ON APPLICATIONS OF DIFFERENTIAL SUBORDINATION

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Abstract: Using a generalized differential operator we define certain subclasses of analytic functions and study about their inclusion relationships using differential subordination.

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1. Introduction

Let \( A \) denote the class of functions of the form

\[
f(z) := z + \sum_{k=2}^{\infty} a_k z^k, \quad a_k \geq 0 \tag{1.1}
\]

which are analytic in the open unit disk \( U = \{ z \in \mathbb{C} : |z| < 1 \} \). Let \( S, S^*(\alpha) \) and \( K(\alpha) \) denote the subclasses of \( A \) consisting of functions that are univalent, starlike of order \( \alpha \) and convexlike of order \( \alpha \) respectively. Also \( S^*(0) = S^* \) and \( K(0) = K \) are the class of starlike and convex functions defined on \( U \) respectively. For two functions \( f(z) \) given by (1.1) and \( g(z) = z + \sum_{k=2}^{\infty} b_k z^k \),

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the Hadamard product or convolution of $f$ and $g$ is denoted by $(f * g)(z)$, defined as
\[
(f * g)(z) := z + \sum_{k=2}^{\infty} a_k b_k z^k.
\]

For complex numbers $\alpha_1, \alpha_2, \ldots, \alpha_q$ and $\beta_1, \beta_2, \ldots, \beta_s$; $(\beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^-; \mathbb{Z}_0^- = \{0, -1, -2, \ldots\}$ for $j = 1, 2, \ldots, s$), we define the generalized hypergeometric function as
\[
_q F_s(\alpha_1, \alpha_2, \ldots, \alpha_q; \beta_1, \beta_2, \ldots, \beta_s; z) := \sum_{k=0}^{\infty} \frac{(\alpha_1)_k (\alpha_2)_k \ldots (\alpha_q)_k}{(\beta_1)_k (\beta_2)_k \ldots (\beta_s)_k k!} z^k,
\]
where $\mathbb{N}$ denotes the set of all positive integers and $(x)_k$ is the Pochhammer symbol defined in terms of gamma function, as
\[
(x)_k = \frac{\Gamma(x + k)}{\Gamma(x)} = \begin{cases} 1 & \text{if } k = 0 \\ x(x+1) \ldots (x+k-1) & \text{if } k \in \mathbb{N}. \end{cases}
\]

Corresponding to the function $g_{q,s}(\alpha_1, \beta_1; z)$, defined by
\[
g_{q,s}(\alpha_1, \beta_1; z) := z_q F_s(\alpha_1, \alpha_2, \ldots, \alpha_q; \beta_1, \beta_2, \ldots, \beta_s; z),
\]
recently in [9] an operator $\mathcal{D}^m_{\lambda, \mu}(\alpha_1, \beta_1)f(z) : \mathcal{A} \to \mathcal{A}$ has been defined by
\[
\mathcal{D}^0_{\lambda, \mu}(\alpha_1, \beta_1)f(z) := f(z) * g_{q,s}(\alpha_1, \beta_1; z), \\
\mathcal{D}^1_{\lambda, \mu}(\alpha_1, \beta_1)f(z) := (1 - \lambda + \mu)(f(z) * g_{q,s}(\alpha_1, \beta_1; z)) \\
+ (\lambda - \mu)z(f(z) * g_{q,s}(\alpha_1, \beta_1; z))' \\
+ \lambda \mu z^2(f(z) * g_{q,s}(\alpha_1, \beta_1; z))'',
\]
\[
\mathcal{D}^m_{\lambda, \mu}(\alpha_1, \beta_1)f(z) := \mathcal{D}^1_{\lambda, \mu}(\mathcal{D}^{m-1}_{\lambda, \mu}(\alpha_1, \beta_1)f(z)),
\]
where $0 \leq \mu \leq \lambda \leq 1$ and $m \in \mathbb{N}_0$. By using the above definition, we can find that
\[
\mathcal{D}^m_{\lambda, \mu}(\alpha_1, \beta_1)f(z) = z \\
+ \sum_{k=2}^{\infty} [1 + (k - 1)(\lambda - \mu + k \mu \lambda)]^m \frac{(\alpha_1)_{k-1}(\alpha_2)_{k-1} \ldots (\alpha_q)_{k-1}}{(\beta_1)_{k-1}(\beta_2)_{k-1} \ldots (\beta_s)_{k-1}(k-1)!} a_k z^k.
\]
For brevity, let us take
\[
B_k = \frac{(\alpha_1)_{k-1}(\alpha_2)_{k-1} \ldots (\alpha_q)_{k-1}}{(\beta_1)_{k-1}(\beta_2)_{k-1} \ldots (\beta_s)_{k-1}(k-1)!}.
\]
Hence we have
\[ \mathcal{D}_{\lambda,\mu}^m (\alpha_1, \beta_1) f(z) = z + \sum_{k=2}^{\infty} [1 + (k-1)(\lambda - \mu + k\mu\lambda)]^m B_k a_k z^k. \]

