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CHAINABILITY IN TOPOLOGICAL SPACES THROUGH CONTINUOUS FUNCTIONS

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Abstract: In this paper concept of $\varepsilon - chainability$ in topological spaces has been introduced using continuity. Several well known results prove perquisites to results established in the paper. In case of metric spaces the equivalence of $\varepsilon - chainability$ and $function - \varepsilon - chainability$ is also established in the paper.

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For the definitions of ε-chain, uniform-ε-chainability, uniform chainability refer [1].

Throughout this paper $[0, \infty)$ has usual metric topology and ε is positive real number unless mentioned otherwise.

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Definition 1. (see [2]) A map f from a metric space X into another metric space Y is said to be an $\varepsilon -$ *chain preserving map* if and only if for every $\varepsilon-$chain between any two points in X there is an $\varepsilon-$chain between images of these two points under f in Y where $\varepsilon > 0$.

Definition 2. (see [2]) A map f from a metric space X into another metric space Y is said to be an *strongly* $\varepsilon-$ *chain preserving map* if and only if for every ε – chain $x_0, x_1, \ldots, x_n$ between any two points x_0 and x_n of X, $f(x_0), f(x_1), \ldots, f(x_n)$ is an ε – chain between $f(x_0)$ and $f(x_n)$.

Definition 3. (see [2]) A map f from a metric space X into another metric space Y is said to be *chain preserving* if and only if f is $\varepsilon -$ *chain* preserving for every $\varepsilon > 0$.

Definition 4. (see [2]) A map f from a metric space X into another metric space Y is said to be *strongly chain preserving* if and only if f is strongly ε-chain preserving for every $\varepsilon > 0$.

Definition 5. A topological space (X, τ) is said to be $function - f - \varepsilon - chainable$ if for $\varepsilon > 0$ there exists a non-constant continuous function $f : X \to [0, \infty)$ such that for every pair of elements x, y of X, there is a sequence $x = x_0, x_1, \ldots, x_n = y$ of elements in X with

$$|f(x_i) - f(x_{i-1})| < \varepsilon ; \quad 1 \leq i \leq n.$$

Definition 6. Let (X, τ) be a topological space and let there exist a non-constant continuous function $f : X \to [0, \infty)$ such that X is function $- f - \varepsilon$ - chainable for every $\varepsilon > 0$. Then X is said to be $function - f - chainable$.

Definition 7. Let (X, τ) be a topological space and $A \subset X$. Then A is said to be $function - f - \varepsilon - chainable$ if (A, τ_A) is function $- f - \varepsilon-$ chainable subspace of X.

Definition 8. Let (X, τ) be a topological space and $A \subset X$. Let there exist a non-constant continuous function $f : A \to [0, \infty)$ such that (A, τ_A) is function $-f-\varepsilon$-chainable for every $\varepsilon > 0$. Then A is said to be $function - f - chainable$.

Definition 9. A topological space (X, τ) is said to be $uniformlyfunction- f - \varepsilon - chainable$ if for $\varepsilon > 0$ there exists a positive integer $l_\varepsilon(f)$ and a non-constant continuous function $f : X \to [0, \infty)$ such that for every pair of elements x, y of X there is a sequence $x = x_0, x_1, \ldots, x_n = y$ in X with $n \leq l_\varepsilon(f)$ and $|f(x_i) - f(x_{i-1})| < \varepsilon ; \quad 1 \leq i \leq n$.

Definition 10. Let (X, τ) be a topological space and let there exist a non-constant continuous function $f : X \to [0, \infty)$ such that X is uniformly function $-f - \varepsilon-$ chainable for every $\varepsilon > 0$. Then X is said to be *uniformly function-f-chainable.*

Theorem 1. *For every $\varepsilon-$ chainable metric space (X, d) with non - constant metric d , there exist a non-negative real valued continuous function f on X such that X is function $-f-\varepsilon-$ chainable. Moreover f is a strong $\varepsilon-$ chain preserving function.*

Proof. Let $\overline{x} \in X$. Define a continuous function $f : X \to [0, \infty)$ by $f(x) = d(x, \overline{x})$ for all $x \in X$.

