International Journal of Pure and Applied Mathematics
Volume 84 No. 3 2013, 153-162
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v84i3.2

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THE L^2 NORM ERROR ESTIMATE FOR THE DIV-CURL LEAST-SQUARES METHOD FOR 3D-STOKES EQUATIONS

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Abstract: This paper studies L^2 norm error estimate for the div-curl least-squares finite element method for Stokes equations with homogenous velocity boundary condition. The analysis using a different way from that in [11] shows that, without the divergence of the vorticity, the L^2 norm error bound of the velocity is $O(h^{\frac{3}{2}})$ in the standard linear element method.

AMS Subject Classification: 65N30
Key Words: error estimate, Stokes equations, least-squares finite element method

1. Introduction

The purpose of this paper is to present a short proof for the L^2 norm error estimate for the div-curl least-squares finite element method for Stokes equations with homogenous velocity boundary condition. It is well known that least-squares finite element methods applied the system lead to a minimization problem solving symmetric and positive definite system. The approximation spaces do not require the inf-sup condition and any conforming finite element space including piecewise continuous polynomial spaces can be used as approximation spaces. In this paper the divergence of vorticity is dropped in the

Received: June 17, 2012

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first-order least-squares finite element method for 3D-Stokes equations. In this case, we can show the L^2 error bound for velocity is $O(h^{\frac{3}{2}})$ in the standard linear element method. We establish the L^2 norm error estimate of velocity through a refined duality argument. The key step of the duality argument is to express the the L^2 norm error of velocity in terms of the bilinear form. The L^2 norm error estimates for div least-squares method is studied in [5], in this paper we study L^2 norm error estimates for div-curl least-squares method.

Many least-squares finite element methods for Stokes equations have been proposed and analyzed, let us just mention those by Z.Cai, T.Manteuffel and S.McCormick [7], V.Girault and P.A.Raviart [14], B.N.Jiang [15], J.M.Deang [25], P.B.Bochev [26], C.L.Chang [27] and H.Y.Duan [11]. But only a few of them are discussed without the divergence of vorticity for 3D-Stokes equations, especially the L^2 norm error estimate of the velocity. The L^2 norm error estimate of the velocity is derived from a constructive approach in [11], here we get the L^2 norm error estimate of velocity through a refined duality argument directly.

The paper is organized as follows: In section 2, we recall some Hilbert spaces, Green's formulae and the regularities to classic problem. In section 3, the div-curl least-squares formulation and its finite element approximation for 3D-Stokes equations are described. we state the L^2 norm error bound of velocity and provide new proof in section 4.

2. Hilbert Spaces, Green's Formulae and the Regularities

First we recall some notations. Let $\Omega \subset R^3$ is an open bounded domain with boundary $\Gamma = \partial\Omega$, $\mathbf{n}$ is unit normal vector to Γ . We introduce the following Hilbert spaces

$$
\begin{aligned}
&L^2(\Omega) = \{v; \int_\Omega v^2 < \infty\},\\
&H^m(\Omega) = \{\partial^\gamma v \in L^2(\Omega),\ 0 \le |\gamma| \le m\},\ (m \ge 1),\\
&H_0^1(\Omega) = \{v \in L^2(\Omega);\ \nabla v \in (L^2(\Omega))^3,\ v_{|\Gamma} = 0\},\\
&H(\mathbf{curl};\Omega) = \{\mathbf{u} \in (L^2(\Omega))^3;\ \mathbf{curl}\ \mathbf{u} \in (L^2(\Omega))^3\},\\
&H(div;\Omega) = \{\mathbf{u} \in (L^2(\Omega))^3;\ div\mathbf{u} \in L^2(\Omega)\},
\end{aligned}
$$

In addition, $H^{-1}(\Omega)$ is the dual space of $H_0^1(\Omega)$. The norm and the product in $H^s(\Omega)$ or $(H^s(\Omega))^3$ will be denoted by $\|\cdot\|_s$ and $(\cdot,\cdot)_s$, for $s = 0$, $H^s(\Omega)$

or $(H^s(\Omega))^3$ coincides with $L^2(\Omega)$ or $(L^2(\Omega))^3$. In this case the norm and the product will be denoted by $\|\cdot\|_0$ and $(\cdot,\cdot)$.

