that for every pair of elements $x, y$ of $X$ there is a sequence $x = x_0, x_1, \ldots, x_n = y$ in $X$ with $n \leq l_\varepsilon(f)$ and $|f(x_i) - f(x_{i-1})| < \varepsilon; \quad 1 \leq i \leq n$. 
Definition 10. Let \((X, \tau)\) be a topological space and let there exist a non-constant continuous function \(f : X \rightarrow [0, \infty)\) such that \(X\) is uniformly function \(-f - \varepsilon-\) chainable for every \(\varepsilon > 0\). Then \(X\) is said to be uniformly function-\(f\)-chainable.

Theorem 1. For every \(\varepsilon-\) chainable metric space \((X, d)\) with non-constant metric \(d\), there exist a non-negative real valued continuous function \(f\) on \(X\) such that \(X\) is function \(-f - \varepsilon-\) chainable. Moreover \(f\) is a strong \(\varepsilon-\) chain preserving function.

Proof. Let \(\overline{x} \in X\). Define a continuous function \(f : X \rightarrow [0, \infty)\) by \(f(x) = d(x, \overline{x})\) for all \(x \in X\).

Let \(x, y \in X\) and consider an \(\varepsilon\) - chain \(x = x_0, x_1, \ldots, x_n = y\) between them.

Now
\[
|f(x_i) - f(x_{i-1})| = |d(x_i, \overline{x}) - d(x_{i-1}, \overline{x})| < d(x_i, x_{i-1}) < \varepsilon, \quad 1 \leq i \leq n.
\]

Hence \(X\) is function \(-f - \varepsilon-\)chainable.

Obviously, \(f\) is strong \(\varepsilon-\) chain preserving.

Corollary 1. For every chainable metric space \((X, d)\) with non-constant metric \(d\), there exist a non-negative real valued continuous function \(f\) on \(X\) such that \(X\) is function \(-f - \varepsilon-\) chainable. Moreover \(f\) is a strong chain preserving function.

Theorem 2. Let \((X, \tau)\) be a function \(-f - \varepsilon-\) chainable space and let \(f\) be one-to-one function from \(X\) to \([0, \infty)\). Then there exists a metric \(d\) on \(X\) such that \((X, d)\) is \(\varepsilon-\) chainable space and the induced metric topology is coarser than \(\tau\).

Proof. Define a function \(d(x, y) = |f(x) - f(y)|\) for all \(x, y \in X\).

Then as \(f\) is one-to-one, \(d\) is a metric on \(X\).

Let \(x, y \in X\). Then there is a sequence of elements \(x = x_0, x_1, \ldots, x_n = y\) of \(X\) such that
\[
|f(x_i) - f(x_{i-1})| < \varepsilon, \quad 1 \leq i \leq n.
\]
or
\[
d(x_i, x_{i-1}) < \varepsilon, \quad 1 \leq i \leq n.
\]
or
\[
x \text{ and } y \text{ are } \varepsilon\text{-chainable.}
\]
Let $\tau_d$ be the induced topology by metric $d$.

Now $f^{-1}\{(f(x) - \epsilon, f(x) + \epsilon)\} = \{y \in X : d(x, y) < \epsilon\}$, or $\tau_d \subset \tau$ using continuity of $f$ in $(X, \tau)$.

Hence the result is proved.

**Corollary 2.** Let $(X, \tau)$ be a function - $f$ - $\epsilon$ - chainable space. If $f$ is a homeomorphism from $X$ to $[0, \infty)$ then $X$ is metrizable $\epsilon$ - chainable space.

**Proof.** Since $f$ is a homeomorphism from $X$ to $[0, \infty)$, every inverse image of every open set in $[0, \infty)$ is open in both $\tau$ as well as in $\tau_d$, it follows that $\tau = \tau_d$.

Hence $(X, \tau)$ is metrizable $\epsilon$ - chainable space.

**Theorem 3.** Let $(X, \tau)$ be a topological space and $f : X \rightarrow [0, \infty)$ be continuous onto function then $X$ is function $-f$-chainable.

**Proof.** Let $x, x' \in X$ and $\epsilon > 0$. Let $n$ be the least positive integer greater than $(|f(x) - f(x'|))\epsilon$.

Without loss of generality, let $f(x') > f(x)$.