Let $x, y \in X$ and consider an ε - chain $x = x_0, x_1, \ldots, x_n = y$ between them.

Now

$$|f(x_i) - f(x_{i-1})| = |d(x_i, \overline{x}) - d(x_{i-1}, \overline{x})|$$
$$< d(x_i, x_{i-1}) < \varepsilon, \quad 1 \leq i \leq n.$$

Hence X is function - $f - \varepsilon-$chainable.

Obviously, f is strong ε - chain preserving.

Corollary 1. *For every chainable metric space (X, d) with non - constant metric d, there exist a non-negative real valued continuous function f on X such that X is function - f - chainable. Moreover f is a strong chain preserving function.*

Theorem 2. *Let (X, τ) be a function - f - ε - chainable space and let f be one - to - one function from X to $[0, \infty)$. Then there exists a metric d on X such that (X, d) is ε - chainable space and the induced metric topology is coarser than τ.*

Proof. Define a function $d(x, y) = |f(x) - f(y)|$ for all $x, y \in X$.

Then as f is one - one, d is a metric on X.

Let $x, y \in X$. Then there is a sequence of elements $x = x_0, x_1, \ldots, x_n = y$ of X such that

$$|f(x_i) - f(x_{i-1})| < \varepsilon, \quad 1 \leq i \leq n.$$

or

$$d(x_i, x_{i-1}) < \varepsilon, \quad 1 \leq i \leq n .$$

or

x and y are ε-chainable.

Let τ_d be the induced topology by metric d.

Now $f^{-1}\{(f(x)-\varepsilon, f(x)+\varepsilon)\} = \{y \in X : d(x,y) < \varepsilon\}$, or $\tau_d \subset \tau$ using continuity of f in (X,τ).

Hence the result is proved.

Corollary 2. *Let (X,τ) be a function - f - ε - chainable space. If f is a homeomorphism from X to $[0,\infty)$ then X is metrizable ε - chainable space.*

Proof. Since f is a homeomorphism from X to $[0,\infty)$, every inverse image of every open set in $[0,\infty)$ is open in both τ as well as in τ_d, it follows that $\tau = \tau_d$.

Hence (X,τ) is metrizable ε - chainable space.

Theorem 3. *Let (X,τ) be a topological space and $f : X \to [0,\infty)$ be continuous onto function then X is function $-f$-chainable.*

Proof. Let $x, x' \in X$ and $\varepsilon > 0$. Let n be the least positive integer greater than $(|f(x) - f(x')|)\varepsilon$.

Without loss of generality, let $f(x') > f(x)$.

Choose

$$
\begin{aligned}
y_0 =& f(x),\\
y_1 =& y_0 + \frac{|f(x) - f(x')|}{n},\\
y_2 =& y_0 + \frac{2\,(|f(x) - f(x')|)}{n},\\
&\vdots\\
y_n =& f\left(x'\right) \text{ in } [0,\infty).
\end{aligned}
$$

Also $|y_i - y_{i-1}| < \varepsilon$, $1 \le i \le n$.

Then there exists a sequence $x,\ x_1,\ \ x_2,\ \ldots\ ,\ x_{n-1},\ x_n$ in X such that

$$|f(x_i) - f(x_{i-1})| = |y_i - y_{i-1}| < \varepsilon,$$

or X is function-f-chainable.

Theorem 4. *The relation of function - f - ε - chainability in a topological space is an equivalence relation.*

Proof. Obvious.

Theorem 5. *Let (X,τ) be a topological space and $A \subset X$. If for every $\varepsilon > 0$ there exists a continuous function $f : X \to [0,\infty)$ such that A is function $- f_A - \varepsilon$ -chainable, then $\overline{A}$ is function $- f_{\overline{A}} - \varepsilon$ -chainable.*

Proof. Let $x, y \in \overline{A}$.