Two Green's formulae of integration by parts (cf. [14]) are as follows:

$$(\mathbf{u}, \nabla q) + (div\mathbf{u}, q) = \langle \mathbf{u}\cdot\mathbf{n},\ q\rangle_\Gamma, \forall \mathbf{u} \in H(div;\Omega), \quad \forall\ q \in H^1(\Omega). \quad (1)$$

$$(\mathbf{curl\ w}, \mathbf{v}) - (\mathbf{w}, \mathbf{curl v}) = \langle \mathbf{v}\times\mathbf{n},\ \mathbf{w}\rangle_\Gamma,\ \forall\ \mathbf{v} \in H(\mathbf{curl};\Omega), \quad \forall\ \mathbf{w} \in (H^1(\Omega))^3. \quad (2)$$

Proposition 2.1 [14] *Assume that $\Omega \subset R^3$ is a simply-connected and bounded domain with a Lipschitz continuous boundary Γ. Given $\mathbf{f} \in (H^{-1}(\Omega))^3$ and $g \in L_0^2(\Omega)$, there exist a unique solution $(\mathbf{u}, p) \in (H_0^1(\Omega))^3 \times (H^1(\Omega) \bigcap L_0^2(\Omega))$ to the Stokes problem*

$$-\triangle\mathbf{u} + \nabla p = \mathbf{f}, \quad div\mathbf{u} = g, \quad \mathbf{u}_{|\Gamma} = \mathbf{0},$$

the solution of which satisfies

$$\|\mathbf{u}\|_1 + \|p\|_0 \le C(\|\mathbf{f}\|_{-1} + \|g\|_0).$$

If additionally $\Gamma \in C^{r+2}$ and $\mathbf{f} \in (H^r(\Omega))^3$ and $g \in H^{r+1}(\Omega)$ with $r = 0, 1$, we have

$$\|\mathbf{u}\|_{r+2} + \|p\|_{r+1} \le C(\|\mathbf{f}\|_r + \|g\|_{r+1}).$$

3. Stokes Equations, Least-Squares Formulation and Finite Element Approximation

Consider the following Stokes problem in three dimensional space

$$-\triangle\mathbf{u} + \nabla p = \mathbf{f}, \quad in\ \Omega, \quad (3)$$

$$div\ \mathbf{u} = 0, \quad in\ \Omega, \quad (4)$$

$$\mathbf{u} = \mathbf{0}, \quad on\ \ \Gamma, \quad (5)$$

where $\mathbf{u} \in (H_0^1(\Omega))^3$ and $p \in L_0^2(\Omega)$ are velocity and pressure respectively, and $\mathbf{f} \in (L^2(\Omega))^3$ is the given function, and $\Omega \subset R^3$ is the domain occupied by the flow, with boundary $\Gamma = \partial\Omega$, $\mathbf{n}$ is unit normal vector to Γ .

Let us introduce a new unknown-vorticity $\mathbf{w} = \mathbf{curl\ u}$. In light of

$$\mathbf{curlcurl\ u} = -\triangle\mathbf{u} + \nabla div\ \mathbf{u} \text{ and } div\ \mathbf{u} = 0,$$

the Stokes problem(3)-(5) may be rewritten as the following first-order partial differential system:

$$\mathbf{curl}\ \mathbf{w} + \nabla p = \mathbf{f}, \quad in\ \Omega, \tag{6}$$

$$\mathbf{w} - \mathbf{curl}\ \mathbf{u} = \mathbf{0}, \quad in\ \Omega, \tag{7}$$

$$div\ \mathbf{u} = 0, \quad in\ \Omega, \tag{8}$$

where $\mathbf{u}_{|\Gamma} = \mathbf{0}, \int_\Omega pdx = 0$.