For suitable values of \( \alpha, s, \beta, q, s, \lambda \) and \( \mu \) we can deduce several operators \([1, 6, 14]\) as a special case of this operator. Also a simple computation shows that
\[
(1-\gamma)\mathcal{D}_{\lambda,\mu}^m (\alpha_1, \beta_1) f(z) + \gamma z [\mathcal{D}_{\lambda,\mu}^m (\alpha_1, \beta_1) f(z)]' = \gamma \alpha_1 \mathcal{D}_{\lambda,\mu}^m (\alpha_1 + 1, \beta_1) f(z) \tag{1.2}
\]

\[-(\gamma \alpha_1 - 1) \mathcal{D}_{\lambda,\mu}^m (\alpha_1, \beta_1) f(z).\]

Let \( f(z) \) and \( g(z) \) be analytic in the unit disc \( U \). Then \( f(z) \) is said to be subordinate to \( g(z) \) in \( U \), if there exists a Schwarz function \( w(z) \), analytic in \( U \) with \( w(0) = 0 \) and \( |w(z)| < 1 \) for \( z \in U \), such that \( f(z) = g(w(z)) \). We denote it as \( f \prec g \). Further if \( g(z) \) is univalent then we write \( f \prec g \) if \( f(0) = g(0) \) and \( f(U) \subset g(U) \).

**Definition 1.1.** Let \( h(z) \) be an analytic convex univalent function in \( U \) with \( h(0) = 1 \) and \( \Re\{h(z)\} > 0 \) for \( z \in U \). Let \( A(\alpha, \beta, \gamma, \lambda, \mu, m, h) \) denote the subclass of \( A \) consisting of functions \( f(z) \) which satisfy the condition
\[
z [\mathcal{D}_{\lambda,\mu}^m (\alpha_1, \beta_1) f(z)]' + \gamma z^2 [\mathcal{D}_{\lambda,\mu}^m (\alpha_1, \beta_1) f(z)]'' < h(z)
\]
for some \( \gamma(0 \leq \gamma \leq 1) \) and for all \( z \in U \).

**Definition 1.2.** Let \( h(z) \) be an analytic convex univalent function in \( U \) with \( h(0) = 1 \) and \( \Re\{h(z)\} > 0 \) for \( z \in U \). Let \( B(\alpha, \beta, \gamma, \lambda, \mu, m, h) \) denote the subclass of \( A \) consisting of functions \( f(z) \) which satisfy the condition
\[
(1 - \gamma) \frac{\mathcal{D}_{\lambda,\mu}^m (\alpha_1, \beta_1) f(z)}{z} + \gamma [\mathcal{D}_{\lambda,\mu}^m (\alpha_1, \beta_1) f(z)]' < h(z)
\]
for some \( \gamma(0 \leq \gamma \leq 1) \) and for all \( z \in U \).