As $f\left(\overline{A}\right) \subset \overline{f\left(A\right)}$, $f\left(x\right),\ f\left(y\right) \in \overline{f(A)}$ or there exist $x',\ y' \in A$ such that $|f\left(x\right) - f\left(x'\right)| < \varepsilon$ and $|f\left(y\right) - f\left(y'\right)| < \varepsilon$.

Hence there exist a sequence of elements $x' = x_1,\ x_2\ ,\ \ldots\ ,\ x_{n-1} = y'$ in A such that

$$|f_A\left(x_i\right) - \ f_A\left(x_{i-1}\right)| = |f\left(x_i\right) - \ f\left(x_{i-1}\right)| < \varepsilon\ ,\ \ 2 \le i \le n-1,$$

or there exist a sequence of elements $x = x_0,\ x' = x_1,\ x_2\ ,\ \ldots\ ,\ x_{n-1} = y',\ x_n = y$ in $\overline{A}$ such that

$$|f_{\overline{A}}\ \left(x_i\right) - \ f_{\overline{A}}\ \left(x_{i-1}\right)| = |f\left(x_i\right) - \ f\left(x_{i-1}\right)| < \varepsilon\ ,\ \ 1 \le i \le n.$$

Hence, we obtain the result.

Theorem 6. *Let A be a dense subset of a topological space (X, τ) and for every $\varepsilon > 0$ and let there exist a continuous function $f\ :X\ \to [0, \infty)$ such that A is function $-f_A - \varepsilon$-chainable. Then X is function $-f - \varepsilon$-chainable.*

Proof. Follows from Theorem 5.

Theorem 7. *Let $f\ : (X, \tau) \to (Y, u)$ be an onto continuous function. If Y is function $-g - \varepsilon-$ chainable then X is function $-gof - \varepsilon-$ chainable.*

Proof. Let $x, x' \in X$. Then there is a sequence of elements $f\left(x\right) = y_0,\ y_1,\ \ldots\ ,\ y_n = f(x')$ in Y such that $\left|g\left(y_i\right) - g\left(y_{i-1}\right)\right| < \varepsilon\ ;\ \ 1 \le i \le n$, or there is a sequence of elements $x = x_0,\ x_1,\ \ldots\ ,\ x_n = x'$ in X such that

$$f\left(x_i\right) = y_i\ ; 1 \le i \le n$$

and $|gof\left(x_i\right) - gof\left(x_{i-1}\right)| < \varepsilon\ ;\ \ 1 \le i \le n$.

Hence X is function - gof - ε - chainable.

Theorem 8. *Let $f\ : (X, \tau) \to (Y, u)$ be one-one onto open map. If X is function - g - ε - chainable then Y is function - gof^{-1} - ε - chainable.*

Proof. Let $f\left(x\right),\ \ f(x') \in Y$ where $x, x' \in X$.Now there is a sequence of elements $x = x_0,\ x_1,\ \ldots\ ,\ x_n = x'$ in X such that

$$\left|g\left(x_i\right) - g\left(x_{i-1}\right)\right| < \varepsilon\ ;\ \ 1 \le i \le n.$$

Or there is a sequence of elements $f\left(x\right) = f\left(x_0\right),\ \ f(x_1),\ \ldots\ ,\ f(x_n) = f(x')$ in Y such that

$$\left|gof^{-1}\left(f(x_i)\right) - gof^{-1}\left(f\left(x_{i-1}\right)\right)\right| = |g\left(x_i\right) - g\left(x_{i-1}\right)| < \varepsilon\ ;\ \ 1 \le i \le n.$$

or Y is function – gof^{-1} – ε – chainable.

Theorem 9. *Let $f\ :(X,\tau)\to(Y,u)$ be a homeomorphism. Then X is function - g - ε - chainable iff Y is function - gof^{-1} – ε - chainable.*

Proof. Follows from Theorems 7 and 8.

