A NEW BOUND ON POISSON APPROXIMATION
FOR INDEPENDENT GEOMETRIC VARIABLES

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Abstract: The Stein-Chen method is used to improve the bound in [3] to be more appropriate for measuring the accuracy of Poisson approximation with mean \( \lambda = \sum_{i=1}^{n} q_i \) for a sum of independently distributed geometric random variables.

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1. Introduction

Let \( X_1, ..., X_n \) be \( n \) independently distributed geometric random variables, each with probability \( P(X_i = k) = (1 - p_i)^k p_i \) for \( k = 0, 1, ..., \) and let \( W = \sum_{i=1}^{n} X_i \). If \( p_i \)'s are identical to \( p \), then \( W \) has the negative binomial distribution with parameters \( n \in \mathbb{N} \) and \( p \). It is well-known that if all \( q_i = (1 - p_i) \) are small, the distribution of \( W \) can be approximated by the Poisson distribution with mean \( \lambda = E(W) = \sum_{i=1}^{n} q_i p_i^{-1} \). Correspondingly, the distribution function of \( W \) can also be approximated by the Poisson distribution function with mean \( \lambda \). Let
\( P(W \leq w_0) = P(W \leq w_0) \) and \( P_\lambda(w_0) = \sum_{k=0}^{w_0} \frac{\lambda^k e^{-\lambda}}{k!} \) be the distribution function of \( W \) and the Poisson distribution function at \( w_0 \in \mathbb{N} \cup \{0\} \), respectively. In this case, Teerapabolarn and Wongkasem [5] used the Stein-Chen method to give a non-uniform bound for the difference of \( P(W \leq w_0) \) and \( P_\lambda(w_0) \) as follows:

\[
|P(W \leq w_0) - P_\lambda(w_0)| \leq \frac{\lambda^{-1} (e^\lambda - 1)}{\sum_{i=1}^{n} \min \left\{ 1, \frac{1}{p_i(w_0 + 1)} \right\} q_i^2 p_i^{-1}}.
\] (1.1)

Later, Teerapabolarn [3] used this method to obtain a better result of (1.1),

\[
|P(W \leq w_0) - P_\lambda(w_0)| \leq \frac{\lambda^{-1} (e^\lambda - 1)}{\sum_{i=1}^{n} \min \left\{ 1, \frac{1}{p_i(w_0 + 1)} \right\} q_i^2},
\] (1.2)

for \( \lambda = \sum_{i=1}^{n} q_i \) and \( w_0 \in \mathbb{N} \cup \{0\} \). In this study, we focus on improving the non-uniform bound of (1.2) to be more appropriate for measuring the accuracy of this approximation.

## 2. Method

Stein’s method was originally formulated for normal approximation by Stein [2]. It was adapted and applied to the Poisson case by Chen [1], which is referred to as the Stein-Chen method. Following Teerapabolarn [3], Stein’s equation of the Poisson cumulative distribution function with parameter \( \lambda > 0 \) is of the form

\[
h_{w_0}(w) - P_\lambda(w_0) = \lambda f_{w_0}(w + 1) - w f_{w_0}(w),
\] (2.1)

where \( w_0, w \in \mathbb{N} \cup \{0\} \), and for \( h_{w_0}(w) = 1 \) if \( w \leq w_0 \) and \( h_{w_0}(w) = 0 \) if \( w > w_0 \), the solution \( f_{w_0} \) is

\[
f_{w_0}(w) = \begin{cases} (w - 1)! e^{-\lambda} [P_\lambda(w) - \lambda^{-1} (1 - P_\lambda(w))] & \text{if } w \leq w_0, \\ (w - 1)! e^{-\lambda} [P_\lambda(w) - \lambda^{-1} (1 - P_\lambda(w))] & \text{if } w > w_0, \\ 0 & \text{if } w = 0. \end{cases}
\] (2.2)

Note that \( f_{w_0}(w) \geq 0 \) for every \( x_0, x \in \mathbb{N} \cup \{0\} \). The following lemma gives a non-uniform bound of (2.2), which is used to determine the main result.