Least-squares functional is defined as the sum of the L^2 norm of the residual of the first-order system in (6)-(8). Least-squares variational problem is then the minimization of the least-squares functional in appropriate solution spaces.

Define the following least-squares functional

$$G(\mathbf{u}, p, \mathbf{w}; \mathbf{f}) = \|\mathbf{curl}\ \mathbf{w} + \nabla p - \mathbf{f}\|_0^2 + \|\mathbf{w} - \mathbf{curl}\ \mathbf{u}\|_0^2 + \|div\ \mathbf{u}\|_0^2 \tag{9}$$

for all $(\mathbf{u}, p, \mathbf{w}) \in (H_0^1(\Omega))^3 \times (H^1(\Omega) \bigcap L_0^2(\Omega)) \times H(\mathbf{curl}; \Omega)$.

The following theorem was proved in [11].

Theorem 3.1 *Assume that $\Omega \subset R^3$ is a simply-connected and bounded domain with $C^{1,1}$boundary Γ, or is a bounded and convex polyhedron, then there holds*

$$G(\mathbf{v}, q, \mathbf{z}; \mathbf{0}) \geq C\{\|\mathbf{v}\|_1^2 + \|q\|_0^2 + \|\mathbf{z}\|_0^2\}.$$

for all $(\mathbf{v}, q, \mathbf{z}) \in (H_0^1(\Omega))^3 \times (H^1(\Omega) \bigcap L_0^2(\Omega)) \times H(\mathbf{curl}; \Omega)$.

The variational problem corresponding to the L^2 norm least-squares functional is to minimize functional (9) over $(H_0^1(\Omega))^3 \times (H^1(\Omega) \bigcap L_0^2(\Omega)) \times H(\mathbf{curl}; \Omega)$. That is , to find $(\mathbf{u}, p, \mathbf{w}) \in (H_0^1(\Omega))^3 \times (H^1(\Omega) \bigcap L_0^2(\Omega)) \times H(\mathbf{curl}; \Omega)$ such that

$$G(\mathbf{u}, p, \mathbf{w}; \mathbf{f}) = \inf G(\mathbf{v}, q, \mathbf{z}; \mathbf{f}). \tag{10}$$

Now we define a bilinear form $b(\cdot, \cdot)$ on

$$((H_0^1(\Omega))^3 \times (H^1(\Omega) \bigcap L_0^2(\Omega)) \times H(\mathbf{curl}; \Omega)) \times ((H_0^1(\Omega))^3 \times (H^1(\Omega) \bigcap L_0^2(\Omega)) \times H(\mathbf{curl}; \Omega))$$

by

$$b(\mathbf{u}, p, \mathbf{w};\ \mathbf{v}, q, \mathbf{z}) = (\mathbf{curl}\ \mathbf{w} + \nabla p,\ \mathbf{curl}\ \mathbf{z} + \nabla q) + (\mathbf{w} - \mathbf{curl}\ \mathbf{u},\ \mathbf{z} - \mathbf{curl}\ \mathbf{v}) + (div\ \mathbf{u},\ div\ \mathbf{v})$$

and a linear form $F(\cdot)$ on $(H_0^1(\Omega))^3 \times H^1(\Omega) \bigcap L_0^2(\Omega) \times H(\mathbf{curl}; \Omega)$ by

$$F(\mathbf{v}, q, \mathbf{z}) = (\mathbf{f},\ \mathbf{curl}\ \mathbf{z} + \nabla q).$$

Then (10) can be written in the equivalent form as follows: find $(\mathbf{u}, p, \mathbf{w}) \in (H_0^1(\Omega))^3 \times (H^1(\Omega) \bigcap L_0^2(\Omega)) \times H(\mathbf{curl}; \Omega)$ such that

$$b(\mathbf{u}, p, \mathbf{w};\ \mathbf{v}, q, \mathbf{z}) = F(\mathbf{v}, q, \mathbf{z}) \tag{11}$$

for all $(\mathbf{v}, q, \mathbf{z}) \in (H_0^1(\Omega))^3 \times (H^1(\Omega) \bigcap L_0^2(\Omega)) \times H(\mathbf{curl}; \Omega)$.