**Definition 1.3.** Let \( h(z) \) be an analytic convex univalent function in \( U \) with \( h(0) = 1 \) and \( \Re\{h(z)\} > 0 \) for \( z \in U \). Let \( C(\alpha, \beta, \gamma, \lambda, \mu, m, h) \) denote the subclass of \( A \) consisting of functions \( f(z) \) which satisfy the condition
\[
[\mathcal{D}_{\lambda,\mu}^m (\alpha_1, \beta_1) f(z)]' + \gamma z [\mathcal{D}_{\lambda,\mu}^m (\alpha_1, \beta_1) f(z)]'' < h(z)
\]
for some \( \gamma(0 \leq \gamma \leq 1) \) and for all \( z \in U \).
Note that special cases of \( A(\alpha_1, \beta_1, \gamma, \lambda, \mu, m, h) \), \( B(\alpha_1, \beta_1, \gamma, \lambda, \mu, m, h) \) and \( C(\alpha_1, \beta_1, \gamma, \lambda, \mu, m, h) \) include \( S^*, K, S^*(\alpha), K(\alpha) \) and many other subclasses of \( A \) which were defined earlier in [11, 12, 13]. By specializing the parameters we get the corresponding classes containing Hohlov operator, Ruscheweyh operator, fractional calculus operator, Sălăgean derivative operator, Libera-Bernardi-Livingston integral operator, Dziok-Srivatsava operator and the operator studied in [15].

### 2. Preliminaries

To prove our main results we need the following lemmas.

**Lemma 2.1.** [10, p. 81] Let \( h \) be analytic, univalent and convex in \( U \) with \( h(0) = 1 \) and \( \Re\{\beta h(z) + \gamma\} > 0, (\beta, \gamma \in \mathbb{C}, z \in U) \). If \( p \) is analytic in \( U \) with \( p(0) = h(0) \), then

\[
p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} < h(z)
\]

implies that

\[
p(z) \prec h(z) \quad (z \in U).
\]

**Lemma 2.2.** [10, p. 71] Let \( h \) be analytic, univalent and convex in \( U \) with \( h(0) = 1 \). Also let \( p \) be analytic in \( U \) with \( p(0) = h(0) \). If

\[
p(z) + \frac{zp'(z)}{\gamma} < h(z)
\]

then

\[
p(z) \prec q(z) \prec h(z),
\]

where

\[
q(z) = \frac{\gamma}{z^\gamma} \int_0^z t^{\gamma-1} h(t)dt \quad (z \in U, \Re\{\gamma\} \geq 0; \gamma \neq 0).
\]

### 3. Inclusion Relations

**Theorem 3.1.** For \( \alpha_1 \geq 1 \),

\[
A(\alpha_1 + 1, \beta_1, \gamma, \lambda, \mu, m, h) \subset A(\alpha_1, \beta_1, \gamma, \lambda, \mu, m, h).
\]
Proof. Let \( f(z) \in A(\alpha_1 + 1, \beta_1, \gamma, \lambda, \mu, m, h) \) and let
\[
p(z) := \frac{z[D^m_{\lambda, \mu}(\alpha_1, \beta_1)f(z)]'' + \gamma z^2[D^m_{\lambda, \mu}(\alpha_1, \beta_1)f(z)]''}{(1 - \gamma)D^m_{\lambda, \mu}(\alpha_1, \beta_1)f(z) + \gamma z[D^m_{\lambda, \mu}(\alpha_1, \beta_1)f(z)]''}.
\]
By differentiating (1.2) we have,
\[
p(z) + (a - 1) = \frac{\gamma \alpha_1 z [D^m_{\lambda, \mu}(\alpha_1 + 1, \beta_1)f(z)]' + (1 - \gamma) \alpha_1 D^m_{\lambda, \mu}(\alpha_1, \beta_1)f(z)}{(1 - \gamma)D^m_{\lambda, \mu}(\alpha_1, \beta_1)f(z) + \gamma z[D^m_{\lambda, \mu}(\alpha_1, \beta_1)f(z)]''}.
\]
Taking logarithmic differentiation on both sides we get
\[
p(z) + \frac{zp'(z)}{p(z) + (a - 1)} = \frac{z[D^m_{\lambda, \mu}(\alpha_1 + 1, \beta_1)f(z)]' + \gamma z^2[D^m_{\lambda, \mu}(\alpha_1 + 1, \beta_1)f(z)]''}{(1 - \gamma)D^m_{\lambda, \mu}(\alpha_1, \beta_1)f(z) + \gamma z[D^m_{\lambda, \mu}(\alpha_1, \beta_1)f(z)]''}.
\]
As \( f(z) \in A(\alpha_1 + 1, \beta_1, \gamma, \lambda, \mu, m, h) \) we have
\[
p(z) + \frac{zp'(z)}{p(z) + (a - 1)} \prec h(z).
\]
It follows from Lemma 2.1 that
\[
p(z) \prec h(z)
\]
for \( \alpha_1 \geq 1 \). Thus \( f \in A(\alpha_1, \beta_1, \gamma, \lambda, \mu, m, h) \).