Theorem 10. *Let $f\ :(X,\tau)\to(Y,\tau^*)$ be one-one open map. Let $A\subset X$ and $f(A)=B$. If A is function - g - ε - chainable then B is function - gof_A^{-1} - ε - chainable.*

Proof. Follows from Problem 24, chap. 7 [5] and Theorem 8.

Theorem 11. *Let $f\ :(X,\tau)\to(Y,\tau^*)$ be a homeomorphism. Let $A\subset X$ and $f(A)=B$. Then A is function - g - ε - chainable iff B is function - gof_A^{-1} - ε - chainable.*

Proof. Follows from Problem 25, chap. 7 [5] and Theorem 9.

Theorem 12. *Let $X=A\cup B$ where A and B are closed sets in X. Let A be function - f - ε - chainable and B be function - g - ε - chainable such that $f(x)=g(x)$ for every $x\in A\cap B$.*
Then X is function $-h-\varepsilon$- chainable where

$$h(x)=\begin{cases} f(x), & x\in A\\ g(x), & x\in B\end{cases}$$

Proof. By pasting lemma the function $h\ :X\ \to[0,\infty)$ is continuous.
Then X is function - h - ε - chainable follows directly from definition of h and function - f - ε - chainability of A and function - g - ε - chainability of B.

Theorem 13. *Let (X,τ) be a topological space and $A,B\subset X$ such that $A\sim B$ and $B\sim A$ are separated sets and $X=A\cup B$. Let for every $\varepsilon>0$ there exist a function $f:X\to[0,\infty)$ such that $f_A\ :A\to[0,\infty)$ and $f_B\ :\ B\to[0,\infty)$ are continuous. If A is function $-\ f_A\ -\ \varepsilon$ -chainable and B is function $-\ f_B\ -\ \varepsilon$ -chainable then X is function - $f\ -\ \varepsilon$ - chainable.*

Proof. Now $f_A:A\to[0,\infty)$ and $f_B:B\to[0,\infty)$ are continuous functions.
By Problem B, chap. 3[4] , f is continuous on X.
Again $f(x)=\begin{cases} f_A(x), & x\in A\\ f_B(x), & x\in B\end{cases}$ and $f_A(x)=f_B(x)$ for every $x\in A\cap B$.
Then by Theorem 12, X is function $-f-\varepsilon$-chainable.

Theorem 14. *Let X be a compact space and Y be a Hausdorff space and $f\ :X\ \to\ Y$ be a continuous bijection. Then X is function - g - ε - chainable iff Y is function - gof^{-1} - ε - chainable.*

Proof. Now f is a homeomorphism by Corollary 2.4, Chap. 7 in [3].
The result then follows from Theorem 9.

Theorem 15. *Let X be a topological space and $\{f_n\}$ be a sequence of continuous functions from X to $[0,\infty)$ such that $\{f_n\}$ uniformly converges to a function $f : X \to [0,\infty)$. If X is function - f_n - chainable for each $n \in \mathbb{N}$ then X is function - f- chainable.*

Proof. Now $f : X \to [0,\infty)$ is continuous, by Theorem 4.4 in [3].
By uniform convergence; there is $m \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \varepsilon$ for all $x \in X$ and for all $n \geq m$.
Let $n \geq m$ and let $x, y \in X$.
Then there is a sequence of elements $x = x_0, x_1, \ldots, x_n = y$ such that

$$|f_n(x_i) - f_n(x_{i-1})| < \varepsilon \quad ; \quad 1 \leq i \leq n \; .$$

Also $|f_n(x_i) - f(x_i)| < \varepsilon$; and $|f_n(x_{i-1}) - f(x_{i-1})| < \varepsilon$.
Consequently, $|f(x_i) - f(x_{i-1})| < 3\varepsilon$; $1 \leq i \leq n$.
Hence the result.

Theorem 16. *Let X be a function - f- chainable metric space. Then f is a chain preserving map.*

Proof. Now $f : X \to [0,\infty)$ is a continuous map where $[0,\infty)$ is a chainable metric space with usual metric on it.By theorem 17 [2], f is chain preserving.