Choose

\[ y_0 = f(x), \]
\[ y_1 = y_0 + \frac{|f(x) - f(x')|}{n}, \]
\[ y_2 = y_0 + \frac{2(|f(x) - f(x')|)}{n}, \]
\[ \vdots \]
\[ y_n = f(x') \text{ in } [0, \infty). \]

Also $|y_i - y_{i-1}| < \epsilon$, $1 \leq i \leq n$.

Then there exists a sequence $x, x_1, x_2, \ldots, x_{n-1}, x_n$ in $X$ such that

\[ |f(x_i) - f(x_{i-1})| = |y_i - y_{i-1}| < \epsilon, \]

or $X$ is function-$f$-chainable.

**Theorem 4.** The relation of function - $f$ - $\epsilon$ - chainability in a topological space is an equivalence relation.

**Proof.** Obvious.

**Theorem 5.** Let $(X, \tau)$ be a topological space and $A \subset X$. If for every $\epsilon > 0$ there exists a continuous function $f : X \rightarrow [0, \infty)$ such that $A$ is function $-f_A - \epsilon$ - chainable, then $\bar{A}$ is function $-f_{\bar{A}} - \epsilon$ - chainable.
Proof. Let \( x, y \in \overline{A} \).

As \( f(\overline{A}) \subset \overline{f(A)} \), \( f(x), f(y) \in \overline{f(A)} \) or there exist \( x', y' \in A \) such that

\[
|f(x) - f(x')| < \varepsilon \text{ and } |f(y) - f(y')| < \varepsilon.
\]

Hence there exist a sequence of elements \( x' = x_1, x_2, \ldots, x_{n-1} = y' \) in \( A \) such that

\[
|f_A(x_i) - f_A(x_{i-1})| = |f(x_i) - f(x_{i-1})| < \varepsilon, \quad 2 \leq i \leq n-1,
\]
or there exist a sequence of elements \( x = x_0, x' = x_1, x_2, \ldots, x_{n-1} = y', x_n = y \in \overline{A} \) such that

\[
|f_A(x_i) - f_A(x_{i-1})| = |f(x_i) - f(x_{i-1})| < \varepsilon, \quad 1 \leq i \leq n.
\]

Hence, we obtain the result.

**Theorem 6.** Let \( A \) be a dense subset of a topological space \( (X, \tau) \) and for every \( \varepsilon > 0 \) and let there exist a continuous function \( f : X \to [0, \infty) \) such that \( A \) is function \(-f_A - \varepsilon\)-chainable. Then \( X \) is function \(-f - \varepsilon\)-chainable.

**Proof.** Follows from Theorem 5.

**Theorem 7.** Let \( f : (X, \tau) \to (Y, u) \) be an onto continuous function. If \( Y \) is function \(-g - \varepsilon\)-chainable then \( X \) is function \(-gof - \varepsilon\)-chainable.

**Proof.** Let \( x, x' \in X \). Then there is a sequence of elements \( f(x) = y_0, y_1, \ldots, y_n = f(x') \) in \( Y \) such that \( |g(y_i) - g(y_{i-1})| < \varepsilon ; \quad 1 \leq i \leq n \), or there is a sequence of elements \( x = x_0, x_1, \ldots, x_n = x' \) in \( X \) such that

\[
f(x_i) = y_i ; 1 \leq i \leq n
\]

and \( |gof(x_i) - gof(x_{i-1})| < \varepsilon ; \quad 1 \leq i \leq n \).

Hence \( X \) is function \(-gof - \varepsilon\)-chainable.

**Theorem 8.** Let \( f : (X, \tau) \to (Y, u) \) be one-one onto open map. If \( X \) is function \(-g - \varepsilon\)-chainable then \( Y \) is function \(-gof^{-1} - \varepsilon\)-chainable.

**Proof.** Let \( f(x), f(x') \in Y \) where \( x, x' \in X \). Now there is a sequence of elements \( x = x_0, x_1, \ldots, x_n = x' \) in \( X \) such that

\[
|g(x_i) - g(x_{i-1})| < \varepsilon ; \quad 1 \leq i \leq n.
\]

Or there is a sequence of elements \( f(x) = f(x_0), f(x_1), \ldots, f(x_n) = f(x') \) in \( Y \) such that

\[
|gof^{-1}(x_i) - gof^{-1}(x_{i-1})| = |g(x_i) - g(x_{i-1})| < \varepsilon ; \quad 1 \leq i \leq n.
\]
or $Y$ is function $-gof^{-1}-\varepsilon$ - chainable.