\begin{theorem}
If \( f \in A(\alpha_1, \beta_1, \gamma, \lambda, \mu, m, h) \) for \( \alpha_1 \geq 1 \), then \( F_c(f) \in A(\alpha_1, \beta_1, \gamma, \lambda, \mu, m, h) \), where \( F_c \) is the integral operator defined by
\[
F_c(f) = F_c(f)(z) := \frac{c + 1}{z^c} \int_0^z t^{c-1} f(t) \, dt \quad (c \geq 0).
\]
\end{theorem}

Proof. Let \( f \in A(\alpha_1, \beta_1, \gamma, \lambda, \mu, m, h) \) and
\[
p(z) := \frac{z[D^m_{\lambda, \mu}(\alpha_1, \beta_1)F_c(f(z))]' + \gamma z^2[D^m_{\lambda, \mu}(\alpha_1, \beta_1)F_c(f(z))]''}{(1 - \gamma)D^m_{\lambda, \mu}(\alpha_1, \beta_1)F_c(f(z)) + \gamma z[D^m_{\lambda, \mu}(\alpha_1, \beta_1)F_c(f(z))]''}.
\]
A simple computation using (3.1) yields that
\[
z(F_c(f)(z))' + cF_c(f)(z) = (c + 1)f(z)
\]
and so
\[ D^m_{\lambda, \mu}(\alpha_1, \beta_1)(zF_c(f(z))') + cD^m_{\lambda, \mu}(\alpha_1, \beta_1)F_c(f)(z) = (c + 1)D^m_{\lambda, \mu}(\alpha_1, \beta_1)f(z). \]

By making use of the identity
\[ z[D^m_{\lambda, \mu}(\alpha_1, \beta_1)F_c(f(z))]' = D^m_{\lambda, \mu}(\alpha_1, \beta_1)(zF_c(f(z))'). \]
we get
\[ z[D^m_{\lambda, \mu}(\alpha_1, \beta_1)F_c(f(z))]' + cD^m_{\lambda, \mu}(\alpha_1, \beta_1)F_c(f)(z) = (c + 1)D^m_{\lambda, \mu}(\alpha_1, \beta_1)f(z) \tag{3.2} \]
Differentiating (3.2), we have
\[ p(z) + c \]
\[ = (c + 1) \left( \frac{(1 - \gamma)D^m_{\lambda, \mu}(\alpha_1, \beta_1)f(z) + \gamma z[D^m_{\lambda, \mu}(\alpha_1, \beta_1)f(z)]'}{(1 - \gamma)D^m_{\lambda, \mu}(\alpha_1, \beta_1)F_c(f(z)) + \gamma z[D^m_{\lambda, \mu}(\alpha_1, \beta_1)F_c(f(z))]'} \right). \tag{3.3} \]

