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ON THE DIOPHANTINE EQUATION $23^x + 32^y = z^2$

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Abstract: In this paper, we prove that the Diophantine equation $23^x + 32^y = z^2$ has no non-negative integer solution.

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1. Introduction

In 1844, Catalan [2] conjectures that $(3, 2, 2, 3)$ is a unique solution (a, b, x, y) for the Diophantine equation $a^x - b^y = 1$ where a, b, x and y are integers with $\min\{a, b, x, y\} > 1$. In 2004, Mihailescu [4] gave a proof of the conjecture.

In 2007, Acu [1] proved that $(3, 0, 3)$ and $(2, 1, 3)$ are only two solutions (x, y, z) for the Diophantine equation $2^x + 5^y = z^2$ where x, y and z are non-negative integers.

In 2011, Suvarnamani, Singta and Chotchaisthit [8] proved that the two Diophantine equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$ have no non-negative integer solution.

In 2012, Chotchaisthit [3] solved the Diophantine equation of type $4^x + p^y = z^2$ where p is a positive prime number.

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In the same year, Sroysang [5, 6] showed that (i) $(1, 0, 3)$ is a unique solution (x, y, z) for the Diophantine equation $8^x + 19^y = z^2$ where x, y and z are non-negative integers and (ii) $(1, 0, 2)$ is a unique solution (x, y, z) for the Diophantine equation $3^x + 5^y = z^2$ where x, y and z are non-negative integers.

Moreover, Sroysang [7] proved that the Diophantine equation $31^x + 32^y = z^2$ has no non-negative integer solution.

In this paper, we will show that the Diophantine equation $23^x + 32^y = z^2$ has no non-negative integer solution.

2. Preliminaries

In this section, we present two Lemmas by the Catalan's conjecture.

Proposition 2.1. (see [4]) $(3, 2, 2, 3)$ *is a unique solution* (a, b, x, y) *for the Diophantine equation* $a^x - b^y = 1$ *where* a, b, x *and* y *are integers with* $\min\{a, b, x, y\} > 1$.

Lemma 2.2. *The Diophantine equation* $23^x + 1 = z^2$ *has no non-negative integer solution.*

Proof. Suppose that there are non-negative integers x and z such that $23^x + 1 = z^2$. If $x = 0$, then $z^2 = 2$ which is impossible. Then $x \geq 1$. Thus, $z^2 = 23^x + 1 \geq 23^1 + 1 = 24$. Then $z \geq 5$. Now, we consider on the equation $z^2 - 23^x = 1$. By Proposition 2.1, we have $x = 1$. Then $z^2 = 24$. This is a contradiction. Hence, the equation $23^x + 1 = z^2$ has no non-negative integer solution. □

Lemma 2.3. (see [7]) *The Diophantine equation* $1 + 32^y = z^2$ *has no non-negative integer solution.*

3. Results

In this section, we show that the Diophantine equation $23^x + 32^y = z^2$ has no non-negative integer solution.

Theorem 3.1. *The Diophantine equation* $23^x + 32^y = z^2$ *has no non-negative integer solution.*

Proof. Suppose that there are non-negative integers x, y and z such that $23^x + 32^y = z^2$. By Lemma 2.2, we have $y \geq 1$. Thus, z is odd. Then $z = 2t + 1$

for some a non-negative integer t. Thus, $23^x + 32^y = 4(t^2 + t) + 1$. This implies that $23^x \equiv 1 \pmod 4$. Since $23 \equiv 3 \pmod 4$, we have $3^x \equiv 1 \pmod 4$. Then x is even. By Lemma 2.3, we have $x \geq 2$. Then $x = 2k$ for some a positive integer k. Then $32^y = z^2 - 23^{2k}$. Then $2^{5y} = (z - 23^k)(z + 23^k)$. Thus, $z - 23^k = 2^u$ for some a non-negative integer u. Then $z + 23^k = 2^{5y-u}$. Thus, $2(23^k) = 2^{5y-u} - 2^u = 2^u(2^{5y-2u} - 1)$. Then $u = 1$. Then $2^{5y-2} - 1 = 23^k$. Then $2^{5y-2} - 23^k = 1$. Note that $5y - 2 \geq 3$. By Proposition 2.1, we have $k = 1$. Then $2^{5y-2} = 24 = 3(2^3)$. Thus, $32^{y-1} = 3$. This is a contradiction. Therefore, the equation $23^x + 32^y = z^2$ has no non-negative integer solution. □

Corollary 3.2. *The Diophantine equation $23^x + 32^y = w^4$ has no non-negative integer solution.*

Proof. We set $z = w^2$. By Theorem 3.1, the equation $23^x + 32^y = z^2$ has no non-negative integer solution. Hence, the equation $23^x + 32^y = w^4$ has no non-negative integer solution. □

4. Open Problem

We may note that $23 = 2(10)+3$ and $32 = 3(10)+2$. Let $a, b \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. If $m = a(10) + b$ and $n = b(10) + a$, then we may ask what is the set of all solutions (x, y, z) for the Diophantine equation $m^x + n^y = z^2$ where x, y and z are non-negative integers.

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