**Theorem 9.** Let $f : (X, \tau) \to (Y, u)$ be a homeomorphism. Then $X$ is function $-g-\varepsilon$ - chainable iff $Y$ is function $-gof^{-1}-\varepsilon$ - chainable.

**Proof.** Follows from Theorems 7 and 8.

**Theorem 10.** Let $f : (X, \tau) \to (Y, \tau^*)$ be one-one open map. Let $A \subset X$ and $f(A) = B$. If $A$ is function $-g-\varepsilon$ - chainable then $B$ is function $-gof^{-1}_A-\varepsilon$ - chainable.

**Proof.** Follows from Problem 24, chap. 7 [5] and Theorem 8.

**Theorem 11.** Let $f : (X, \tau) \to (Y, \tau^*)$ be a homeomorphism. Let $A \subset X$ and $f(A) = B$. Then $A$ is function $-g-\varepsilon$ - chainable iff $B$ is function $-gof^{-1}_A-\varepsilon$ - chainable.

**Proof.** Follows from Problem 25, chap. 7 [5] and Theorem 9.

**Theorem 12.** Let $X = A \cup B$ where $A$ and $B$ are closed sets in $X$. Let $A$ be function $-f-\varepsilon$ - chainable and $B$ be function $-g-\varepsilon$ - chainable such that $f(x) = g(x)$ for every $x \in A \cap B$.

Then $X$ is function $-h-\varepsilon$ - chainable where

$$h(x) = \begin{cases} f(x), & x \in A \\ g(x), & x \in B \end{cases}$$

**Proof.** By pasting lemma the function $h : X \to [0, \infty)$ is continuous.

Then $X$ is function $-h-\varepsilon$ - chainable follows directly from definition of $h$ and function $-f-\varepsilon$ - chainability of $A$ and function $-g-\varepsilon$ - chainability of $B$.

**Theorem 13.** Let $(X, \tau)$ be a topological space and $A, B \subset X$ such that $A \sim B$ and $B \sim A$ are separated sets and $X = A \cup B$. Let for every $\varepsilon > 0$ there exist a function $f : X \to [0, \infty)$ such that $f_A : A \to [0, \infty)$ and $f_B : B \to [0, \infty)$ are continuous. If $A$ is function $-f_A-\varepsilon$ - chainable and $B$ is function $-f_B-\varepsilon$ - chainable then $X$ is function $-f-\varepsilon$ - chainable.

**Proof.** Now $f_A : A \to [0, \infty)$ and $f_B : B \to [0, \infty)$ are continuous functions. By Problem B, chap. 3[4] , $f$ is continuous on $X$.

Again $f(x) = \begin{cases} f_A(x), & x \in A \\ f_B(x), & x \in B \end{cases}$ and $f_A(x) = f_B(x)$ for every $x \in A \cap B$.

Then by Theorem 12, $X$ is function $-f-\varepsilon$-chainable.

**Theorem 14.** Let $X$ be a compact space and $Y$ be a Hausdorff space and $f : X \to Y$ be a continuous bijection. Then $X$ is function $-g-\varepsilon$ - chainable iff $Y$ is function $-gof^{-1}-\varepsilon$ - chainable.
Proof. Now $f$ is a homeomorphism by Corollary 2.4, Chap. 7 in [3]. The result then follows from Theorem 9.

**Theorem 15.** Let $X$ be a topological space and $\{f_n\}$ be a sequence of continuous functions from $X$ to $[0, \infty)$ such that $\{f_n\}$ uniformly converges to a function $f : X \to [0, \infty)$. If $X$ is function - $f_n$ - chainable for each $n \in \mathbb{N}$ then $X$ is function - $f$ - chainable.

Proof. Now $f : X \to [0, \infty)$ is continuous, by Theorem 4.4 in [3].

By uniform convergence; there is $m \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \varepsilon$ for all $x \in X$ and for all $n \geq m$.

Let $n \geq m$ and let $x, y \in X$.

Then there is a sequence of elements $x = x_0, x_1, \ldots, x_n = y$ such that

$$|f_n(x_i) - f_n(x_{i-1})| < \varepsilon \quad ; \quad 1 \leq i \leq n .$$

Also $|f_n(x_i) - f(x_i)| < \varepsilon$ ; and $|f_n(x_{i-1}) - f(x_{i-1})| < \varepsilon$ .

Consequently, $|f(x_i) - f(x_{i-1})| < 3\varepsilon \quad ; \quad 1 \leq i \leq n$ .

Hence the result.