Taking logarithmic differentiation of (3.3), we get
\[ p(z) + \frac{zp'(z)}{p(z) + c} = \frac{z[D^m_{\lambda, \mu}(\alpha_1, \beta_1)f(z)]' + \gamma z^2[D^m_{\lambda, \mu}(\alpha_1, \beta_1)f(z)]''}{(1 - \gamma)D^m_{\lambda, \mu}(\alpha_1, \beta_1)f(z) + \gamma z[D^m_{\lambda, \mu}(\alpha_1, \beta_1)f(z)]'}. \tag{3.4} \]

By applying Lemma 2.1 in (3.3), it follows that
\[ p(z) \prec h(z) \quad (z \in U). \]

Hence \( F_c(f) \in A(\alpha_1, \beta_1, \gamma, \lambda, \mu, m, h). \)

Special cases of Theorems 3.1 and 3.2 include the results which were given in [2, 11, 12, 13]. Interestingly for \( q = 2, s = 1, m = 0, \alpha_1 = \beta_1 = \alpha_2 = 1, h(z) = \frac{1 + z}{1 - z} \) and \( \gamma = 0 \) in Theorem 3.1 we obtain \( K \subset S^* \).

**Theorem 3.3.** \( f \in A(\alpha_1, \beta_1, \gamma, \lambda, \mu, m, h) \) if and only if \( \gamma zf' + (1 - \gamma)f \in A(\alpha_1, \beta_1, 0, \lambda, \mu, m, h). \)

**Proof.** Let \( f \in A(\alpha_1, \beta_1, \gamma, \lambda, \mu, m, h) \) and \( g(z) = \gamma zf' + (1 - \gamma)f \). Using the definition of \( D^m_{\lambda, \mu}(\alpha_1, \beta_1)f(z) \) and a property of the Hadamard product, we find that \( g \in A(\alpha_1, \beta_1, 0, \lambda, \mu, m, h) \). Converse is obvious.

Special cases of the Theorem 3.3 includes results which were in [11, 12].

**Theorem 3.4.** If \( f \in A(\alpha_1, \beta_1, \gamma, \lambda, \mu, m, h) \) then \( \gamma f + (1 - \gamma) \int_0^z \frac{f(t)}{t} dt \in A(\alpha_1, \beta_1, 1, \lambda, \mu, m, h). \)
Proof. Let \( f \in A(\alpha_1, \beta_1, \gamma, \lambda, \mu, m, h) \) then from Theorem 3.3
\[
g(z) = zf' + (1 - \gamma)f \in A(\alpha_1, \beta_1, 0, \lambda, \mu, m, h).
\]
It can be easily seen that \( f \in A(\alpha_1, \beta_1, 1, \lambda, \mu, m, h) \) if and only if \( zf' \in A(\alpha_1, \beta_1, 0, \lambda, \mu, m, h) \). Applying this result for \( g(z) \), we see that
\[
\gamma f + (1 - \gamma) \int_0^z \frac{f(t)}{t} dt \in A(\alpha_1, \beta_1, 1, \lambda, \mu, m, h).
\]

\[\square\]

Theorem 3.5.
\[
B(\alpha_1 + 1, \beta_1, \gamma, \lambda, \mu, m, h) \subset B(\alpha_1, \beta_1, \gamma, \lambda, \mu, m, h).
\]

Proof. Let \( f \in B(\alpha_1 + 1, \beta_1, \gamma, \lambda, \mu, m, h) \) and
\[
p(z) = (1 - \gamma)[D_{\lambda, \mu}^m(\alpha_1, \beta_1)f(z)]' + \gamma[D_{\lambda, \mu}^m(\alpha_1, \beta_1)f(z)]'.
\]
Taking \( \gamma = 1 \) in (1.2) we get
\[
z[D_{\lambda, \mu}^m(\alpha_1, \beta_1)f(z)]' = \alpha_1 D_{\lambda, \mu}^m(\alpha_1 + 1, \beta_1)f(z) - (\alpha_1 - 1) D_{\lambda, \mu}^m(\alpha_1, \beta_1)f(z). \tag{3.5}
\]
Using (3.5) and the differentiation of (3.5), we get
\[
p(z) + \frac{zp'(z)}{\alpha_1} = (1 - \gamma)\frac{D_{\lambda, \mu}^m(\alpha_1 + 1, \beta_1)f(z)}{z} + \gamma[D_{\lambda, \mu}^m(\alpha_1 + 1, \beta_1)f(z)]'. \tag{3.6}
\]
By applying Lemma 2.2 in (3.6), we obtain
\[
p(z) \prec q(z) \quad (z \in U).
\]
Hence the result follows. \(\square\)

