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THE SOLUTION OF EULER-CAUCHY EQUATION EXPRESSED BY DIFFERENTIAL OPERATOR USING LAPLACE TRANSFORM

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Abstract: It is well known fact that the Laplace transform is useful in solving linear ordinary differential equations with constant coefficients such as free/forced oscillations, but in the case of differential equation with variable coefficients is not. In here, we would like to propose the Laplace transform of Euler-Cauchy equation with variable coefficients, and find the solution of Euler-Cauchy equation represented by the differential operator using Laplace transform. The purpose of this research is to make an application to its difference equation and oscillation.

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1. Introduction

Euler-Cauchy equations are ODEs of the form $t^2y'' + aty' + by = 0$ with given constants a and b and unknown $y(t)$ (see [8]). The equation appears in a numbers of physics and engineering applications, such as when solving Laplace's equation in a polar coordinates, describing time-harmonic vibrations of a thin

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elastics rod, boundary value problem in spherical coordinates and so on.

Several researches have been pursued for the stability, the oscillation (see [1]-[4], [11]) and Laplace transform (see [5], [9]-[10]) of differential equations with variable coefficients. Through many trials, they found that the Laplace method is not appropriate for ODEs with variable coefficients, but many applications to its difference equation, the discrete equivalent of differential equations, are still stayed. Using the property of $L\{tf(t)\} = -F'(s)$ for $L(f) = F(S)$ and $L(f)$ is Laplace transform of a function f, we would like to propose the Laplace transform of Euler-Cauchy equation, and find the solution of Euler-Cauchy equation represented by differential operator using Laplace transform. The purpose of this research is to make an application to its difference equation/z transform and oscillation.

In this paper, we have showed that a solution of Euler-Cauchy equation can be expressed by

$$y = my(0)e^{\frac{a}{2}t} cos\sqrt{\frac{b}{m} - \frac{a^2}{4}}\ t$$
$$+ \{ay(0)(m/2-1) + my'(0)\}\ (\ 1/\sqrt{\frac{b}{m} - \frac{a^2}{4}}\ )e^{\frac{a}{2}t} sin\sqrt{\frac{b}{m} - \frac{a^2}{4}}\ t$$

where $d/ds = m$.

2. The Solution of Euler-Cauchy Equation using Laplace Transform

Euler-Cauchy equations have the form $t^2y'' + aty' + by = 0$, and let us check the Laplace transform of Euler-Cauchy equation represented by the second derivative. Remark that

$$L(ty') = -Y - s\frac{dY}{ds}$$

and

$$\begin{aligned} L(t^2y'') =& \frac{d^2}{ds^2}[s^2Y - sy(0) - y'(0)] \\ =& -\frac{d}{ds}[2sY + s^2\frac{dY}{ds} - y(0)] \\ =& 2Y + 4s\frac{dY}{ds} + s^2\frac{d^2Y}{ds^2}. \end{aligned}$$

Hence, the Laplace transform of Euler-Cauchy equation is expressed by

$$s^2\frac{d^2Y}{ds^2} + (4s - as)\frac{dY}{ds} + (b - a + 2)Y = 0 \qquad (*)$$

where $Y = L(y) = F(s)$ and $L(y)$ is the Laplace transform of y.

By the Frobenius Method (see [7], p. 182), the Euler-Cauchy equation has a basis of the form $x^{r_1}a_0$ and $x^{r_2}b_0$ where a_0 and b_0 are constants, and $r^2 + (3 - a)r + (b - a + 2) = 0$.

Next we would like to apply the method of reduction of order credited J. L. Lagrange to the equation (*).

Lemma 1. *The equation (*) has the form of a basis y_1 and*

$$y_1 \int s^{a-4}/y_1^2 \, ds$$

of the solutions. In particular, if $b = (a-1)^2/4$, *then (*) has bases*

$$s^{\frac{a-3}{2}}, \; s^{a-3/2} \, ln|s|.$$

Proof. We assume a solution y_1 of the equation on an open interval I to be known and want to find a basis. Let us find a second solution y_2. We substitute

$$y = y_2 = uy_1, \; y' = u'y_1 + uy_1', \; y'' = u''y_1 + 2u'y_1' + uy_1''$$

into the equation (*). This gives

$$u''y_1s^2 + u'\{2y_1's^2 + (4-a)sy_1\} + u\{y_1''s^2 + (4-a)sy_1' + (b-a+2)y_1\} = 0.$$

As we divide by y_1s^2 and put $u' = v$ for $y_1''s^2 + (4-a)sy_1' + (b-a+2)y_1 = 0$, we have

$$v' + v \, \{\frac{2y_1'}{y_1} + \frac{4-a}{s}\} = 0.$$

By the the simple calculation, we get

$$u = \int s^{a-4}/y_1^2 \, ds,$$

and so

$$y_2 = y_1u = y_1 \int s^{a-4}/y_1^2 \, ds.$$

Since $u > 0$, the quotient $y_2/y_1 = u = \int vds$ cannot be constant, so that y_1 and y_2 form a basis of solutions.