**Theorem 16.** Let $X$ be a function - $f$ - chainable metric space. Then $f$ is a chain preserving map.

Proof. Now $f : X \to [0, \infty)$ is continuous map where $[0, \infty)$ is a chainable metric space with usual metric on it. By theorem 17 [2], $f$ is chain preserving.

**Theorem 17.** For every $\varepsilon > 0$, a normal space $X$ is function - $f - \varepsilon$ - chainable for some function $f$ on $X$.

Proof. Choose two non-negative real numbers $a$ and $b$ such that $b - a < \varepsilon$.

Let $A$ and $B$ be disjoint closed subsets of $X$.

By Urysohns Lemma, there is a continuous function $f : X \to [a, b]$ where $f(x) = a$ for all $x \in A$ and $f(x) = b$ for all $x \in B$.

Or $|f(x) - f(y)| \leq b - a < \varepsilon$ for all $x, y \in X$.

As $f : X \to [0, \infty)$ is continuous, $X$ is $f - \varepsilon$ - chainable.

**Theorem 18.** Let $X$ be a compact uniformly function - $f$ - chainable space for some positive real valued function $f$ on $X$. Then there exists a positive real number $e$ such that

$$l_e(f) + 1 > \frac{f(\overline{x})}{f(\overline{y})} \quad \text{for some} \quad \overline{x}, \overline{y} \in X.$$
Proof. Let \( x, y \in X \) such that
\[
    f(x) = \inf_{x \in X} f(x) \quad \text{and} \quad f(y) = \sup_{x \in X} f(x)
\]

By Problem A(b) and (c), chap. 5 [4], there is an \( e > 0 \) such that
\[
    f(x) > e \quad \text{for all} \quad x \in X.
\]

Now there is a sequence of elements \( x_0, x_1, \ldots, x_n = y \) in \( X \) with
\[
|f(x_i) - f(x_{i-1})| < \varepsilon \quad \text{and} \quad n \leq l_e(f)
\]
or
\[
    f(y) - f(x) < l_e(f) \varepsilon.
\]

Setting \( f(y) = k f(x) \) for some \( k > 1 \),
\[
    f(x) < \frac{e l_e(f)}{k - 1} \neq e.
\]

Hence
\[
    l_e(f) + 1 > \frac{f(x)}{f(y)}.
\]

Examples

1. Let \( X \) be an odd-even topology which is partition topology generated by
\[
P = \{ \{1, 2\}, \{3, 4\}, \{5, 6\}, \ldots \}\]

   (a) Let \( f : X \to [0, \infty) \) defined by \( f(2k) = k \), \( f(2k - 1) = k \) is continuous.
   
   Then \( X \) is function - \( f - \varepsilon \) - chainable for \( \varepsilon > 1 \).

   (b) Let \( f : X \to [0, \infty) \) defined by \( f(2k) = 1/k \), \( f(2k - 1) = 1/k \) is continuous.
   
   Then \( X \) is function - \( f - \varepsilon \) - chainable for \( \varepsilon > 0.5 \).

   (c) Let \( \varepsilon > 0 \) choose \( n \in \mathbb{N} \) with \( n > 1/\varepsilon \).

   Define \( f_\varepsilon : X \to [0, \infty) \) by \( f_\varepsilon(2k) = 1/n^k \), \( f_\varepsilon(2k - 1) = 1/n^k \) is continuous. Then \( X \) is function - \( f_\varepsilon - \varepsilon \) - chainable for any \( \varepsilon > 0 \).
2. Let $\tau$ be a discrete topology on space $X = [0, \infty)$ and let identity map $i : X \to [0, \infty)$ be continuous. 
Then $(X, \tau)$ is function $-i$-chainable.

3. Let $(X, \tau)$ be a discrete topological space where $X = [0, 1)$.
Let $f : X \to [0, \infty)$ defined by $f(x) = x/(1 - x)$ be continuous.
Then $(X, \tau)$ is function $-f$-chainable.

4. Let $\tau$ be a partition topology on space $X \subset [0, \infty) \times \mathbb{R}$ generated by the sets 
$A_{\alpha} = \{(\alpha, \beta) : \beta \in \mathbb{R} \} ; \alpha \geq 0$.
Then $X$ is function $-\pi$-chainable where $\pi$ is the projection map given by 
$\pi(\alpha, \beta) = \alpha ; (\alpha, \beta) \in [0, \infty) \times \mathbb{R}$.

References