Theorem 3.6. If \( f \in B(\alpha_1, \beta_1, \gamma, \lambda, \mu, m, h) \), then
\[
F_c(f) \in B(\alpha_1, \beta_1, \gamma, \lambda, \mu, m, h).
\]

Proof. Assume \( f \in B(\alpha_1, \beta_1, \gamma, \lambda, \mu, m, h) \) and
\[
p(z) = (1 - \gamma)\frac{D_{\lambda, \mu}^m(\alpha_1, \beta_1)f_c(f(z))}{z} + \gamma[D_{\lambda, \mu}^m(\alpha_1, \beta_1)f_c(f(z))]'.
\]
Differentiating (3.2) we have

$$p(z) + \frac{zp'(z)}{c+1} = (1 - \gamma) \frac{D^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z)}{z} + \gamma[D^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z)]'. \tag{3.7}$$

By applying Lemma 2.2 in (3.7) we get

$$p(z) \prec h(z)$$

and hence

$$F_c(f) \in B(\alpha_1, \beta_1, \gamma, \lambda, \mu, m, h).$$

\[\square\]

**Theorem 3.7.** $C(\alpha_1 + 1, \beta_1, \gamma, \lambda, \mu, m, h) \subseteq C(\alpha_1, \beta_1, \gamma, \lambda, \mu, m, h)$.

**Proof.** Let $f \in C(\alpha_1 + 1, \beta_1, \gamma, \lambda, \mu, m, h)$ and

$$p(z) = [D^m_{\lambda,\mu}(\alpha_1 + 1, \beta_1)f(z)]' + \gamma z[D^m_{\lambda,\mu}(\alpha_1 + 1, \beta_1)f(z)]''. \tag{3.8}$$

Differentiating (3.5), we have

$$p(z) + \frac{zp'(z)}{\alpha_1} = [D^m_{\lambda,\mu}(\alpha_1 + 1, \beta_1)F_c(f(z))]' + \gamma z[D^m_{\lambda,\mu}(\alpha_1 + 1, \beta_1)F_c(f(z))]''. \tag{3.9}$$

Applying Lemma 2.2 in (3.8), we get

$$p(z) \prec h(z) \quad (z \in U)$$

and the result now follows. \[\square\]

**Theorem 3.8.** If $f \in C(\alpha_1, \beta_1, \gamma, \lambda, \mu, m, h)$, then

$$F_c(f) \in C(\alpha_1, \beta_1, \gamma, \lambda, \mu, m, h).$$

**Proof.** Let $f \in C(\alpha_1, \beta_1, \gamma, \lambda, \mu, m, h)$ and

$$p(z) = [D^m_{\lambda,\mu}(\alpha_1, \beta_1)F_c(f(z))]' + \gamma z[D^m_{\lambda,\mu}(\alpha_1, \beta_1)F_c(f(z))]''.$$

Differentiating (3.2) we get

$$p(z) + \frac{zp'(z)}{c+1} = [D^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z)]' + \gamma z[D^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z)]''. \tag{3.9}$$

A simple application of Lemma 2.2 will give the desired result. \[\square\]
Theorem 3.9. $f \in C(\alpha_1, \beta_1, \gamma, \lambda, \mu, m, h)$ if and only if $zf' \in B(\alpha_1, \beta_1, \gamma, \lambda, \mu, m, h)$.