Theorem 17. *For every $\varepsilon > 0$, a normal space X is function - $f - \varepsilon$ - chainable for some function f on X.*

Proof. Choose two non -negative real numbers a and b such that $b - a < \varepsilon$.
Let A and B be disjoint closed subsets of X.
By Urysohns Lemma, there is a continuous function $f : X \to [a,b]$ where $f(x) = a$ for all $x \in A$ and $f(x) = b$ for all $x \in B$.
Or $|f(x) - f(y)| \leq b - a < \varepsilon$ for all $x, y \in X$.
As $f : X \to [0,\infty)$ is continuous, X is $f - \varepsilon -$ chainable.

Theorem 18. *Let X be a compact uniformly function - f - chainable space for some positive real valued function f on X. Then there exists a positive real number e such that*

$$l_e(f) + 1 > \frac{f(\overline{x})}{f(\overline{y})} \quad \text{for some} \quad \overline{x},\ \overline{y} \in X.$$

Proof. Let $\overline{x}, \overline{y} \in X$ such that

$$f(\overline{x}) = \inf_{x \in X} f(x) \text{ and } f(\overline{y}) = \sup_{x \in X} f(x)$$

By Problem A(b) and (c), chap. 5 [4], there is an $e > 0$ such that $f(x) > e$ for all $x \in X$.

Now there is a sequence of elements $\overline{x} = x_0, x_1, \ldots, x_n = \overline{y}$ in X with

$$|f(x_i) - f(x_{i-1})| < \varepsilon \ \ and \ \ n \leq l_e(f)$$

or

$$f(\overline{y}) - f(\overline{x}) < l_e(f)\, e.$$

Setting $f(\overline{y}) = k\ f(\overline{x})$ for some $k > 1$,

$$f(\overline{x}) < \frac{e\, l_e(f)}{k-1} \nleq e.$$

Hence

$$l_e(f) + 1 > \frac{f(\overline{x})}{f(\overline{y})}.$$

Examples

1. Let X be an odd-even topology which is partition topology generated by

$$P = \{\{1,2\}, \{3,4\}, \{5,6\}, \ldots\}$$

(a) Let $f : X \to [0, \infty)$ defined by $f(2k) = k$, $f(2k-1) = k$ is continuous.
Then X is function - f - ε - chainable for $\varepsilon > 1$.

(b) Let $f : X \to [0, \infty)$ defined by $f(2k) = 1/k$, $f(2k-1) = 1/k$ is continuous.
Then X is function - f - ε - chainable for $\varepsilon > 0.5$.

(c) Let $\varepsilon > 0$ choose $n \in \mathbb{N}$ with $n > 1/\varepsilon$.
Define $f_\varepsilon : X \to [0, \infty)$ by $f_\varepsilon(2k) = 1/n^k$, $f_\varepsilon(2k-1) = 1/n^k$ is continuous. Then X is function $-f_\varepsilon - \varepsilon-$ chainable for any $\varepsilon > 0$.

2. Let τ be a discrete topology on space $X = [0, \infty)$ and let identity map $i : X \to [0, \infty)$ be continuous.
   Then (X, τ) is function $-i$-chainable.

3. Let (X, τ) be a discrete topological space where $X = [0, 1)$.
   Let $f : X \to [0, \infty)$ defined by $f(x) = x/(1 - x)$ be continuous.
   Then (X, τ) is function $-f$-chainable.

4. Let τ be a partition topology on space $X \subset [0, \infty) \times \mathbb{R}$ generated by the sets
   $A_\alpha = \{(\alpha, \beta) : \beta \in \mathbb{R}\}$; $\alpha \geq 0$.
   Then X is function - π - chainable where π is the projection map given by
   $\pi(\alpha, \beta) = \alpha$; $(\alpha, \beta) \in [0, \infty) \times \mathbb{R}$.

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