**Lemma 2.1.** For \( w_0 \in \mathbb{N} \cup \{0\} \) and \( k \in \mathbb{N} \setminus \{1\} \), let \( p_\lambda(w_0) = \frac{e^{-\lambda} \lambda^{w_0}}{w_0!} \). Then the following inequality holds:

\[
\sup_{w \geq k} f_{w_0}(w) \leq \frac{P_\lambda(w_0)(1 - P_\lambda(w_0))}{p_\lambda(w_0 + 1)} \min \left\{ \frac{1}{w_0 + 1}, \frac{1}{k} \right\}.
\] (2.3)
Proof. The first bound of (2.3) is directly obtained from [4]. Thus, we shall show that the second bound of (2.3) holds. For \( k \leq w \leq w_0 \), we have

\[
f_{w_0}(w) \leq \frac{p_{\lambda}(w_0)(1 - p_{\lambda}(w_0))}{p_{\lambda}(w_0 + 1)(w_0 + 1)} \leq \frac{p_{\lambda}(w_0)(1 - p_{\lambda}(w_0))}{p_{\lambda}(w_0 + 1)(k + 1)}. \tag{2.4}
\]

For \( w > w_0 \) and \( w \geq k \), we obtain

\[
f_{w_0}(w) = \mathbb{P}_{\lambda}(w_0)(w - 1)! \sum_{j=w}^{\infty} \frac{\lambda^j}{j!}
\]

\[
\leq \mathbb{P}_{\lambda}(w_0) \left\{ \frac{1}{w} + \frac{\lambda}{w(w + 1)} + \frac{\lambda^2}{w(w + 1)(w + 2)} + \cdots \right\}
\]

\[
\leq \mathbb{P}_{\lambda}(w_0)(w_0 + 1)! \left\{ \frac{1}{(w_0 + 1)!} + \frac{\lambda}{(w_0 + 2)!} + \frac{\lambda^2}{(w_0 + 3)!} + \cdots \right\}
\]

\[
\leq \frac{\mathbb{P}_{\lambda}(w_0)(w_0 + 1)!}{k} \sum_{j=w_0+1}^{\infty} \frac{\lambda^j}{j!}
\]

\[
= \frac{\mathbb{P}_{\lambda}(w_0)(1 - \mathbb{P}_{\lambda}(w_0))}{p_{\lambda}(w_0 + 1)k}. \tag{2.5}
\]

Hence, from (2.4) and (2.5), the second bound of (2.3) holds. \( \Box \)

3. Result

The following theorem presents a new non-uniform bound on the distance \( |\mathbb{P}W(w_0) - \mathbb{P}_{\lambda}(w_0)| \), which can be obtained by using the Stein-Chen method.

**Theorem 3.1.** For \( w_0 \in \mathbb{N} \cup \{0\} \), if \( \lambda = \sum_{i=1}^{n} q_i \) then we have the following:

\[
|\mathbb{P}W(w_0) - \mathbb{P}_{\lambda}(w_0)| \leq \frac{\mathbb{P}_{\lambda}(w_0)(1 - \mathbb{P}_{\lambda}(w_0))}{p_{\lambda}(w_0 + 1)} \sum_{i=1}^{n} \min \left\{ \frac{1}{p_i(w_0 + 1)}, 1 \right\} q_i^2. \tag{3.1}
\]

**Proof.** Teerapabolarn[3] showed that

\[
|\mathbb{P}W(w_0) - \mathbb{P}_{\lambda}(w_0)| \leq \sum_{i=1}^{n} \sum_{k=2}^{(k-1)p_i q_i^k \sup_{w \geq k} f_{w_0}(w)}.
\]

With Lemma 2.1, we have that

\[
|\mathbb{P}W(w_0) - \mathbb{P}_{\lambda}(w_0)| \leq \frac{\mathbb{P}_{\lambda}(w_0)(1 - \mathbb{P}_{\lambda}(w_0))}{p_{\lambda}(w_0 + 1)} \sum_{i=1}^{n} \sum_{k=2}^{(k-1)p_i q_i^k} \min \left\{ \frac{1}{w_0 + 1}, \frac{1}{k} \right\}
\]
\[
\leq \frac{\mathbb{P}_\lambda(w_0)(1 - \mathbb{P}_\lambda(w_0))}{p_\lambda(w_0 + 1)} \sum_{i=1}^{n} \min \left\{ \frac{1}{p_i(w_0 + 1)}, 1 \right\} q_i^2.
\]

Hence, the theorem is proved. \(\square\)

Remark. It is seen that
\[
\frac{\mathbb{P}_\lambda(w_0)(1 - \mathbb{P}_\lambda(w_0))}{p_\lambda(w_0 + 1)} \leq \mathbb{P}_\lambda(w_0) \lambda^{-1}(e^\lambda - 1) < \lambda^{-1}(e^\lambda - 1)
\]
for every \(w_0 \in \mathbb{N} \cup \{0\}\). Therefore, the bound in (3.1) is sharper than the bound in (1.2).

4. Conclusion

The non-uniform bound in the Theorem 3.1 was obtained by improving the bound in (1.2) using the Stein-Chen method. It provides a new general criteria for measuring the accuracy of Poisson approximation to the distribution function of a sum of independently distributed geometric random variables with Poisson mean \(\lambda = \sum_{i=1}^{n} q_i\).

References