Now let us consider the finite element method.

Let T_h be the regular triangulation of Ω into finite elements of tetrahedrons(cf.[9]). Define

$$V_h = \{v \in H^1(\Omega); v_{|K} \in P_1(K), \forall K \in T_h\}$$

where $P_1(K)$ is the space of linear polynomials. Let $\widetilde{v} \in V_h$ be the standard interpolation to $v \in H^2(\Omega)$, from the standard interpolation theory in [9, 1], we have

$$\|v - \widetilde{v}\|_0 + h\|v - \widetilde{v}\|_1 \leq Ch^2\|v\|_2. \tag{12}$$

Define

$$U_h = (V_h \cap H_0^1(\Omega))^3,\ Q_h = V_h \cap L_0^2(\Omega),\ W_h = (V_h)^3$$

The finite element method to (11) is find $(\mathbf{u}_h, p_h, \mathbf{w}_h) \in U_h \times Q_h \times W_h$ such that

$$b(\mathbf{u}_h, p_h, \mathbf{w}_h;\ \mathbf{v}, q, \mathbf{z}) = F(\mathbf{v}, q, \mathbf{z}) \tag{13}$$

for all $(\mathbf{v}, q, \mathbf{z}) \in U_h \times Q_h \times W_h$

The following error estimate in the energy norm was also provided in [11].

Theorem 3.2 *Under the same conditions as in Theorem 3.1, let* $(\mathbf{u}, p, \mathbf{w})$ *and* $(\mathbf{u}_h, p_h, \mathbf{w}_h)$ *be the solutions of (11) and (13) respectively. If* $(\mathbf{u}, p, \mathbf{w}) \in (H^2(\Omega))^3 \times H^2(\Omega) \times (H^2(\Omega))^3$, *then*

$$\|\mathbf{u} - \mathbf{u}_h\|_1 + \|p - p_h\|_0 + \|\mathbf{w} - \mathbf{w}_h\|_0 \leq Ch\{\|\mathbf{u}\|_2 + \|p\|_2 + \|\mathbf{w}\|_2\}. \tag{14}$$

4. The L^2 Norm Error Estimate

This section presents the main result of this paper on the L^2 norm error estimate of velocity $\mathbf{u}$. Although the divergence of vorticity is dropped in the first-order least-squares finite element method , we still can show the L^2 error bound for velocity is $O(h^{\frac{3}{2}})$ in the standard linear element method. We get the result with the duality argument. The key step of the duality argument is to express the the L^2 norm error of velocity in terms of the bilinear form. While this is straightforward for the Galerkin finite element method, it is a non-trivial matter for the div-curl least-squares method.

Lemma 4.1 *Let* $(\mathbf{u}, p, \mathbf{w})$ *and* $(\mathbf{u}_h, p_h, \mathbf{w}_h)$ *be the solutions of (11) and (13) respectively. Assume that the regularity estimates in Proposition 2.1 hold. Then* there exist (α, β, γ) such that

$$\|\mathbf{u} - \mathbf{u}_h\|_0^2 = b(\mathbf{u} - \mathbf{u}_h, p - p_h, \mathbf{w} - \mathbf{w}_h;\ \alpha, \beta, \gamma) \tag{15}$$

and that

$$\|\alpha\|_2 \leq C\|\mathbf{u} - \mathbf{u}_h\|_0. \ \ \|\beta\|_2 + \|\gamma\|_2 \leq C\|\mathbf{u} - \mathbf{u}_h\|_1. \tag{16}$$