Proof. Using the equality
\[ z[D^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z)]' = D^m_{\lambda,\mu}(\alpha_1, \beta_1)(zf'(z)). \]

We see that
\[
(1 - \gamma) \frac{D^m_{\lambda,\mu}(\alpha_1, \beta_1)zf'(z)}{z} + \gamma[D^m_{\lambda,\mu}(\alpha_1, \beta_1)zf'(z)]' = [D^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z)]' + \gamma z[D^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z)]''
\]
which implies the required result.

Theorem 3.10. $C(\alpha_1, \beta_1, \gamma, \lambda, \mu, m, h) \subset B(\alpha_1, \beta_1, \gamma, \lambda, \mu, m, h)$.

Proof. Let $f \in C(\alpha_1, \beta_1, \gamma, \lambda, \mu, m, h)$ and
\[ p(z) = (1 - \gamma) \frac{D^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z)}{z} + \gamma[D^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z)]'. \]

Hence
\[ p(z) + zp'(z) = [D^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z)]' + \gamma z[D^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z)]'' \]
and the result follows as an application of Lemma 2.2.

Theorem 3.11. For $\gamma > \delta \geq 0$,
\[ B(\alpha_1, \beta_1, \gamma, \lambda, \mu, m, h) \subset B(\alpha_1, \beta_1, \delta, \lambda, \mu, m, h). \]

Proof. Let $f \in B(\alpha_1, \beta_1, \gamma, \lambda, \mu, m, h)$ and $p(z) = \frac{D^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z)}{z}$.

When $\delta = 0$, we have
\[
p(z) + zp'(z) = (1 - \gamma) \frac{D^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z)}{z} + \gamma[D^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z)]' \quad (3.10)
\]
Hence the result follows as an application of Lemma 2.2 in (3.10), when $\delta = 0$. Suppose $\delta \neq 0$. Since $f \in B(\alpha_1, \beta_1, \gamma, \lambda, \mu, m, h)$, we have
\[
(1 - \gamma) \frac{D^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z)}{z} + \gamma[D^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z)]' \in h(U) \quad (z \in U).
\]
But \( \frac{D^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z)}{z} \) ∈ \( h(U) \) and \( h(U) \) is convex. Also

\[
(1-\delta) \frac{D^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z)}{z} + \delta[D^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z)]' = (1-\frac{\delta}{\gamma}) \frac{D^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z)}{z}
\]

\[
+ \frac{\delta}{\gamma} \left[(1-\gamma) \frac{D^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z)}{z} \gamma[D^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z)]' \right].
\]

Therefore we have

\[
(1-\delta) \frac{D^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z)}{z} + \delta[D^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z)]' \in h(U).
\]

Hence the result follows.

\[\square\]

**Theorem 3.12.** For \( \gamma > \delta \geq 0 \),

\[
C(\alpha_1, \beta_1, \gamma, \lambda, \mu, m, h) \subset C(\alpha_1, \beta_1, \delta, \lambda, \mu, m, h).
\]

**Proof.** Let \( f(z) \in C(\alpha_1, \beta_1, \gamma, \lambda, \mu, m, h) \) and \( p(z) = [D^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z)]' \).

When \( \delta = 0 \), we have

\[
p(z) + \gamma z p'(z) = [D^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z)]' + \gamma z[D^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z)]''.
\]

Hence the result follows as an application of Lemma 2.2, when \( \delta = 0 \).

Suppose \( \delta \neq 0 \). Then

\[
[D^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z)]' + \gamma z[D^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z)]'' \in h(U) \quad (z \in U).
\]

Note that

\[
[D^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z)]' + \delta z[D^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z)]''
\]

\[
= (1-\frac{\delta}{\gamma})[D^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z)]' + \frac{\delta}{\gamma} [D^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z)]' + \gamma z[D^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z)]''.
\]

As \( h(U) \) is convex and \( \frac{\delta}{\gamma} < 1 \) we have

\[
[D^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z)]' + \delta z[D^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z)]'' \in h(U) \quad (z \in U)
\]

and hence the result follows. \[\square\]
References