Next, the second statement is immediately followed from

$$y_1 = s^{\frac{a-3}{2}}.$$

□

Recall that

$$L^{-1}\{\frac{1}{(s-a)^2+w^2}\} = \frac{1}{w}e^{at}sinwt$$

and

$$L^{-1}\{\frac{s-a}{(s-a)^2+w^2}\} = e^{at}coswt$$

for L^{-1} is the inverse Laplace transform. Now that let us go to the theorem 2.

Theorem 2. *The solution of Euler-Cauchy equation* $t^2y'' + aty' + by = 0$ *can be expressed by*

$$y = my(0)e^{\frac{a}{2}t}cos\sqrt{\frac{b}{m} - \frac{a^2}{4}}\ t$$
$$+ \{ay(0)(m/2-1) + my'(0)\}\ (\ 1/\sqrt{\frac{b}{m} - \frac{a^2}{4}}\ )e^{\frac{a}{2}t}sin\sqrt{\frac{b}{m} - \frac{a^2}{4}}\ t$$

where $d/ds = m$*.*

Proof. Taking Laplace transform on both sides of the above equation, we get

$$L\{t^2\frac{dy^2}{dt^2}\} + aL\{t\frac{dy}{dt}\} + bL(y) = 0.$$

Using the derivative property of Laplace transform, we have

$$\frac{d^2}{ds^2}L(\frac{dy^2}{dt^2}) - a\frac{d}{ds}L(\frac{dy}{dt}) + bL(y) = 0.$$

By the Laplace transform of derivatives, we obtain

$$\frac{d^2}{ds^2}[s^2Y - sy(0) - y'(0)] - a\frac{d}{ds}[sY - y(0)] + bY = 0$$

for $Y = L(y) = F(s)$. Let us put $d/ds = m$. Then

$$m^2[s^2Y - sy(0) - y'(0)] - am[sY - y(0)] + bY = 0$$

and arranging the equation,

$$Y = \frac{sm^2y(0) + m^2y'(0) - amy(0)}{ms^2 - ams + b}$$

for $ms^2 - ams + b \neq 0$. What we want to find is $y(t)$ for all values of t. We have

$$y = L^{-1}(Y)$$

$$
\begin{aligned}
=&L^{-1}[\frac{sm^2y(0)+m^2y'(0)-amy(0)}{ms^2-ams+b}]\\
=&L^{-1}[\frac{sm^2y(0)+m^2y'(0)}{ms^2-ams+b}]+L^{-1}[\frac{-amy(0)}{ms^2-ams+b}]\\
=&L^{-1}[\frac{smy(0)+my'(0)}{s^2-as+b/m}]-aL^{-1}[\frac{y(0)}{s^2-as+b/m}]\\
=&L^{-1}[\frac{(s-\frac{a}{2})my(0)+\frac{am}{2}y(0)+my'(0)}{(s-\frac{a}{2})^2+\frac{b}{m}-\frac{a^2}{4}}]\\
&-ay(0)L^{-1}[\frac{1}{(s-\frac{a}{2})^2+\frac{b}{m}-\frac{a^2}{4}}].
\end{aligned}
$$

As we scan a table of Laplace transforms, we see that

$$
\begin{aligned}
y =&my(0)e^{\frac{a}{2}t}cos\sqrt{\frac{b}{m}-\frac{a^2}{4}}\;t\\
&+m\{\frac{a}{2}y(0)+y'(0)\}\;(\;1/\sqrt{\frac{b}{m}-\frac{a^2}{4}}\;)e^{\frac{a}{2}t}sin\sqrt{\frac{b}{m}-\frac{a^2}{4}}\;t\\
&-ay(0)(\;1/\sqrt{\frac{b}{m}-\frac{a^2}{4}}\;)e^{\frac{a}{2}t}sin\sqrt{\frac{b}{m}-\frac{a^2}{4}}\;t,
\end{aligned}
$$

and hence, we have

$$
\begin{aligned}
y =\;& my(0)e^{\frac{a}{2}t}cos\sqrt{\frac{b}{m}-\frac{a^2}{4}}\;t\\
&+\;\{ay(0)(m/2-1)+my'(0)\}\;(\;1/\sqrt{\frac{b}{m}-\frac{a^2}{4}}\;)e^{\frac{a}{2}t}sin\sqrt{\frac{b}{m}-\frac{a^2}{4}}\;t. \quad \square
\end{aligned}
$$

If the continuous signal $f(t)$ is sampled at intervals of time, T, we obtain a sequence of sampled values $f(k)$, $k \in N$. Since the alternated representation of the equation (*) is

$$
Y=\frac{sm^2y(0)+m^2y'(0)-amy(0)}{ms^2-ams+b}\;,\;(d/ds=m) \qquad (**)
$$

it is clear that the equation

$$
(**) \approx Tf^*(t)=T\sum_{k=0}^{\infty} f(k)\delta(t-kT)
$$

where $f^*(t)$ is a weighted impulses. On the other hand, the result of (see [6]) shows that if the Euler-Cauchy equation has a complex characteristic roots, it is oscillatory on T.

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