Proof. Let $\mathbf{e}_h = \mathbf{u} - \mathbf{u}_h, q_h = p - p_h, E_h = \mathbf{w} - \mathbf{w}_h$. Consider the following dual problem: let $(\mathbf{x}, r) \in (H_0^1(\Omega))^3 \times L_0^2(\Omega)$ be the solution of the Stokes equations in (3)-(5) with the right-hand sides $\mathbf{f} = \mathbf{u} - \mathbf{u}_h$. That is

$$-\triangle \mathbf{x} + \nabla r = \mathbf{e}_h, \quad in\ \Omega, \tag{17}$$

$$div\ \mathbf{x} = 0, \quad in\ \Omega, \tag{18}$$

$$\mathbf{x} = \mathbf{0}, \quad on\ \ \Gamma. \tag{19}$$

It follows from Proposition 2.1 that

$$\|\mathbf{x}\|_2 + \|r\|_1 \leq C\|\mathbf{e}_h\|_0, \quad \|\mathbf{x}\|_3 + \|r\|_2 \leq C\|\mathbf{e}_h\|_1. \tag{20}$$

Using the above dual problem we can get

$$\|\mathbf{e}_h\|_0^2 = (E_h - \mathbf{curl}\ \mathbf{e}_h, -\mathbf{curl}\ \mathbf{x}) + (\mathbf{curl}\ E_h + \nabla q_h, \mathbf{x}) + (div\ \mathbf{e}_h, -r). \tag{21}$$

In fact, from (17), we have

$$\|\mathbf{e}_h\|_0^2 = (\mathbf{e}_h, -\triangle \mathbf{x} + \nabla r) = (\mathbf{e}_h, \mathbf{curl}\ \mathbf{curl}\ \mathbf{x} + \nabla r) \\ = (\mathbf{e}_h, \mathbf{curl}\ \mathbf{curl}\ \mathbf{x}) + (\mathbf{e}_h, \nabla r),$$

from Green's formulae (1),(2) and $\mathbf{e}_h = \mathbf{0}$ on Γ, we get

$$\|\mathbf{e}_h\|_0^2 = (\mathbf{curl}\ \mathbf{e}_h, \mathbf{curl}\ \mathbf{x}) - (div\ \mathbf{e}_h, r) = (-\mathbf{curl}\ \mathbf{e}_h, -\mathbf{curl}\ \mathbf{x}) - (div\ \mathbf{e}_h, r) \\ = (E_h - \mathbf{curl}\ \mathbf{e}_h, -\mathbf{curl}\ \mathbf{x}) - (div\ \mathbf{e}_h, r) + (E_h, \mathbf{curl}\ \mathbf{x}),$$

with Green's formulae (1) and (2) again and $div\ \mathbf{x} = 0$, we get

$$\begin{aligned}
&\|\mathbf{e}_h\|_0^2 = (E_h - \mathbf{curl}\ \mathbf{e}_h, -\mathbf{curl}\ \mathbf{x}) - (div\ \mathbf{e}_h, r) + (\mathbf{curl}\ E_h, \mathbf{x}) \\
&= (E_h - \mathbf{curl}\ \mathbf{e}_h, -\mathbf{curl}\ \mathbf{x}) - (div\ \mathbf{e}_h, r) + (\mathbf{curl}\ E_h + \nabla q_h, \mathbf{x}) + (q_h, div\ \mathbf{x}) \\
&= (E_h - \mathbf{curl}\ \mathbf{e}_h, -\mathbf{curl}\ \mathbf{x}) + (div\ \mathbf{e}_h, -r) + (\mathbf{curl}\ E_h + \nabla q_h, \mathbf{x}).
\end{aligned}$$

Thus we get (21).

Next we want to find (α, β, γ) such that

$$\begin{aligned}
\gamma - \mathbf{curl}\ \alpha &= -\mathbf{curl}\ \mathbf{x} && (22)\\
\mathbf{curl}\ \gamma + \nabla\beta &= \mathbf{x} && (23)\\
div\ \alpha &= -r. && (24)
\end{aligned}$$

To do so, let (α, β) be the solution of

$$\begin{aligned}
-\triangle\alpha + \nabla\beta &= \mathbf{x} - \triangle\mathbf{x} + \nabla r, \quad in\ \Omega, && (25)\\
div\ \alpha &= -r, \quad in\ \Omega, && (26)\\
\alpha &= \mathbf{0}, \quad on\ \ \Gamma. && (27)
\end{aligned}$$

With proposition 2.1, we have

$$\|\alpha\|_3 + \|\beta\|_2 \leq C\|\mathbf{x} - \triangle\mathbf{x}\|_1 + \|r\|_2 \leq C\|\mathbf{x}\|_3 + \|r\|_2 \leq C\|\mathbf{e}_h\|_1 \qquad (28)$$

From (20) and (28), (16) is proved. Let $\gamma = \mathbf{curl}\ \alpha - \mathbf{curl}\ \mathbf{x}$, with equation (25) and (26) we can easily prove that (α, β, γ) is the solution of (22)-(24). The Lemma is proved.

With Lemma 4.1 the L^2 norm error estimate of the velocity is given by

Theorem 4.1 *Under the same conditions as in Theorem 3.2, If* $(\mathbf{u}, p, \mathbf{w}) \in (H^2(\Omega))^3 \times H^2(\Omega) \times (H^2(\Omega))^3$, *then*

$$\|\mathbf{u} - \mathbf{u}_h\|_0 \leq Ch^{\frac{3}{2}}\{\|\mathbf{u}\|_2 + \|p\|_2 + \|\mathbf{w}\|_2\}. \qquad (29)$$

Proof. It follows from Lemma 4.1 , the error orthogonality property, Cauchy-Schwarz inequality and the approximation property (12) that for any $(\alpha_h, \beta_h, \gamma_h) \in U_h \times Q_h \times W_h$

$$\begin{aligned}
\|\mathbf{u} - \mathbf{u}_h\|_0^2 \ &= \ b(\mathbf{u} - \mathbf{u}_h, p - p_h, \mathbf{w} - \mathbf{w}_h;\ \alpha, \beta, \gamma) \\
&= b(\mathbf{u} - \mathbf{u}_h, p - p_h, \mathbf{w} - \mathbf{w}_h;\ \alpha - \alpha_h, \beta - \beta_h, \gamma - \gamma_h) \\
&\leq Cb(\mathbf{u} - \mathbf{u}_h, p - p_h, \mathbf{w} - \mathbf{w}_h; \mathbf{u} - \mathbf{u}_h, p - p_h, \mathbf{w} - \mathbf{w}_h) \\
&\times b(\alpha - \alpha_h, \beta - \beta_h, \gamma - \gamma_h; \alpha - \alpha_h, \beta - \beta_h, \gamma - \gamma_h)
\end{aligned}$$

$$
\begin{aligned}
&\leq Chb(\mathbf{u}-\mathbf{u}_h, p-p_h, \mathbf{w}-\mathbf{w}_h;\\
&\qquad \mathbf{u}-\mathbf{u}_h, p-p_h, \mathbf{w}-\mathbf{w}_h)\|\mathbf{u}-\mathbf{u}_h\|_1\\
&\leq Ch^2\{\|\mathbf{u}\|_2+\|p\|_2+\|\mathbf{w}\|_2\}\|\mathbf{u}-\mathbf{u}_h\|_1
\end{aligned}
$$

With Theorem 3.2 we get

$$\|\mathbf{u}-\mathbf{u}_h\|_0^2 \leq Ch^3\{\|\mathbf{u}\|_2+\|p\|_2+\|\mathbf{w}\|_2\}$$

This completes the proof of the theorem.

Acknowledgments

This research is supported by the National Natural Science Foundation of China(11271106), the natural science foundation of Hebei Province (F20112010 63).

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